Asymptotic stability of 2-domain walls for the Landau-Lifshitz-Gilbert equation in a nanowire with Dzyaloshinskii-Moriya interaction

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Abstract

We consider a ferromagnetic nanowire, with an energy functional E with easy-axis in the direction e_1, and which takes into account the Dzyaloshinskii-Moriya interaction. We consider configurations of the magnetization which are perturbations of two well separated domain wall, and study their evolution under the Landau-Lifshitz-Gilbert flow associated to E.

Our main result is that, if the two walls have opposite speed, these configurations are asymptotically stable, up to gauges intrinsic to the invariances of the energy E. Our analysis builds on the framework developed in [4], taking advantage that it is amenable to space localisation.

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1. INTRODUCTION

1.1. A model for a ferromagnetic nanowire

We model a ferromagnetic nanowire by a straight line $\mathbb{R}e_1 \subset \mathbb{R}^3$ (of infinite length) where

$$e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

is the canonical basis of $\mathbb{R}^3$. The magnetization $m = (m_1, m_2, m_3) : \mathbb{R} \to \mathbb{S}^2$ of this nanowire takes its values into the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, and we associate to it the energy functional

$$E_\gamma(m) = \frac{1}{2}\int_{\mathbb{R}} |\partial_x m|^2 + 2\gamma\partial_x m \cdot (e_1 \wedge m) + (1 - m_1^2)\, dx, \tag{1.1}$$

where x is the variable in direction e_1 of the nanowire and $\gamma \in \mathbb{R}$ is a given constant with $|\gamma| < 1$; it will be convenient to denote

$$\Gamma := \sqrt{1-\gamma^2}.$$

Here, $\cdot$ and $\wedge$ are the scalar and cross product in $\mathbb{R}^3$. The term with γ accounts for the Dzyaloshinskii-Moriya interaction. We refer to [4] where this model was derived from the full 3D system by Γ-convergence in a special regime.

We are interested in the evolution of the magnetization under the Landau-Lifshitz-Gilbert flow associated to E_γ, that is the equation:

$$\partial_t m = m \wedge H(m) - \alpha m \wedge (m \wedge H(m)), \tag{LLG}$$

where now $m : I \times \mathbb{R} \to \mathbb{S}^2$ is the time dependent magnetization (I is interval of time of $\mathbb{R}$), $\alpha > 0$ is the damping coefficient, and the magnetic field H is given by

$$H(m) = -\delta E_\gamma(m) + h(t)e_1.$$

$\delta E_\gamma(m)$ is the variation of the energy, which writes

$$\delta E_\gamma(m) = -\partial^2_{xx} m - 2\gamma e_1 \wedge \partial_x m + m_2 e_2 + m_3 e_3.$$

(recall that $m_1^2 + m_2^2 + m_3^2 = 1$). Finally, the function $h : I \to \mathbb{R}$ is the (given) intensity of an applied external field, which we stress that it depends solely on the time variable t, and is oriented on the axis e_1.

The (LLG) flow is equivariant under the following set of transformations:

- translations in space $\tau_y m(x) = m(x - y)$ for $y \in \mathbb{R}$, and
- rotations $R_\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$ about the axis e_1 and angle $\phi \in \mathbb{R}$.

There is another symmetry: if m solves (LLG) with parameter γ, then $^\sharp m(t, x) := m(t, -x)$ solves (LLG) with parameter $-\gamma$. We nonetheless leave this last symmetry aside (it does not play any role in modulation theory for example), and we are lead to define the group

$$G := \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$$

which naturally acts on function $w : \mathbb{R} \to \mathbb{R}^3$ as follows: if $g = (y, \phi) \in G$, $g.w := R_\phi \tau_y w$. The action of G preserves $\mathbb{S}^2$ valued functions, and so acts on magnetizations; it also extends naturally to functions of space and time, for which it preserves solutions to (LLG). Also, we endow G with the natural quotient distance over $\mathbb{R}^2$:

$$\forall g = (y, \phi) \in G, \quad |g| := |y| + \inf\{|\phi + 2k\pi|, k \in \mathbb{Z}\}.$$

Our main object of interest here are (precessing) domain walls. These are explicit solutions studied in [4] (to which we refer for further details): given $\sigma = (\sigma_1, \sigma_2) \in \{\pm 1\}^2$ (we equivalently use the notation $\pm$ instead of ± 1), denote

$$\forall x \in \mathbb{R}, \quad w_*^\sigma(x) := \begin{pmatrix} \cos(\theta_*(\sigma_1 x)) \\ \sigma_2 \sin(\theta_*(\sigma_1 x)) \cos(\gamma x) \\ \sigma_1\sigma_2 \sin(\theta_*(\sigma_1 x)) \sin(\gamma x) \end{pmatrix} \quad \text{with} \quad \theta_*(x) := 2\arctan(e^{-\Gamma x}), \tag{1.2}$$

and

$$g_*^\sigma := (\sigma_1 y_*, \phi_*^{\sigma_2}) \quad \text{where} \tag{1.3}$$

$$\text{for } t \geq 0, \quad y_*(t) := -\frac{\alpha}{\Gamma}\int_0^t h(s)\,\mathrm{d}s \quad \text{and} \quad \phi_*^\pm(t) := \left(-1 \pm \frac{\alpha\gamma}{\Gamma}\right)\int_0^t h(s)\,\mathrm{d}s. \tag{1.4}$$

Then

$$(t, x) \mapsto g_*^\sigma(t).w_*^\sigma(x) \tag{1.5}$$

is a solution to (LLG), which we call a domain wall.

Recall that w_*^σ are the only solutions, up to a gauge in G, to the static equation

$$w \wedge \delta E_\gamma(w) = 0.$$

Moreover, they satisfy $\delta E(w_*^\sigma) = \beta_* w_*^\sigma$ where

$$\beta_* := 2\Gamma^2 \sin^2\theta_*. \tag{1.6}$$

They connect $-\sigma_1 e_1$ at $-\infty$ to $\sigma_1 e_1$ at $+\infty$. The case $\gamma = 0$ (i.e., absence of DMI) corresponds to (in-plane) static domain walls where a rotation in θ_* of $180°$ takes place along the nanowire axis e_1; these transitions are called Bloch walls (see e.g. [3, 8]). For future reference, we note that $\theta_* : \mathbb{R} \to (0, \pi)$ solves the first order ODE

$$\partial_x \theta_* = -\Gamma \sin\theta_*, \quad \theta_*(-\infty) = \pi,\ \theta_*(+\infty) = 0, \tag{1.7}$$

and w_*^σ satisfies the system of first order ODEs:

$$\partial_x w_*^\sigma = \sigma_1 \Gamma w_*^\sigma \wedge (e_1 \wedge w_*^\sigma) - \gamma e_1 \wedge w_*^\sigma. \tag{1.8}$$

Note also that formulas (1.5), (1.2) and (1.4) make sense for all $\alpha \in \mathbb{R}$; however the condition $\alpha > 0$ is the physically relevant one, and will be required in all the following analysis.

1.2. Functional spaces and Cauchy problem

We denote H^s (and L^p) for the Sobolev space $H^s(\mathbb{R},\mathbb{R}^3)$ with $s \geq 0$ (and the Lebesgue space $L^p(\mathbb{R},\mathbb{R}^3)$ with $p \in [1,\infty]$, respectively). We also denote $\dot{H}^s$ for the homogeneous Sobolev space whose seminorm is given via Fourier transform:

$$\|m\|_{\dot{H}^s}^2 := \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{m}(\xi)|^2|\xi|^{2s}d\xi, \quad \text{where} \quad \hat{m}(\xi) = \int_{\mathbb{R}} e^{-ix\xi}m(x)\,dx. \tag{1.9}$$

(In particular, $\|m\|_{\dot{H}^2} = \|\partial_{xx}m\|_{L^2}$). We define for $s \geq 1$, the spaces

$$\begin{aligned}\mathcal{H}^s &:= \{m = (m_1,m_2,m_3) \in \mathcal{C}(\mathbb{R},\mathbb{S}^2) : \|m\|_{\mathcal{H}^s} < +\infty\} \\ \text{with} \quad & \|m\|_{\mathcal{H}^s} := \|m_2\|_{L^2} + \|m_3\|_{L^2} + \|m\|_{\dot{H}^s}.\end{aligned} \tag{1.10}$$

The $\mathcal{H}^s$ spaces are modelled on the usual Sobolev spaces H^s, but adapted to the geometry of the target manifold $\mathbb{S}^2$ and to the energy functional E: the main point is that $|m_1| \to 1$ at $\pm\infty$ so that $m_1 \notin L^2$. $\mathcal{H}^1$ corresponds to the set of finite energy configurations $E(m) < +\infty$ in which case, the energy gradient $\delta E(m) \in H^{-1}$. Also if $m, \tilde{m} \in \mathcal{H}^1$ with $m(\pm\infty) = \tilde{m}(\pm\infty)$, then $m - \tilde{m} \in H^1$. Moreover, if w_* is a domain wall (1.2), then every configuration $m \in \mathcal{H}^1$ with $\|m - w_*\|_{\mathcal{H}^1}$ small enough is actually close to w_* in H^1 with Lipschitz bounds, i.e., $\|m - w_*\|_{H^1} \lesssim \|m - w_*\|_{\mathcal{H}^1}$ (we refer to [4] for details and proofs).

Note that all the derivates of w_*^σ of order $k \geq 1$ are exponentially localised, so that $w_*^\sigma \in \mathcal{H}^k$ for all $k \geq 1$.

We use the following well posedness result, quoted from [4]: see section 4 there, and the reference therein for more comments.

Theorem 1.1 (Local well-posedness in $\mathcal{H}^s$)**.** *Let $\alpha > 0$, $\gamma \in (-1,1)$ and $h \in L^\infty([0,+\infty),\mathbb{R})$. Assume $s \geq 1$ and $m_0 \in \mathcal{H}^s$. Then there exist a maximal time $T_+ = T_+(m_0) \in (0,+\infty]$ and a unique solution $m \in \mathcal{C}([0,T_+),\mathcal{H}^s)$ to* (LLG) *with initial data m_0.*

Moreover,

1. *if $T_+ < +\infty$, then $\|m(t)\|_{\mathcal{H}^1} \to +\infty$ as $t \uparrow T_+$;*
2. *for $T < T_+$ (with T_+ finite or infinite), the map $\tilde{m}_0 \in \mathcal{H}^s \to \tilde{m} \in \mathcal{C}([0,T],\mathcal{H}^s)$ is continuous in a small $\mathcal{H}^s$ neighbourhood of m_0 (for every initial data $\tilde{m}_0$ in that neighborhood, the maximal time of the corresponding solution $\tilde{m}$ satisfies $T_+(\tilde{m}_0) > T$);*
3. *if $s \geq 2$, one has the energy dissipation identity : $t \mapsto E(m(t))$ is a locally Lipschitz function in $[0,T_+)$ (even $\mathcal{C}^1$ provided h is continuous) and for all $t \in [0,T_+)$,*

$$\frac{d}{dt}E(m) = -\alpha\int(|\delta E(m)|^2 - |m\cdot\delta E(m)|^2)\,dx + \alpha h(t)\int(m\wedge e_1)\cdot(m\wedge\delta E(m))\,dx. \tag{1.11}$$

1.3. Statement of the main result

In [4], the flow of (LLG) around the domains wall (1.5) was studied: for small H^1 perturbation, and under a small applied field h (in $L_t^\infty((0,+\infty))$, domains walls were proved to be (exponentially) asymptotically stable, up to a gauge. This work thus extended previous results in two directions: in the absence of Dzyaloshinskii-Moriya interaction (case $\gamma = 0$), precessing domain walls were reported in [6], and their linear asymptotic stability was proved in [5] (it however completely disregards the gauge involved); nonlinear stability was also checked numerically in [5]. We can also mention earlier studies of stability for Bloch or Walker wall (which are travelling fronts, not precessing) under some variant of (LLG) (the DMI interaction is not taken into account in the energy E_γ): we refer for example to [3, 7, 1, 2, 11].

In the present paper, we are further interested to study the dynamics of solutions to (LLG) in the presence of several domain walls. This question is not only academically relevant for the long time dynamics, but also motivated by application of this model to data storage: domain walls encode information, and their stability property is important for the persistence of this storage over time.

The simplest case to tackle is the interaction of domain walls decoupling with time: in view of y_*, there is essentially one such configuration, where the speeds are opposite (the transition of these domains wall are centered at $y_*(t)$ and $-y_*(t)$ respectively, up to a fixed translation). This corresponds to studying the evolution of a perturbation of

$$g_*^{(1,\sigma_2)}(t).w_*^+(x) + g_*^{(-1,\sigma_2')}(t).w_*^-(x) \tag{1.12}$$

(given $\sigma_2, \sigma_2' \in \{\pm 1\}$), where $w_*^+ := w_*^{(1,\sigma_2)}$ and $w_*^- := w_*^{(-1,\sigma_2')}$. We will also note now $g_*^+ := g_*^{(1,\sigma_2)}$ and $g_*^- := g_*^{(-1,\sigma_2')}$

In the decomposition of a $\mathbb{S}^2$ magnetisation around two decoupled domain walls, it is interesting to consider gauges in G with large translation parameter, which motivate the notation, given $L > 0$,

$$G_{>L} := \{(y,\phi) \in G : y > L\}, \qquad G_{<-L} := \{(y,\phi) \in G : y < -L\}.$$

Theorem 1.2. *There exist $L_0, \delta_0 > 0$ and $C, \lambda > 0$ such that the following holds. Assume that h satisfies*

$$\|h\|_{L^\infty((0,\infty))} < \delta_0, \tag{1.13}$$

and that

$$\int_0^\infty \sqrt{q(2y_*(t))}\,\mathrm{d}t < +\infty \quad \text{where, for } r \in \mathbb{R}, \quad q(r) := (1+|r|)e^{-\Gamma r}. \tag{1.14}$$

Denote for $t \geq 0$,

$$\kappa(t) = e^{-\Gamma y_*(t)} + \left(\int_0^t e^{-2\lambda(t-s)} q(2y_*(s)) ds \right)^{1/2}. \tag{1.15}$$

Let $m_0 \in \mathcal{H}^1$ such that that there exist $L \geq L_0$ and $\zeta^+, \zeta^- \in G_{>L}$ with

$$\delta := \left\| m_0 - \left(\zeta^+.w_*^+ + \zeta^-.w_*^- + e_1 \right) \right\|_{H^1} < \delta_0. \tag{1.16}$$

Then the solution m to (LLG) *is global for forward times and there exist 2 gauges $g^+, g^- \in W^{1,\infty}(\mathbb{R}_+, G)$, such that,*

$$\forall t \geq 0, \quad \left\| m(t) - \left(g^+.w_*^+ + g^-.w_*^- + e_1 \right) \right\|_{H^1} \leq C(\delta + \sqrt{q(2L)})e^{-\lambda t} + C\sqrt{q(2L)})\kappa(t). \tag{1.17}$$

Moreover, there exist two gauges $g_\infty^\pm \in G$ such that

$$\forall t \geq 0, \quad \sum_{\iota \in \{\pm\}} |g^\iota(t) - (g_*^\iota(t) + g_\infty^\iota)| \leq C(\delta + \sqrt{q(2L)})e^{-\lambda t} + C\sqrt{q(2L)} \int_t^{+\infty} \kappa(s) ds. \tag{1.18}$$

As it will be seen from Lemma 1.3, $\kappa \to 0$ as $t \to +\infty$ and is integrable in time, so that the estimates (1.17)-(1.18) yield convergence results. Notice that κ depends on h alone, whereas λ is essentially a coercivity constant, which depends on γ (it is related to the closeness of $|\gamma|$ to 1). The decay functions $e^{-\lambda t}$ and $\kappa(t)$ are therefore unrelated, even though in most cases (for example, as soon as $h \to 0$), $\kappa(t) \gg e^{-\lambda t}$.

Theorem 1.2 therefore quantifies how and under which condition the structure made of two decoupled domain walls persists over time. Assumption (1.13) ensure that the external magnetic field is not too strong: this is required even for configuration with one domain wall not to be destroyed. Our second assumption (1.14) states that the free evolution of the center of the domain wall should separate them indefinitely: in order to have asymptotic stability (that is convergence of the gauge $g^\pm$), a requirement of the type $y_* \to +\infty$ is in order. It turns out that, for our analysis to work, we need a somewhat stronger integrability condition, which however remains rather mild (see Lemma 1.3).

This result is a stability statement for well prepared data, which bear some resemblance with the stability of the sum of decoupled solitons for non linear dispersive model: we refer for example to [9] for the generalised Korteweg-de Vries equation, or to [10] for the nonlinear Schrödinger equation. An important difference though, is that in these settings, each soliton bears its own dynamic, which is leading order (solitons are assumed to have distinct speeds), whereas in the present context, the dynamics is determined by the external magnetic field represented by h.

Our analysis relies on the framework developed in [4], which combines modulation techniques to split the evolution between some geometric parameters (the gauge) and a remainder term; energy estimates to control the remainder; and dynamical arguments (consequence of energy dissipation) for the gauge.

An important point of this paper, and a novelty with respect to [4], is that this framework is amenable to space localization, and is therefore suitable to study the interactions of domain walls: we believe that is much less so for earlier methods and results (referred to at the beginning of this paragraph), which relied on spectral properties of the linearized (LLG) flow around domain walls. We localize the coercivity properties of the energy around each domain walls, as well as the energy dissipation equality. For these two results to make sense, one must first modulate around a sum of two domain walls. These three results are stated at the beginning of section 2, and proven in sections 4, 5 and 6 respectively. Section 3 gives some preliminary results, in particular about a frame adapted to the domain wall and the control of the nonlinearity, and first introduced in [4].

The proof of stability is done in section 2, and consists in a bootstrap argument: on a time interval on which one can modulate the magnetization around two domain walls, and one has sufficient control on the gauges involved and the remainder terms, we combine the localized energy dissipation and coercivity to improve these controls. We give a special attention to the decay of the remainder term in order to make the assumption on the external field h as mild as possible: this is a delicate part of the analysis. Before going to the proof of the main results, we conclude this section by giving some consequences of the assumptions (1.17)-(1.18) for h, on the behavior of y_*.

1.4. On the assumptions on the external field h

We recall that the distance between the two domains walls is essentially $2y_*(t)$, and it will turn out that $q(2y_*(t))$ measures correctly the interaction between them.

The assumptions on h are relatively mild: apart from uniform smallness (required to ensure stability, even for one domain wall), some oscillation and decay are allowed as long as the external field still pushes the domain walls away, so that their interaction enjoys some integrability in time. This is quantified in the simple computation below.

Lemma 1.3. *1) Assume that h satisfies* (1.13) *and* (1.14)*. Then if $t, \tau \ge 0$ are such that $|t-\tau| \le 1$, $|y_*(\tau) - y_*(t)| \le 1$. As a consequence, $y_*(t) \to +\infty$ as $t \to +\infty$, and there exists $C > 0$ such that for all $t, \tau \ge 0$ with $|t-\tau| \le 1$,*

$$q(2y_*(\tau)) \le Cq(2y_*(t)). \tag{1.19}$$

2) If $\liminf_{t\to+\infty} \dfrac{y_*(t)}{\ln t} > \dfrac{1}{\Gamma}$ then (1.14) *is fulfilled. This is in particular the case if $\limsup_{t\to+\infty} th(t) < -\dfrac{1}{\alpha}$.*

Proof. 1) Recall that $\delta_0 \le 1$, so that the first bound is immediate from the mean value theorem. Assume that for some, $R \ge 1$, there exists $t_n \to +\infty$ such that $y_*(t_n) \le R$. We can assume that $R \ge 1/\Gamma$ is so large that q is decreasing on $[R-1, +\infty)$, and that $t_{n+1} \ge t_n + 1$ for all n. Then, in view of the Lipschitz bound on y_* induced by (1.13), $y_*(t) \in [R-1, R+1]$ for all $t \in [t_n, t_n+1]$ so that

$$\int_0^\infty q(y_*(t))dt \ge \sum_{n\ge 0} \int_{t_n}^{t_{n+1}} q(y_*(t))dt \ge \sum_n q(R+1) = +\infty,$$

a contradiction with (1.14). Hence $y_*(t) \to +\infty$ as $t \to +\infty$.

Now for any $r \ge 1$ and $h \in [-1, 1]$,

$$|q(r+h) - q(r)| = |(r+h)e^{-\Gamma(r+h)} - re^{-\Gamma r)}| \le re^{-\Gamma r}e^{|h|} + he^{-\Gamma(r-1)} \le C|h|q(r).$$

In particular,

$$\sup_{h\in[-1,1]} q(r+h) \le Cq(r).$$

Together with the fact that for $|t-\tau| \le 1$, $|y_*(t) - y_*(\tau)| \le \delta$, yields (1.19).

2) The assumption on y_* writes that for some $a > 1$, and some $T \ge 2$,

$$\forall t \ge T, \quad y_*(t) \ge \frac{a}{\Gamma} \ln t.$$

As q is eventually decreasing to 0,

$$\int_T^{+\infty} \sqrt{q(2y_*(t))}\,\mathrm{d}t \lesssim \int_T^{+\infty} \sqrt{1+\ln t}\frac{\mathrm{d}t}{t^a} < +\infty.$$

The condition on h implies that on y_* by direct integration. ☐

Lemma 1.4. *The function κ defined in* (1.15) *has the properties:*

$$\kappa(t) \to 0 \quad \textit{as} \quad t \to +\infty \quad \textit{and} \quad \int_0^{+\infty} \kappa(\tau)d\tau < +\infty.$$

Proof. We already saw that $y_* \to +\infty$ so that $e^{-\Gamma y_*(t)} \to 0$ and is integrable on $[0, +\infty)$. For the integral term, this is merely a convolution: as we made the hypothesis that $\sqrt{q(2y_*)}$ is integrable, convergence to zero is straightforward:

$$\begin{aligned}\int_0^t e^{-2\lambda(t-s)}q(2y_*(s))ds &\le \|q(2y^*)\|_{L^\infty([0,+\infty))} \int_0^{t/2} e^{-2\lambda(t-s)} + \int_{t/2}^t q(2y_*(s))ds \\ &\le e^{-\lambda t}\|q(2y^*)\|_{L^\infty([0,+\infty))} + \|\sqrt{q(2y_*)}\|_{L^\infty([t/2,+\infty))} \int_{t/2}^{+\infty} \sqrt{q(2y_*(s))}ds \to 0.\end{aligned}$$

For the integrability, we need an extra ingredient: the previous Lemma 1.3 allows to relate to a series. To avoid side effect, first observe that there is no integrability issue on $[0, 1]$:

$$\int_0^1 \sqrt{\int_0^t e^{-2\lambda(t-s)}q(2y_*(s))dsdt} \le \|\sqrt{q(2y_*)}\|_{L^\infty([0,1])}.$$

If $t \in [n, n+1)$ for some integer $n \geq 1$, using (1.19) there hold

$$\int_0^t e^{-2\lambda(t-s)} q(2y_*(s)) ds \leq C \int_0^{n+1} e^{-2\lambda(n-s)} q(2y_*(s)) ds \leq C \sum_{k=0}^{n} e^{-2\lambda(n-k)} q(2y_*(k))$$

Hence, using that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, we infer

$$\sqrt{\int_0^t e^{-2\lambda(t-s)} q(2y_*(s)) ds} \leq C \sum_{k=0}^{n} e^{-\lambda(n-k)} \sqrt{q(2y_*(k))} \leq C \int_0^t e^{-\lambda(t-s)} \sqrt{q(2y_*(s))} ds.$$

(We used again (1.19) on each interval $[k, k+1]$ for $k \leq n-2$, and one last time on $[n-1, t]$ which is of length $t-n+1 \leq 2 \leq 2t$; this is where $t \geq 1$ is useful). Therefore,

$$\int_1^{+\infty} \sqrt{\int_0^t e^{-2\lambda(t-s)} q(2y_*(s)) ds dt} \leq C \int_{t \geq 1} \int_{0 \leq s \leq t} e^{-\lambda(t-s)} \sqrt{q(2y_*(s))} ds dt.$$

For this last integral we split the integration domain

$$\{(s,t) : t \geq 1, 0 \leq s \leq t\} = \{(s,t) : 0 \leq s \leq 1 \leq t\} \cup \{(s,t) : 1 \leq s \leq t\}.$$

In both subdomains, we integrate first in t: there hold

$$\iint_{0 \leq s \leq 1 \leq t} e^{-\lambda(t-s)} \sqrt{q(2y_*(s))} ds dt = \frac{1}{\lambda} \int_0^1 e^{-\lambda(1-s)} \sqrt{q(2y_*(s))} ds < +\infty,$$

and

$$\iint_{1 \leq s \leq t} e^{-\lambda(t-s)} \sqrt{q(2y_*(s))} ds dt = \frac{1}{\lambda} \int_1^{+\infty} \sqrt{q(2y_*(s))} ds < +\infty,$$

by assumption. □

2. Proof of the stability

2.1. Preliminary results

Lemma 2.1 (Decomposition of the magnetization)**.** *There exist $\delta_1 > 0$, $L_1 \geq 1$ and $C_1 > 0$ such that the following holds. Let $T > 0$, $h \in L^\infty((0,T))$ and $m \in \mathscr{C}([0,T], \mathcal{H}^2)$ solution to* (LLG)*, assume that for all $t \in [0,T]$, and for some $L \geq L_1$,*

$$\delta := \inf_{\zeta^+ \in G_{>L+1},\, \zeta^- \in G_{<-L-1}} \left\| m(t) - \left(\zeta^+ . w_*^+ + \zeta^- . w_*^- + e_1 \right) \right\|_{H^1} < \delta_1$$

Then there exists three functions :

- $g^+ = (y^+, \phi^+) : [0,T] \to G_{>L}$ *Lipschitz,*
- $g^- = (y^-, \phi^-) : [0,T] \to G_{<-L}$ *Lipschitz,*
- $\varepsilon : [0,T] \to H^2$ *continuous,*

such that, for $w^+ = g^+ . w_*^+$ and $w^- = g^- . w_*^-$,

- $m = w^+ + w^- + e_1 + \varepsilon$*,*
- ε *satisfies for $\iota \in \{\pm 1\}$*
$$\int \varepsilon \cdot \partial_x w^\iota \, \mathrm{d}x = \int \varepsilon \cdot (e_1 \wedge w^\iota) \, \mathrm{d}x = 0, \tag{2.1}$$
- *the following bounds hold for all $t \in [0,T]$ and $\iota \in \{\pm\}$:*

$$|\dot{g}^\iota(t) - \dot{g}_*^\iota(t)| \leq C_1 \Big(\|\varepsilon(t)\|_{H^1} + q(y^+ - y^-) \Big), \tag{2.2}$$

$$\|\varepsilon(t)\|_{H^1} \leq C_1 \Big(\delta + q(y^+ - y^-) \Big). \tag{2.3}$$

Remark 2.2. This decomposition also holds with $T = +\infty$, *mutatis mutandis*.

This result will be proved in Section 6.

The proof of the stability relies on two main estimates. The first one shows an equivalence between the energy E and the norm of ε which will be defined thanks to Lemma 2.1. For this, we recall that $E_\gamma(w_*^+) = E_\gamma(w_*^-)$, denoted by $E(w_*)$ hereafter.

Proposition 2.3 (Coercivity of the energy)**.** *There exists* $0 < \delta_2 \le \delta_1/3$, $L_2 \ge L_1$, $C_2 > 0$ *and* $\lambda_2 > 0$ *such that the following holds. Under the assumptions (and notations) of Lemma 2.1, assuming further* $\delta \le \delta_2$, $L \ge L_2$, *there hold, for any* $0 < R \le L/2$ *and for all* $t \in [0, T]$

$$C_2\Big(\|\varepsilon\|_{H^1}^2 + (e^{2\Gamma(R-y^+)} + e^{2\Gamma(R+y^-)})\Big) \ge E(m) - 2E(w_*)$$
$$\ge \Big(4\lambda_2 - \frac{C_2}{R^2}\Big)\|\varepsilon\|_{H^1}^2 - C_2\Big(\|\varepsilon\|_{H^1}^3 + (e^{2\Gamma(R-y^+)} + e^{2\Gamma(R+y^-)})\Big). \quad (2.4)$$

The second result is an estimate of the evolution of the energy. It shows that, up to some quantities which are negligible enough in some sense, the energy is almost decreasing.

Proposition 2.4 (Localised energy dissipation)**.** *There exists* $0 < \delta_3 \le \delta_1/3$, $L_3 \ge L_1$, $C_3 > 0$ *and* λ_3 *such that the following holds. Under the same assumptions and notations as Proposition 2.3, assuming further* $\delta \le \delta_3$, $L \ge L_3$ *and for any* $0 < R \le L/2$, *there holds, for all* $t \in [0, T]$

$$\frac{\mathrm{d}}{\mathrm{d}t}E(m) + \left(4\alpha\lambda_3 - \frac{C_3}{R^2}\right)\|\varepsilon\|_{H^2}^2 \le C_3\Big((|h| + e^{\Gamma(R-y^+)} + e^{\Gamma(R+y^-)})\|\varepsilon\|_{H^1}^2 + \|\varepsilon\|_{H^1}\|\varepsilon\|_{H^2}^2\Big)$$
$$+ C_3\Big(e^{2\Gamma(R-y^+)} + e^{2\Gamma(R+y^-)} + q(y^+ - y^-)\Big) \quad (2.5)$$

Proof of Theorem 1.2 assuming Lemma 2.1 and Propositions 2.3 and 2.4. We assume in the following that $m_0 \in \mathcal{H}^2$, so that all the computations are justified. When $m_0 \in \mathcal{H}^1$, one can use a limiting argument as in [4]; we will not develop it further here.

Step 1. Main boostrap

Let $T_+(m)$ be the maximum time of existence of the solution m to (LLG). Let $\delta_0, L_0 > 0$, and $M \ge 2$ be an extra large parameter to be fixed later. Assume already that $M\delta_0 \le \min(\delta_2, \delta_3)$, and $L_0 > L_1$ so large that $q(2L_0) \le 1$ and q is non increasing on $[L_0, +\infty)$. Define T_1 as the supremum of

$$\left\{T \in (0, T_+(m)) : \forall t \in [0, T], \inf_{\zeta^+ \in G_{>L+2},\, \zeta^- \in G_{<-L-2}} \big\|m(t) - (\zeta^+.w_*^+ + \zeta^-.w_*^- + e_1)\big\|_{H^1} < M(\delta + \sqrt{q(2L)})\right\},$$

By continuity of the flow of (LLG) and the assumption on the initial data as $M > 1$ and $L \ge L_0 > L_1$, the above set is non empty and $T_0 > 0$. We aim at proving that $T_1 = T_+(m) = +\infty$.

As $M\delta_0 \le \delta_1$, m satisfies the assumptions of Lemma 2.1 on $[0, T]$ for any $T \in (0, T_1)$: it provides us with the functions $g^+ = (y^+, \phi^+)$, $g^- = (y^-, \phi^-)$ and ε satisfying its conclusions, on the interval $[0, T_1)$. Also, at time 0, we have the improved bound:

$$\|\varepsilon(0)\|_{H^1} \le C_1(\delta + q(2L)). \quad (2.6)$$

We recall that the domain walls with initial data $g^\pm(0).w_*^\pm$ have center $y_*^\pm(t) = \pm y_*(t) + y^\pm(0)$ where

$$y_*(t) = -\frac{\alpha}{\Gamma}\int_0^t h(s)\,\mathrm{d}s.$$

Let T_2 be the supremum of

$$\{T' \in (0, T_1) : \forall t \in [0, T'], \forall \iota \in \{\pm\},\ |y^\iota(t) - (y^\iota(0) + \iota y_*(t))| < 1\}.$$

By continuity of y_* and $y^\pm$, we know that $T_2 > 0$ (we will show that $T_2 = T_1$ as well). We choose $T \in (0, T_2]$, and we work on the interval $[0, T]$.

Step 2. Deriving convenient bounds on ε *and* $g^\pm$.

First observe that for $t \in [0, T]$, $y^+(t) \ge L + y_*(t)$ and $y^-(t) \le -L - y_*(t)$ so that

$$e^{-2\Gamma y^+} \le e^{-2\Gamma(L+y_*)}, \quad e^{2\Gamma y^-} \le e^{-2\Gamma(L+y_*)}, \quad (2.7)$$
$$q(y^+ - y^-) \le q(2y_* + 2L) \le e^{-2\Gamma L}q(2y_*) + q(2L_0)e^{-2\Gamma y_*} \le 2q(2L)q(2y_*). \quad (2.8)$$

This allows to take care of the terms in $y^\pm$ in the estimates.

We now choose R so that we gain some coercivity in (2.4) and (2.5). For this we impose that

$$\delta_0 \le \delta_4 := \min\left(\frac{1}{4\sqrt{\lambda_2 C_2}}, \frac{1}{4\sqrt{\alpha\lambda_3 C_3}}\right) \quad \text{and} \quad R = \frac{1}{2\delta_4}, \tag{2.9}$$

so that $\frac{C_3}{R^2} \le \alpha\lambda_3$ and $\frac{C_2}{R^2} \le \lambda_2$.

Hence for $C_4 = \max(C_2, C_3)R^2 e^{2\Gamma R}$ (which depends only on C_2, C_3 and λ_2, λ_3), for all $t \in [0, T_1]$ there hold

$$E(m) - 2E(w_*) \ge 2\lambda_2 \|\varepsilon\|_{H^1}^2 - C_4\Big(\|\varepsilon\|_{H^1}^3 + e^{-2\Gamma L}e^{-2\Gamma y_*}\Big),$$
$$E(m) - 2E(w_*) \le C_4\Big(\|\varepsilon\|_{H^1}^2 + e^{-2\Gamma L}e^{-2\Gamma y_*}\Big),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}E(m) + 3\alpha\lambda_3\|\varepsilon\|_{H^2}^2 \le C_4(|h|\|\varepsilon\|_{H^1}^2 + \|\varepsilon\|_{H^1}\|\varepsilon\|_{H^2}^2) + q(2L)q(2y_*)\Big).$$

We also want to make use of the smallness of h and ε to get rid of terms which are cubic or higher in (ε, h). We therefore assume that

$$\delta_0 \le \frac{\alpha\lambda_3}{C_3}, \tag{2.10}$$

so that for all $t \ge 0$, $C_3|h(t)| \le \alpha\lambda_3$. Recall that $y_* \to +\infty$, so that $\inf y_* > -\infty$: we choose L_0 such that

$$L_0 \ge -2\inf y_*.$$

Then on $[0, T]$, $y^+ - y^- \ge 2(L + y_*) \ge L \ge L_0$ and as q is decreasing on $[L_0, +\infty)$, $q(y^+ - y^-) \le q(L)$. Thus, due to (2.3)

$$\|\varepsilon\|_{H^1} \le C_1 M(\delta + \sqrt{q(2L)}) + C_1 q(L).$$

We therefore assume that $\delta_0 \le \delta_5$ and $L_0 \ge L_5$ where $\delta_5 > 0$ and $L_5 > 0$ are such that

$$\delta_5 + q(L_5) \le \frac{\min(\lambda_2, \alpha\lambda_3)}{MC_1C_4} \tag{2.11}$$

and we infer that on $[0, T]$

$$C_4\|\varepsilon\|_{H^1} \le \min(\lambda_2, \alpha\lambda_3).$$

Therefore, we obtained that

$$E(m) - 2E(w_*) \ge \lambda_2\|\varepsilon\|_{H^1}^2 - C_4 e^{-2\Gamma L}e^{-2\Gamma y_*} \tag{2.12}$$
$$E(m) - 2E(w_*) \le C_4\Big(\|\varepsilon\|_{H^1}^2 + e^{-2\Gamma L}e^{-2\Gamma 2 y_*}\Big) \tag{2.13}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}E(m) + \alpha\lambda_3\|\varepsilon\|_{H^2}^2 \le C_4 q(2L)q(2y_*). \tag{2.14}$$

Step 3. Decay of ε.

Let $\tau, t \in [0, T]$ such that $\tau \le t$. Integrating (2.14) on $[\tau, t]$, we infer

$$E(m(t)) + \alpha\lambda_3\int_\tau^t \|\varepsilon\|_{H^2}^2 \le E(m(\tau)) + C_4 q(2L)\int_\tau^t q(2y_*(s))ds.$$

From there, together with (2.12) and (2.13), we infer

$$\begin{aligned}\lambda_2\|\varepsilon(t)\|_{H^1}^2 + \alpha\lambda_3\int_\tau^t \|\varepsilon(s)\|_{H^2}^2 ds &\le C_4\left(\|\varepsilon(\tau)\|_{H^1}^2 + e^{-2\Gamma L}(e^{-2\Gamma y_*(\tau)} + e^{-2\Gamma y_*(t)}) + q(2L)\int_\tau^t q(2y_*(s))ds\right)\\ &\le 2C_4(\|\varepsilon(\tau)\|_{H^1}^2 + q(2L)\kappa_0(\tau, t)).\end{aligned} \tag{2.15}$$

where for $\tau \le t$,

$$\kappa_0(\tau, t) := e^{-2\Gamma y_*(\tau)} + e^{-2\Gamma y_*(t)} + \int_\tau^t q(2y_*(s))ds.$$

In particular, with $\tau = 0$, we obtain a uniform bound:

$$\lambda_2\|\varepsilon(t)\|_{H^1}^2 + \alpha\lambda_3\int_0^t \|\varepsilon(s)\|_{H^2}^2 ds \le 2C_4\left(\|\varepsilon(0)\|_{H^1}^2 + q(2L)\kappa_0(0, +\infty)\right), \tag{2.16}$$

$$\text{where}\quad \kappa_0(0,+\infty) = 1 + \int_0^{+\infty} q(2y_*(s))ds < +\infty. \tag{2.17}$$

Going back to (2.15), fixing for now t and seeing τ as a variable, we have

$$\begin{aligned}\frac{\partial}{\partial\tau}\left(e^{\frac{\alpha\lambda_3}{C_4}\tau}\int_\tau^t \|\varepsilon(s)\|_{H^1}^2 ds\right) &= e^{\frac{\alpha\lambda_3}{C_4}\tau}\left(\frac{\alpha\lambda_3}{C_4}\int_\tau^t \|\varepsilon(s)\|_{H^1}^2 ds - \|\varepsilon(\tau)\|_{H^1}^2\right)\\ &\le 2C_4 e^{\frac{\alpha\lambda_3}{C_4}\tau} q(2L)\kappa_0(\tau,t).\end{aligned}$$

Now integrate this estimate on $[0,\tau]$ (for $\tau \le t$) to get

$$e^{\frac{\alpha\lambda_3}{C_4}\tau}\int_\tau^t \|\varepsilon(s)\|_{H^1}^2 ds \le \int_0^\tau \|\varepsilon(s)\|_{H^1}^2 ds + 2C_4 q(2L)\int_0^\tau e^{\frac{\alpha\lambda_3}{C_4}s}\kappa_0(s,t)ds.$$

Assume for now that $t \ge 1$. In view of (2.16) with $\tau = t-1$, we infer that for $\lambda = \frac{\alpha\lambda_3}{2C_4}$,

$$\int_{t-1}^t \|\varepsilon(s)\|_{H^1}^2 ds \le e^{-2\lambda t}\frac{1}{\lambda}(\|\varepsilon(0)\|_{H^1}^2 + q(2L_0)\kappa_0(0,+\infty)) + 2C_4 q(2L)\int_0^t e^{-2\lambda(t-s)}\kappa_0(s,t)ds.$$

Let $\tau \in [t-1,t]$ such that

$$\|\varepsilon(\tau)\|_{H^1}^2 \le \int_{t-1}^t \|\varepsilon(s)\|_{H^1}^2 ds.$$

(we use the mean value theorem). Then (2.15) now writes

$$\lambda_2\|\varepsilon(t)\|_{H^1}^2 \le e^{-2\lambda_4 t}\frac{C_4}{\lambda_4}(\|\varepsilon(0)\|_{H^1}^2 + q(2L)\kappa_0(0,+\infty)) + 2C_4 q(2L)\int_0^t e^{-2\lambda(t-s)}\kappa_0(s,t)ds. \tag{2.18}$$

Observe that

$$\begin{aligned}\int_0^t e^{-2\lambda(t-s)}\kappa_0(s,t)ds &= \int_0^t e^{-2\lambda(t-s)}\left(e^{-2\Gamma y_*(s)} + e^{-2\Gamma y_*(t)} + \int_s^t q(2y_*(u)du\right)ds\\ &\le \frac{1}{2\lambda}e^{-2\Gamma y_*(t)} + \int_0^t e^{-2\lambda(t-s)}q(2y_*(s))ds + \iint_{0\le s\le u\le t} e^{-2\lambda(t-s)}q(2y_*(u))duds\end{aligned}$$

After integrating in s, notice that the last double integral is bounded by

$$\frac{1}{2\lambda}\int_{u=0}^t e^{-2\lambda(t-u)}q(2y_*(u))du.$$

We can therefore simplify (2.18): recall the initial estimate (2.6) on $\varepsilon(0)$, we also use that $\sqrt{a+b} \le \sqrt{a}+\sqrt{b}$ for $a,b\ge 0$, and $C_1 \ge 1$, $q(2L)\le 1$. For $C_5 = \sqrt{\frac{C_1}{2\lambda_2}\max\left(\frac{C_4}{\lambda_4}(1+\kappa_0(0,+\infty)), C_4(1+\frac{1}{\lambda})\right)}$, we obtain the bound

$$\forall t\in[0,T],\quad \|\varepsilon(t)\|_{H^1} \le C_5 e^{-\lambda t}\delta + C_5\sqrt{q(2L)}(e^{-\lambda t} + \kappa(t)), \tag{2.19}$$

where κ is defined in (1.15). Let us also recall at this point that due to Lemma 1.4, $\kappa \to 0$ at ∞ and is integrable on $[0,+\infty)$. In particular, κ is bounded.

Step 4. $T_1 = T_2 = +\infty$.

Let us first define M and choose δ_0 and L_0. For this, we recall (2.2) and (2.8): there hold for $\iota \in \{\pm\}$ and $t \in [0,T_1]$

$$|y^\iota(t) - (y^\iota(0) + \iota y_*(t))| \le C_1\int_0^t \|\varepsilon(s)\|_{H^1}ds + 2C_1 q(2L)\int_0^t q(2y_*(s))ds. \tag{2.20}$$

Define

$$M_1 = 2C_1C_5\left(\frac{1}{\lambda} + \int_0^{+\infty} q(2y_*(t))dt + \int_0^{+\infty}\kappa(t)dt\right), \tag{2.21}$$

and

$$M = 2C_5(1 + \|\kappa\|_{L^\infty([0,+\infty))}).$$

Now that M has been determined, choose δ_0 and L_0 so as to satisfies the constraints in the previous steps, namely (2.9), (2.10) and (2.11), and also so that

$$\delta_0 \leq 1/(4M_1) \quad \text{and} \quad \sqrt{2q(L_0)} \leq 1/(4M_1).$$

From (2.20) and the definition of M_1 (2.21), we obtain $t \in [0,T]$

$$|y^\pm(t) - (y^\pm(0) \pm y_*(t))| \leq C_1 M_1(\delta + \sqrt{2q(L)}) \leq \frac{1}{2} < 1.$$

Using a continuity argument, this last bound implies that $T_2 = T_1$. Now, from (2.19) and the definition of M, we get that

$$\forall t \in [0,T], \quad \|\varepsilon(t)\|_{H^1} \leq \frac{M}{2}(\delta + \sqrt{q(2L)}).$$

Again, a continuity argument yields that $T_1 = +\infty$.

Step 5. Convergence of m and g.

The bound (2.19) now holds for all $t \geq 0$, and (together with the estimate on $\|\varepsilon(0)\|_{H^1}$) gives a rate of convergence of $\varepsilon \to 0$, which is precisely (1.17).

Finally, (2.2) writes

$$\forall t, \geq 0, \quad |\dot{g}^{[i]} - \dot{g}_*^{[i]}| \leq C_1(\|\varepsilon(t)\|_{H^1} + q(2L_0)q(2y_*(t)))$$

so that as $\|\varepsilon\|_{H^1}$ and $q(2y_*)$ are integrable in time, $g^{[i]}(t) - \dot{g}_*^{[i]}(t)$ admits a limit $g_\infty^{[i]}$ as $t \to +\infty$, and

$$\begin{aligned}
|g^{[i]}(t) - (g_\infty^{[i]} + g_*^{[i]}(t))| &\leq C_1 \int_t^{+\infty} (\|\varepsilon(s)\|_{H^1} + q(2L)q(2y_*(s)))ds \\
&\leq \frac{C_1 C_5}{\lambda}(\delta + \sqrt{q(2L)})e^{-\lambda t} \\
&\quad + C_1 C_5 (1 + \sqrt{q(2L)} \|\sqrt{q(2y_*)}\|_{L^\infty([0,+\infty))}) \sqrt{q(2L)} \int_t^{+\infty} \kappa(s) ds.
\end{aligned}$$

and (as q is bounded) this gives (1.18). □

3. Localisation and associated basis

In order to prove Propositions 2.3 and 2.4, we need to introduce a localisation function ψ_R and the basis related to w_*^+ and w_*^-. This is the content of this section, along with several miscellaneous notations and results. From now on, C will be a universal positive constant which may change from line to line.

3.1. Interaction of domain walls

From the explicit formula of the domain walls, there holds the following.

Lemma 3.1. *There exists $C > 0$ which does not depend on $\gamma \in (-1,1)$ such that, for all $j \in \{0,1,2\}$*

$$\left|\partial_x^j (w_*^{(1,\sigma_2)} - e_1)(x)\right| \leq C\, e^{-\Gamma x} \qquad \text{if } x \geq 0,$$
$$\left|\partial_x^j (w_*^{(1,\sigma_2)} + e_1)(x)\right| \leq C\, e^{-\Gamma |x|} \qquad \text{if } x \leq 0.$$

Similar estimates for w_*^σ follow, *mutatis mutandis*.

Corollary 3.2. *There exists $C > 0$ such that for all $g^+ \in G_{>0}$ and $g^- \in G_{<0}$, for all $x \in \mathbb{R}$ and $j \in \{0,1,2\}$, there holds*

$$\left|\partial_x^j (g^+.w_*^{(1,\sigma_2)} - e_1)\right|\left|\partial_x^j (g^-.w_*^{(-1,\sigma_2')} + e_1)\right| \leq C e^{-\Gamma(2x - y^+ - y^-)} \qquad \text{if } x \geq y^+,$$

$$\left|\partial_x^j (g^+.w_*^{(1,\sigma_2)} + e_1)\right|\left|\partial_x^j (g^-.w_*^{(-1,\sigma_2')} + e_1)\right| \leq C e^{-\Gamma(y^+ - y^-)} \qquad \text{if } y^- \leq x \leq y^+,$$

$$\left|\partial_x^j (g^+.w_*^{(1,\sigma_2)} + e_1)\right|\left|\partial_x^j (g^-.w_*^{(-1,\sigma_2')} - e_1)\right| \leq e^{\Gamma(2x - y^+ + y^-)} \qquad \text{if } x \leq y^-,$$

$$\left(\left|\partial_x(g^+.w_*^{(1,\sigma_2)})\right|+\left|e_1\wedge g^+.w_*^{(1,\sigma_2)}\right|\right)\left(\left|\partial_x(g^-.w_*^{(-1,\sigma_2')})\right|+\left|e_1\wedge(g^-.w_*^{(-1,\sigma_2')})\right|\right)$$
$$\leq Ce^{-\Gamma(y^+-y^-)}\times\begin{cases}e^{-2\Gamma(x-y^+)} & \text{if } x\geq y^+\\ 1 & \text{if } x\in[y^-,y^+]\\ e^{2\Gamma(x-y^-)} & \text{if } x\leq y^-\end{cases}.$$

$$\left|\partial_x(g^+.w_*^{(1,\sigma_2)})\right|\cdot\left|g^-.w_*^{(-1,\sigma_2')}+e_1\right|\leq Ce^{-\Gamma(y^+-y^-)}\times\begin{cases}e^{-2\Gamma(x-y^+)} & \text{if } x\geq y^+\\ 1 & \text{if } x\in[y^-,y^+]\\ e^{\Gamma(x-y^-)} & \text{if } x\leq y^-\end{cases}.$$

$$\left|g^+.w_*^{(1,\sigma_2)}+e_1\right|\left|g^-.(\beta_*w_*^{(-1,\sigma_2')})\right|\leq Ce^{-\Gamma(y^+-y^-)}\times\begin{cases}e^{-2\Gamma(x-y^+)} & \text{if } x\geq y^+\\ e^{-\Gamma(x-y^-)} & \text{if } x\in[-y^-,y^+]\\ e^{2\Gamma(x-y^-)} & \text{if } x\leq y^-\end{cases}.$$

Lemma 3.3 ([4, Lemma 3.2])**.** *There holds*

$$\|\partial_x w_*^\sigma\|_{L^2}^2=\|e_1\wedge w_*^\sigma\|_{L^2}^2=\frac{2}{\Gamma},$$
$$\int(e_1\wedge w_*^\sigma)\cdot\partial_x w_*^\sigma\,\mathrm{d}x=-\frac{2\gamma}{\Gamma},$$
$$\int(w_*^\sigma\wedge(w_*^\sigma\wedge e_1))\cdot\partial_x w_*^\sigma\,\mathrm{d}x=-2\sigma_1.$$

Lemma 3.4. *For all $g\in G$, there holds*

$$\|g.w_*^\sigma-w_*^\sigma\|_{H^1}\leq C|g|,$$

and, for any $x\in\mathbb{R}$, if $y\geq 0$ and $j=0,1$,

$$\left|\partial_x^j g.w_*^\sigma(x)-\partial_x^j w_*^\sigma(x)\right|\leq C\max(|g|,1)\times\begin{cases}e^{-\Gamma(x-y)} & \text{if } x\geq y\\ 1 & \text{if } x\in[0,y]\\ e^{\Gamma x} & \text{if } x\leq 0\end{cases},$$

and similarly when $y\leq 0$. Moreover there exists $\tilde{\delta}_0>0$ independent of g such that, if $\|g.w_*^\sigma-w_*^\sigma\|_{H^1}\leq\tilde{\delta}_0$, then

$$|g|\leq C\|g.w_*^\sigma-w_*^\sigma\|_{H^1}.$$

Proof. The third point is [4, Claim 4.12]. The first point was also proved in [4], see (4.6) in there. For the second point, we can refine the latter:

$$\begin{aligned}g.w_*^\sigma(x)-w_*^\sigma(x)&=\tau_yR_\phi w_*^\sigma(x)-R_\phi w_*^\sigma(x)+R_\phi w_*^\sigma(x)-w_*^\sigma(x),\\ |g.w_*^\sigma(x)-w_*^\sigma(x)|&\leq|\tau_yR_\phi w_*^\sigma(x)-R_\phi w_*^\sigma(x)|+|R_\phi w_*^\sigma(x)-w_*^\sigma(x)|\\ &\leq|y|\max_{[x-y,x]}|\partial_x w_*^\sigma|+|\phi|_{\mathbb{R}/2\pi\mathbb{Z}}|e_1\wedge w_*(x)|,\end{aligned}$$

and the conclusion comes from Lemma 3.1. On the other hand, one can also estimate in another way for $x\geq 0$:

$$g.w_*^\sigma(x)-w_*^\sigma(x)=g.w_*^\sigma(x)-\sigma_1e_1-(w_*^\sigma(x)-\sigma_1e_1),$$

and we can estimate thanks to Lemma 3.1. A similar computation can be done for $x\leq -y$. Finally, we also have

$$\|g.w_*^\sigma(x)-w_*^\sigma(x)\|_{L^\infty}\leq\|g.w_*^\sigma\|_{L^\infty}+\|w_*^\sigma\|_{L^\infty}\leq 2,$$

which gives the estimate for $x\in[-y,0]$. Similar arguments for the derivative give the conclusion. □

3.2. Localisation

We fix some function ψ which satisfies the following assumptions :

- $\psi\equiv 0$ on $(-\infty,-1]$, $\psi\equiv 1$ on $[1,+\infty)$,
- $0\leq\psi\leq 1$ on $\mathbb{R}$,
- $\forall x\in\mathbb{R}, 1-\psi(x)=\psi(-x)$

- $\sqrt{\psi} \in W^{3,\infty}(\mathbb{R})$.

Then, we take some $R \geq 1$ large to be fixed later and we define a localisation function $\psi_R(x) := \psi\Big(\dfrac{x}{R}\Big)$ and a localised scalar product:

$$(f, g)_{\psi_R} = \int f(x)g(x)\psi_R(x)\,\mathrm{d}x,$$

defined for all $f, g \in L^2(\psi_R(x)\,\mathrm{d}x) := \{h \in L^2_{\mathrm{loc}}(\operatorname{supp}\psi_R), \|h\|^2_{L^2(\psi_R(x)\,\mathrm{d}x)} := (h,h)_{\psi_R} < \infty\}$. We can also define, in a classical way, $H^k(\psi_R(x)\,\mathrm{d}x)$ for all $k \in \mathbb{N}$. Moreover, observe that for all $R \geq 1$ and integer k,there holds

$$\left\|\partial_x^k \psi_R\right\|_{L^\infty} = \frac{1}{R^k}\left\|\partial_x^k \psi\right\|_{L^\infty}, \qquad \left\|\partial_x^k(\sqrt{\psi_R})\right\|_{L^\infty} = \frac{1}{R^k}\left\|\partial_x^k \sqrt{\psi}\right\|_{L^\infty}. \tag{3.1}$$

We also show a result with respect to the localised H^1 and H^2 norms.

Lemma 3.5. *There exists $C > 0$ such that, for any $f \in H^1(\mathbb{R})$ and any y_0 with $\psi_0 := \tau_{y_0}\psi_R$, there holds for $k = 0, 1, 2$*

$$\left|\left\|\sqrt{\psi_0} f\right\|^2_{H^k} - \|f\|^2_{H^k(\psi_0(x)\,\mathrm{d}x)}\right| \leq \frac{C}{R^2}\|f\|^2_{L^2(\operatorname{supp}\partial_x\psi_0)}.$$

Proof. First, it is easy to see that

$$\left\|\sqrt{\psi_0} f\right\|^2_{L^2} = \|f\|^2_{L^2(\psi_0(x)\,\mathrm{d}x)}.$$

As for the homogeneous H^1 (semi-)norm, we compute :

$$\partial_x\Big(\sqrt{\psi_0} f\Big) = \sqrt{\psi_0}\partial_x f + f\partial_x\sqrt{\psi_0}.$$

Therefore,

$$\begin{aligned}
\left\|\partial_x\Big(\sqrt{\psi_0} f\Big)\right\|^2_{L^2} &= \|\partial_x f\|^2_{L^2(\psi_0(x)\,\mathrm{d}x)} + 2\int \sqrt{\psi_0}\partial_x f \cdot f\partial_x\sqrt{\psi_0}\,\mathrm{d}x + \int\Big|f\partial_x\sqrt{\psi_0}\Big|^2\,\mathrm{d}x \\
&= \|\partial_x f\|^2_{L^2(\psi_0(x)\,\mathrm{d}x)} + \frac{1}{2}\int \partial_x|f|^2\partial_x\psi_0\,\mathrm{d}x + \int |f|^2\Big(\partial_x\sqrt{\psi_0}\Big)^2\,\mathrm{d}x \\
&= \|\partial_x f\|^2_{L^2(\psi_0(x)\,\mathrm{d}x)} + \int |f|^2\Big[\Big(\partial_x\sqrt{\psi_0}\Big)^2 - \frac{1}{2}\partial^2_{xx}\psi_0\Big]\,\mathrm{d}x.
\end{aligned}$$

The conclusion easily follows from the estimate of the L^∞ norm of $\Big(\partial_x\sqrt{\psi_0}\Big)^2 - \frac{1}{2}\partial^2_{xx}\psi_0$ with (3.1). Morover, there holds

$$\partial^2_{xx}\Big(\sqrt{\psi_0} f\Big) = \sqrt{\psi_0}\partial^2_{xx} f + 2\partial_x\sqrt{\psi_0}\partial_x f + f\partial^2_{xx}\sqrt{\psi_0},$$

so that

$$\Big(\partial^2_{xx}\Big(\sqrt{\psi_0} f\Big)\Big)^2 - \psi_0(\partial^2_{xx} f)^2 = 2\sqrt{\psi_0}\partial^2_{xx} f\Big(2\partial_x\sqrt{\psi_0}\partial_x f + f\partial^2_{xx}\sqrt{\psi_0}\Big) + \Big(2\partial_x\sqrt{\psi_0}\partial_x f + f\partial^2_{xx}\sqrt{\psi_0}\Big)^2.$$

Expanding the first term of the right-hand side and integrating, we get

$$\begin{aligned}
\int \sqrt{\psi_0}\partial^2_{xx} f\partial^2_{xx}\sqrt{\psi_0} f\,\mathrm{d}x &= \int f\partial^2_{xx} f\Big(\partial^2_{xx}\psi_0 - \Big(\partial_x\sqrt{\psi_0}\Big)^2\Big)\,\mathrm{d}x \\
&= -\int (\partial_x f)^2\Big(\partial^2_{xx}\psi_0 - \Big(\partial_x\sqrt{\psi_0}\Big)^2\Big)\,\mathrm{d}x \\
&\qquad\qquad - \int f\partial_x f\partial_x\Big(\partial^2_{xx}\psi_0 - \Big(\partial_x\sqrt{\psi_0}\Big)^2\Big)\,\mathrm{d}x,
\end{aligned} \tag{3.2}$$

$$\int \sqrt{\psi_0}\partial^2_{xx} f\partial_x\sqrt{\psi_0}\partial_x f\,\mathrm{d}x = \frac{1}{4}\int \partial_x\psi_0\partial_x(\partial_x f)^2\,\mathrm{d}x = -\frac{1}{4}\int \partial^2_{xx}\psi_0(\partial_x f)^2\,\mathrm{d}x, \tag{3.3}$$

and the conclusion follows from obvious estimation. $\square$

3.3. Localized multilinear estimates in Sobolev spaces

Definition 3.6. *For $k \geq 0$ and $\ell \geq 1$, and given a (possibly vector valued) function $f = (f_j)_{1 \leq j \leq J}$, we use the notation*

$$g = O_k^\ell(f)$$

for a (possibly vector valued) function g if each component of g is an homogeneous polynomial of degree ℓ in the components of f and their derivatives such that the total number of derivatives in each term is at most k, and whose coefficients are $\mathscr{C}_b^\infty(\mathbb{R})$ functions. g is then the sum of terms of the form

$$\alpha \prod_{j=1}^{J} \prod_{\kappa=0}^{k} (\partial_x^\kappa f_j)^{\ell_{j,\kappa}}, \qquad \text{where} \quad \sum_{j,\kappa} \ell_{j,\kappa} = \ell, \quad \text{and} \quad \sum_{j,\kappa} l_{j,\kappa}\kappa \leq k, \quad \text{and} \quad \alpha \in \mathscr{C}_b^\infty.$$

Lemma 3.7. 1. *If $k' \geq k$, then $O_k^\ell(f) = O_{k'}^\ell(f)$.*

2. *If $\alpha \in \mathscr{C}_b^\infty$, then $\alpha O_k^\ell(f) = O_k^\ell(f)$.*
3. *$O_k^\ell(f_1) O_{k'}^{\ell'}(f_2) = O_{k+k'}^{\ell+\ell'}(f_1, f_2)$.*
4. *$\partial_x O_k^\ell(f) = O_{k+1}^\ell(f)$,*
5. *$O_k^\ell(f_1 + f_2) = O_k^\ell(f_1, f_2)$.*

This notation has been used in [4] to express pointwise bounds that turn into Sobolev bounds with linear dependence in the highest term. We will generalize these estimates for localised integrations :

Lemma 3.8. 1. *Assume $g = O_k^\ell(f)$. Then there holds if $k \geq 2$*

$$\|g\|_{L^2(\operatorname{supp}\psi_R)} \lesssim \|f\|_{H^k(\operatorname{supp}\psi_R)} \|f\|_{H^{k-1}(\operatorname{supp}\psi_R)}^{\ell-1}.$$

If $k = 1$,

$$\|g\|_{L^2(\operatorname{supp}\psi_R)} \lesssim \|f\|_{H^1(\operatorname{supp}\psi_R)}^{\ell}.$$

2. *If $f \in H^1$, we have for $\ell \geq 2$,*

$$\left| \int O_2^\ell(f) \psi_R(x)\, \mathrm{d}x \right| \lesssim \|f\|_{H^1(\operatorname{supp}\psi_R)}^{\ell},$$

and, if $g \in H^1$,

$$\left| \int O_2^1(f) g(x) \psi_R(x)\, \mathrm{d}x \right| \lesssim \|f\|_{H^1(\operatorname{supp}\psi_R)} \|g\|_{H^1(\operatorname{supp}\psi_R)}.$$

3. *If $f \in H^2$, we have for $\ell \geq 2$*

$$\left| \int O_3^\ell(f) \psi_R(x)\, \mathrm{d}x \right| \lesssim \|f\|_{H^1(\operatorname{supp}\psi_R)}^{\ell-1} \|f\|_{H^2(\operatorname{supp}\psi_R)},$$

$$\left| \int O_4^\ell(f) \psi_R(x)\, \mathrm{d}x \right| \lesssim \|f\|_{H^1(\operatorname{supp}\psi_R)}^{\ell-2} \|f\|_{H^2(\operatorname{supp}\psi_R)}^2.$$

The proof of all these estimates is similar to that of [4], and we refer to it. We emphasize that all the integrals involved are indeed on the support of ψ_R, but also that for all $j \geq 1$, $H^j(\operatorname{supp}\psi_R) \hookrightarrow L^\infty(\operatorname{supp}\psi_R)$ with uniform constant since $\operatorname{supp}\psi_R = (-R, \infty)$ is an unbounded interval.

3.4. Coercivity of a Schrödinger operator

We also define the following operator for $\Gamma = \sqrt{1-\gamma^2}$ which was already used in [4] :

$$L_\Gamma v = -\partial_{xx}^2 v + \Gamma^2(\cos^2\theta_* - \sin^2\theta_*) v.$$

We recall the main properties of this operator.

Lemma 3.9 ([4, Lemma 4.10]). *L_Γ is a self-adjoint operator on $L^2(\mathbb{R})$ with dense domain $H^2(\mathbb{R})$. It admits 0 as a simple eigenvalue with eigenfunction $\sin\theta_*$, and its spectrum is $[\Gamma^2, +\infty)$. As a consequence, there exists $\lambda_0 > 0$ such that, for all $v \in H^1(\mathbb{R})$, $(L_\Gamma v, v) \leq 2\|v\|_{H^1}^2$ and*

$$(L_\Gamma v, v) \geq 4\lambda_0 \|v\|_{H^1} - \frac{1}{\lambda_0}\Big(\int v \sin(\theta_*)\,\mathrm{d}x\Big)^2,$$

and for all $v \in \mathbb{H}^2$,

$$\|L_\Gamma v\|_{L^2} \geq 4\lambda_0 \|v\|_{H^2} - \frac{1}{\lambda_0}\Big(\int v \sin(\theta_*)\,\mathrm{d}x\Big)^2.$$

However, we will not be able to apply directly this lemma on the same functions as in [4]. Indeed, the localisation function needs to be taken into account, as follows.

Lemma 3.10. *There exists $C > 0$ such that, for any $f \in H^2(\mathbb{R})$ and any y_0 with $\psi_0 := \tau_{y_0}\psi_R$,*

$$\begin{aligned} 2\|f\|_{H^1(\psi_0(x)\,\mathrm{d}x)}^2 + \frac{C}{R^2}\|f\|_{L^2(\operatorname{supp}\partial_x\psi_0)}^2 \geq \Big(L_\Gamma f, f\Big)_{\psi_0} \\ \geq 4\lambda_0\|f\|_{H^1(\psi_0(x)\,\mathrm{d}x)}^2 - \frac{1}{\lambda_0}\Big(\int \sqrt{\psi_0} f \sin(\theta_*)\,\mathrm{d}x\Big)^2 - \frac{C}{R^2}\|f\|_{L^2(\operatorname{supp}\partial_x\psi_0)}^2, \end{aligned}$$

and

$$\|L_\Gamma f\|_{L^2(\psi_R^\pm\,\mathrm{d}x)}^2 \geq 4\lambda_0\|f\|_{H^2(\psi_0(x)\,\mathrm{d}x)}^2 - \frac{1}{\lambda_0}\Big(\int \sqrt{\psi_0} f \sin(\theta_*)\,\mathrm{d}x\Big)^2 - \frac{C}{R^2}\|f\|_{H^1\operatorname{supp}\partial_x\psi_0(x)\,\mathrm{d}x}.$$

Proof. First, remark that we constructed ψ_R so that $\sqrt{\psi_0}f \in H^2$ as soon as $f \in H^2$. Then, we also have

$$(L_\Gamma f, f)_{\psi_0} = \Big(\sqrt{\psi_0} L_\Gamma f, \sqrt{\psi_0} f\Big).$$

From the definition of L_Γ, there holds

$$L_\Gamma(\sqrt{\psi_0}f) - \sqrt{\psi_0}L_\Gamma f = -\partial_{xx}^2\Big(\sqrt{\psi_0}f\Big) + \sqrt{\psi_0}\partial_{xx}^2 f = \partial_{xx}^2\sqrt{\psi_0}f + 2\partial_x\sqrt{\psi_0}\partial_x f. \tag{3.4}$$

Thus,

$$\begin{aligned} (L_\Gamma f, f)_{\psi_0} - \Big(L_\Gamma\Big(\sqrt{\psi_0}f\Big), \sqrt{\psi_0}f\Big) &= -\Big(\partial_{xx}^2\sqrt{\psi_0}f, \sqrt{\psi_0}f\Big) - 2\Big(\partial_x\sqrt{\psi_0}\partial_x f, \sqrt{\psi_0}f\Big) \\ &= -\Big(\sqrt{\psi_0}\partial_{xx}^2\sqrt{\psi_0}f, f\Big) - 2\Big(\sqrt{\psi_0}\partial_x\sqrt{\psi_0}\partial_x f, f\Big). \end{aligned}$$

Therefore, after an integration by parts in the last term,

$$\begin{aligned} (L_\Gamma f, f)_{\psi_0} - \Big(L_\Gamma\Big(\sqrt{\psi_0}f\Big), \sqrt{\psi_0}f\Big) &= -\Big(\sqrt{\psi_0}\partial_{xx}^2\sqrt{\psi_0}f, f\Big) + \Big(\partial_x\Big(\sqrt{\psi_0}\partial_x\sqrt{\psi_0}\Big)f, f\Big) \\ &= \Big(\Big(\partial_x\sqrt{\psi_0}\Big)^2 f, f\Big). \end{aligned}$$

Therefore, we get

$$\left|\Big(L_\Gamma f, f\Big)_{\psi_0} - \Big(L_\Gamma\Big(\sqrt{\psi_0}f\Big), \sqrt{\psi_0}f\Big)\right| \leq \frac{C}{R^2}\|f\|_{L^2(\operatorname{supp}\partial_x\psi_0)}^2.$$

The conclusion follows by applying Lemma 3.9 to $\Big(L_\Gamma\Big(\sqrt{\psi_0}f\Big), \sqrt{\psi_0}f\Big)$ and with Lemma 3.5. As for the second estimate, from (3.4), we also get

$$\Big(L_\Gamma\Big(\sqrt{\psi_0}f\Big)\Big)^2 = \Big(L_\Gamma f\Big)^2\psi_0 + 2\sqrt{\psi_0}L_\Gamma f\Big(\partial_{xx}^2\sqrt{\psi_0}f + 2\partial_x\sqrt{\psi_0}\partial_x f\Big) + \Big(\partial_{xx}^2\sqrt{\psi_0}f + 2\partial_x\sqrt{\psi_0}\partial_x f\Big)^2.$$

For the second term, expanding $L_\Gamma f$, we obtain by integrating the following terms

$$\int \sqrt{\psi_0}\partial_{xx}^2 f \partial_{xx}^2\sqrt{\psi_0}f\,\mathrm{d}x = \int f\partial_{xx}^2 f\Big(\partial_{xx}^2\psi_0 - \Big(\partial_x\sqrt{\psi_0}\Big)^2\Big)\,\mathrm{d}x \tag{3.5}$$

$$= -\int (\partial_x f)^2 \Big(\partial^2_{xx}\psi_0 - \Big(\partial_x \sqrt{\psi_0}\Big)^2\Big)\,\mathrm{d}x$$
$$- \int f \partial_x f \partial_x \Big(\partial^2_{xx}\psi_0 - \Big(\partial_x \sqrt{\psi_0}\Big)^2\Big)\,\mathrm{d}x,$$
$$\int \sqrt{\psi_0}\partial^2_{xx} f \partial_x \sqrt{\psi_0}\partial_x f\,\mathrm{d}x = \frac{1}{4}\int \partial_x\psi_0 \partial_x(\partial_x f)^2\,\mathrm{d}x = -\frac{1}{4}\int \partial^2_{xx}\psi_0(\partial_x f)^2\,\mathrm{d}x, \tag{3.6}$$

and also
$$\int \sqrt{\psi_0}\Gamma^2(\cos^2\theta_* - \sin^2\theta_*)\partial^2_{xx}\sqrt{\psi_0} f^2\,\mathrm{d}x \qquad \text{and} \qquad \frac{1}{2}\int \Gamma^2(\cos^2\theta_* - \sin^2\theta_*)\partial_x\psi_0 f \partial_x f\,\mathrm{d}x.$$
From straightforward estimates thanks to (3.1), we get
$$\left| \|L_\Gamma f\|^2_{L^2(\psi_R^\pm\,\mathrm{d}x)} - \left\|L_\Gamma\Big(\sqrt{\psi_R^\pm} f\Big)\right\|^2_{L^2} \right| \le \frac{C}{R}\|f\|^2_{H^1(\operatorname{supp}\partial_x\psi_R^\pm)}.$$

The estimate then comes by applying Lemma 3.9 to $\left\|L_\Gamma\Big(\sqrt{\psi_R^\pm} f\Big)\right\|^2_{L^2}$ and Lemma 3.5 again. □

3.5. Expansion in the associated basis

The computations made in [4] shows that the following frame is better adapted to a $\mathbb{S}^2$-valued magnetisation m close to a domain wall w_*^σ for some $\sigma = (\sigma_1, \sigma_2) \in \{\pm 1\}^2$. Define
$$n_*^\sigma := -\frac{1}{\sin\theta_*} w_*^\sigma \wedge (e_1 \wedge w_*^\sigma), \qquad p_*^\sigma := w_*^\sigma \wedge n_*.$$
$(w_*(x), n_*(x), p_*(x))$ is thus an orthonormal basis in $\mathbb{R}^3$ for all $x \in \mathbb{R}$.

One important observation, which motivates the introduction of this basis, is the following. Let $m = w + \eta \in \mathbb{S}^2$ with η small: if one decomposes
$$\eta = \mu w_* + \nu n_* + \rho p_*,$$
then μ is quadratic in η, whose norm is thus equivalent to that of ν and ρ. This is a pointwise in x, and is can be globalized or localized.

The precise statement is as follows.

Lemma 3.11. *There exists $\delta_3 > 0$ and $C_2 > 0$ such that the following holds. Let $w_* := w_*^\sigma$ for some $\sigma = (\sigma_1, \sigma_2) \in \{\pm 1\}^2$ be a domain wall. Let $m = w_* + \eta : \mathbb{R} \to \mathbb{S}^2$ and $x_0 > 0$ be such that*
$$\|\eta\|_{H^1((-x_0,\infty))} < \delta_3.$$
We decompose η in the (w_*, n_*, p_*) basis pointwise in x:
$$\eta = \mu w_* + \nu n_* + \rho p_* \quad \textit{where} \quad \mu := \eta\cdot w_*, \quad \nu = \eta\cdot n_*, \quad \rho = \eta \cdot p_*.$$
Then $\mu, \nu, \rho \in H^1((-x_0,\infty))$, with
$$\|\mu\|_{H^1((-x_0,\infty))} \le C_2\|\eta\|^2_{H^1((-x_0,\infty))}, \qquad \frac{1}{C_2}\|\eta\|_{H^1((-x_0,\infty))} \le \|(\nu,\rho)\|_{H^1((-x_0,\infty))} \le C_2\|\eta\|_{H^1((-x_0,\infty))}. \tag{3.7}$$
Moreover, as soon as $x_0 \ge R$, there also holds
$$\|\mu\|_{H^1(\psi_R\,\mathrm{d}x)} \le C_2\|\eta\|_{H^1(\psi_R\,\mathrm{d}x)}\|\eta\|_{H^1(\operatorname{supp}\psi_R)}, \qquad \frac{1}{C_2}\|\eta\|_{H^1(\psi_R\,\mathrm{d}x)} \le \|(\nu,\rho)\|_{H^1(\psi_R\,\mathrm{d}x)} \le C_2\|\eta\|_{H^1(\psi_R\,\mathrm{d}x)}. \tag{3.8}$$
In particular, $\mu = \frac{1}{2}|\eta|^2 = O_0^2(\eta)$. If furthermore $\eta \in H^2$, then $\mu, \nu, \rho \in H^2$ and
$$\|(\nu,\rho)\|_{H^2((-x_0,\infty))} \le C_2\|\eta\|_{H^2((-x_0,\infty))}$$
Last, there also hold
$$\rho\sin\theta_* = \eta\cdot(e_1\wedge w_*), \qquad \sigma_1\sin\theta_*\nu = \frac{1}{\Gamma^2}\eta\cdot\partial_x w_* - \gamma\eta\cdot(e_1\wedge w_*) \tag{3.9}$$

Proof. The proof is similar to the first step of the proof of [4, Proposition 4.16]. First, the relations between μ, ν, ρ and η along with Lemma 3.8 give

$$\|\mu\|_{H^k((-x_0,\infty))} + \|\nu\|_{H^k((-x_0,\infty))} + \|\rho\|_{H^k((-x_0,\infty))} \lesssim \|\eta\|_{H^k((-x_0,\infty))}.$$

On the other side, $\eta = \mu w_* + \nu n_* + \rho p_*$ and therefore

$$\|\eta\|_{H^k((-x_0,\infty))} \lesssim \|\mu\|_{H^k((-x_0,\infty))} + \|\nu\|_{H^k((-x_0,\infty))} + \|\rho\|_{H^k((-x_0,\infty))}.$$

$\mu = \frac{1}{2}|\eta|^2$ comes from the expansion of $|w_* + \eta|^2 = 1$, which gives the first inequality of (3.7) with Lemma 3.8. As soon as $\|\eta\|_{H^1((-x_0,\infty))}$ is small enough the second inequality is then straightforward. In a similar way, we also get $\partial_x \mu = \eta \cdot \partial_x \eta$, and the first inequality of (3.8) is then easily proved. If $\operatorname{supp} \psi_R \subset (-x_0, \infty)$, then $\|\eta\|_{H^1(\operatorname{supp} \psi_R)} < \delta_3$ from the assumption and the second inequality of (3.8) is proved similarly.

Eventually, the last equalities comes from the formulas (see (1.8) for the first one)

$$\partial_x w_* = \Gamma^2 \sin\theta_* (\sigma_1 n_* + \gamma p_*), \qquad e_1 \wedge w_* = \sin\theta_* p_*. \qquad \square$$

With this result, the magnetization can be decomposed in a similar way when it is close to a 2-domain wall structure (with the two domain walls far away enough).

Lemma 3.12. *There exists $\delta_2' > 0$ and $L_0 > R$ such that the following holds. Let $L > L_0$, $g^\pm = (y^\pm, \phi^\pm)$ such that $g^+ \in G_{>L}$ and $g^- \in G_{<-L}$. Let $m = w^+ + w^- + e_1 + \varepsilon$ for some $w^+ = g^+.w_*^{(1,\sigma_2)}$ and $w^- = g^-.w_*^{(-1,\sigma_2')}$, with $\varepsilon \in H^1$ and $m \in H^1(\mathbb{R}, \mathbb{S}^2)$. Define also*

$$\eta^\pm := (-g^\pm).m - w_*^\pm = (-g^\pm).(w^\mp + e_1 + \varepsilon).$$

$\eta^\pm$ can be decomposed in the $(w_*^\pm, n^\pm, p^\pm)$ basis associated to $w_*^\pm$:

$$\eta^\pm = \mu^\pm w_*^\pm + \nu^\pm n_*^\pm + \rho^\pm p_*^\pm.$$

Finally, define $\psi_R^\pm(x) = \psi_R(\pm x - y^\pm)$. Then, if $\|\varepsilon\|_{H^1} < \delta_2'$, there hold

$$\|\eta^\pm\|_{H^k(\psi_R^\pm\,\mathrm{d}x)} \leq \|\eta^\pm\|_{H^k(\operatorname{supp}\psi_R^\pm)} \leq \|\varepsilon\|_{H^k} + Ce^{\Gamma(R\pm y^\mp)}, \tag{3.10}$$

$$\|\varepsilon\|_{H^k} \leq \|\eta^+\|_{H^k(\psi_R^+(x)\,\mathrm{d}x)} + \|\eta^-\|_{H^k(\psi_R^-(x)\,\mathrm{d}x)} + C\Big(e^{\Gamma(R-y^+)} + e^{\Gamma(R+y^-)}\Big). \tag{3.11}$$

Moreover, there also holds

$$\frac{1}{C}\|(\nu^\pm, \rho^\pm)\|_{H^k(\operatorname{supp}\psi_R^\pm)} \leq \|\eta^\pm\|_{H^k(\operatorname{supp}\psi_R^\pm)} \leq C\|(\nu^\pm, \rho^\pm)\|_{H^k(\operatorname{supp}\psi_R^\pm)}, \tag{3.12}$$

Remark 3.13. This lemma shows that, as soon as $\|\varepsilon\|_{H^1}$ is small enough, estimating $\|\varepsilon\|_{H^1}$ is equivalent to estimating both $\|(\nu^\pm, \rho^\pm)\|_{H^k(\operatorname{supp}\psi_R^\pm)}$ and $e^{\Gamma(R\pm y^\pm)}$ for $i = 1, 2$. This property will be intensively used in the following.

Proof. The first inequality of (3.10) comes from the fact that $0 \leq \psi_R^\pm \leq 1$. The second one can be easily deduced from the following computation :

$$\begin{aligned}
\|\eta^+\|_{H^k(\operatorname{supp}\psi_R^+)} &= \|(-g^+).(w^- + e_1 + \varepsilon)\|_{H^k(\operatorname{supp}\psi_R^+)} \\
&= \|w^- + e_1 + \varepsilon\|_{H^k(\operatorname{supp}\psi_R)} \\
&\leq \|w^- + e_1\|_{H^k(\operatorname{supp}\psi_R)} + \|\varepsilon\|_{H^k(\operatorname{supp}\psi_R)} \\
&\leq \|g^-.(w_*^- + e_1)\|_{H^k((-R,\infty))} + \|\varepsilon\|_{H^k} \\
&\leq \|w_*^- + e_1\|_{H^k((-R-y^-,\infty))} + \|\varepsilon\|_{H^k},
\end{aligned}$$

and the conclusion with Lemma 3.1. The computations for η^- are similar. We also have

$$\|\varepsilon\|_{H^k}^2 = \|\varepsilon\|_{H^k(\psi_R(x)\,\mathrm{d}x)}^2 + \|\varepsilon\|_{H^k(\psi_R(-x)\,\mathrm{d}x)}^2,$$

and, similarly,

$$\begin{aligned}
\|\varepsilon\|_{H^k(\psi_R(x)\,\mathrm{d}x)} &= \|g^+.\eta^+ - (w^- + e_1)\|_{H^k(\psi_R(x)\,\mathrm{d}x)} \\
&\leq \|g^+.\eta^+\|_{H^k(\psi_R(x)\,\mathrm{d}x)} + \|w^- + e_1\|_{H^k(\psi_R(x)\,\mathrm{d}x)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\|\eta^{+}\right\|_{H^{k}(\psi_R^{+}(x)\,\mathrm{d}x)} + \left\|w^{-}+e_1\right\|_{H^{k}(\operatorname{supp}\psi_R)} \\
&\leq \left\|\eta^{+}\right\|_{H^{k}(\psi_R^{+}(x)\,\mathrm{d}x)} + Ce^{\Gamma(R+y^{-})}.
\end{aligned}
$$

Once again, the computation for $\|\varepsilon\|_{H^k(\psi_R(-x)\,\mathrm{d}x)}$ is similar and symmetric. Eventually, (3.12) comes from Lemma 3.11 and the fact that $\operatorname{supp}\psi_R^{+} \subset [-R-y^{+},\infty)$ and $\operatorname{supp}\psi_R^{-} \subset (-\infty, R-y^{-}]$. □

The goal is to use the previous lemma with the decomposition provided by Lemma 2.1. However, the localisation function will still remain in the integrals we compute. Therefore, we won't be able to get the same vanishing integrals as in [4] when we apply Lemma 3.9. However, the integrals we will obtain are still small enough : the reminiscence of the localisation function gives only negligible terms, as shown in the following lemma.

Lemma 3.14 (Almost orthogonality). *With the same assumptions and notations as in Lemma 2.1, define $\eta^{\pm}$, $\mu^{\pm}$, $\nu^{\pm}$, $\rho^{\pm}$ and $\psi_R^{\pm}$ as in Lemma 3.12. Then there holds*

$$
\left|\int \sqrt{\psi_R^{\pm}}\rho^{\pm}\sin\theta_*\,\mathrm{d}x\right| + \left|\int \sqrt{\psi_R^{\pm}}\nu^{\pm}\sin\theta_*\,\mathrm{d}x\right| \leq C\Big(q(y^{+}-y^{-}) + \|\varepsilon\|_{L^2}e^{\Gamma(R\mp y^{\pm})}\Big)
$$

Proof. From (3.9), we get

$$
\begin{aligned}
\int \sqrt{\psi_R^{\pm}}\rho^{\pm}\sin\theta_*\,\mathrm{d}x &= \int \sqrt{\psi_R^{\pm}}\eta^{\pm}\cdot(e_1\wedge w_*^{\pm})\,\mathrm{d}x, \\
\sigma_1\int \sqrt{\psi_R^{\pm}}\nu^{\pm}\sin\theta_*\,\mathrm{d}x &= \frac{1}{\Gamma}\int \sqrt{\psi_R^{\pm}}\eta^{\pm}\cdot\partial_x w_*^{\pm}\,\mathrm{d}x - \gamma\int \sqrt{\psi_R^{\pm}}\eta^{\pm}\cdot(e_1\wedge w_*^{\pm})\,\mathrm{d}x.
\end{aligned}
$$

On the other hand, by the expression of $\eta^{\pm}$,

$$
\int \sqrt{\psi_R^{\pm}}\eta^{\pm}\cdot(e_1\wedge w_*^{\pm})\,\mathrm{d}x = \int \sqrt{\psi_R(\pm x)}(g^{\mp}.w_*^{\mp}+e_1)\cdot(e_1\wedge g^{\pm}.w_*^{\pm})\,\mathrm{d}x + \int \sqrt{\psi_R(\pm x)}\,\varepsilon\cdot(e_1\wedge g^{\pm}.w_*^{\pm})\,\mathrm{d}x.
$$

For the first term, we can estimate by using the fact that $R < L < \min(y^{+}, -y^{-})$ and with Corollary 3.2:

$$
\begin{aligned}
\left|\int \sqrt{\psi_R}(g^{-}.w_*^{-}+e_1)\cdot(e_1\wedge g^{+}.w_*^{+})\,\mathrm{d}x\right| &\leq \int_{-R}^{\infty}\left|g^{-}.w_*^{-}+e_1\right|\left|e_1\wedge g^{+}.w_*^{+}\right|\mathrm{d}x \\
&\leq C(1+y^{+}-y^{-})e^{-\Gamma(y^{+}-y^{-})}.
\end{aligned}
$$

For the second term, by using the orthogonality conditions (2.1), we get

$$
\begin{aligned}
\left|\int \sqrt{\psi_R}\varepsilon\cdot(e_1\wedge g^{+}.w_*^{+})\,\mathrm{d}x\right| &= \left|\int\Big(\sqrt{\psi_R}-1\Big)\varepsilon\cdot(e_1\wedge g^{+}.w_*^{+})\,\mathrm{d}x\right| \\
&\leq \int_{-\infty}^{R}|\varepsilon|\left|e_1\wedge g^{+}.w_*^{+}\right|\mathrm{d}x \\
&\leq C\|\varepsilon\|_{L^2}e^{\Gamma(R-y^{+})}.
\end{aligned}
$$

Similar estimates hold for $\int\sqrt{\psi_R^{-}}\eta^{-}\cdot(e_1\wedge w_*^{-})\,\mathrm{d}x$ and for $\int\sqrt{\psi_R^{\pm}}\eta^{\pm}\cdot\partial_x w_*^{\pm}\,\mathrm{d}x$, and thus the conclusion □

4. LOCALISED ENERGIES

In this section, we prove Proposition 2.3. For this, we localise the energy thanks to the localisation ψ_R, which is a classical technique for to study multi-solitons for nonlinear dispersive equations.

We define the localised energies :

$$
E^{+}(m) := \frac{1}{2}\int\Big(|\partial_x m|^2 + 2\gamma\partial_x m\cdot(e_1\wedge m) + (1-m_1^2)\Big)\psi_R(x)\,\mathrm{d}x,
$$

$$
E^{-}(m) := \frac{1}{2}\int\Big(|\partial_x m|^2 + 2\gamma\partial_x m\cdot(e_1\wedge m) + (1-m_1^2)\Big)\psi_R(-x)\,\mathrm{d}x
$$

By the properties of ψ_R, we know that $E^{+}+E^{-}=E_\gamma$. Then, we define the following modified energies :

$$
\tilde{E}^{+}(m) := E^{+}(\tau_{y^{+}}m) = \frac{1}{2}\int\Big(|\partial_x m|^2 + 2\gamma\partial_x m\cdot(e_1\wedge m) + (1-m_1^2)\Big)\psi_R^{+}(x)\,\mathrm{d}x
$$

where $\psi_R^+ := \tau_{-y^+}\psi_R = \psi_R(x+y^+)$, and

$$\tilde{E}^-(m) := E^-(\tau_{y^-}m) = \frac{1}{2}\int \Big(|\partial_x m|^2 - 2\gamma\partial_x m\cdot(e_1\wedge m) + (1-m_1^2)\Big)\psi_R^-(x)\,\mathrm{d}x,$$

where $\psi_R^- := \tau_{-y^-}\psi_R(-x) = \psi_R(-x+y^-)$.

4.1. First estimate on the localised energies

First, we want to expand the localised energies defined previously. For this, we define $\eta^\pm$, and then $\mu^\pm$, $\nu^\pm$ and $\rho^\pm$ like in Lemma 3.12. With similar computations as in [4], we show that, up to some additional negligible terms, the expansion of the localised energies gives no term of order 1 and nice terms of order 0 and 2.

Proposition 4.1. *Let the assumptions of Lemma 3.12 be satisfied. Then,*

$$\left|E^\pm(m) - \Big[\tilde{E}^\pm(w_*^\pm) + \frac{1}{2}\Big((L_\Gamma\nu^\pm,\nu^\pm)_{\psi_R^\pm} + (L_\Gamma\rho^\pm,\rho^\pm)_{\psi_R^\pm}\Big)\Big]\right|$$
$$\leq C\Big[\|\varepsilon\|_{H^1}^3 + \frac{1}{R^2}\|\varepsilon\|_{L^2}^2 + e^{2\Gamma(R-y^+)} + e^{2\Gamma(R+y^-)}\Big],$$

Proof. The pointwise estimate of steps 2 and 3 of the proof of [4, Proposition 4.16] still hold, both for η^+ and η^-. In particular, we have :

$$\delta E_\gamma(\eta^\pm) = O_2^2(\eta^\pm) \pm 2(\partial_{xx}^2\theta_*\nu^\pm + \partial_x\theta_*\partial_x\nu^\pm)w_*^\pm + (-\partial_{xx}^2\nu^\pm + \Gamma\nu^\pm)n_*^\pm + (-\partial_{xx}^2\rho^\pm + \Gamma\rho^\pm)p_*^\pm,$$

$$\eta^\pm\cdot\delta E_{\gamma^\pm}(w_*^\pm) = \beta_*\eta^\pm\cdot w_*^\pm = -\frac{1}{2}\beta_*|\eta^\pm|^2, \tag{4.1}$$

$$\eta^\pm\cdot\delta E_{\gamma^\pm}(\eta^\pm) = O_2^3(\eta^\pm) - \nu^\pm\partial_{xx}^2\nu^\pm + \Gamma^2(\nu^\pm)^2 - \rho^\pm\partial_{xx}^2\rho^\pm + \Gamma^2(\rho^\pm)^2 \tag{4.2}$$

Moreover, even if $E^\pm$ consists only in quadratic terms of m, it is not invariant under translation due to the localisation term $\psi_R(\pm x)$, and one should also take care about the integrations by part, so that the relations of the step 4 of the proof of [4, Proposition 4.11] are different :

$$\begin{aligned}
E^\pm(m) &= \tilde{E}^\pm(\eta^\pm + w_*^\pm)\\
&= \tilde{E}^\pm(w_*^\pm) + \int \eta^\pm\cdot\delta E_\gamma(w_*^\pm)\psi_R^\pm(x)\,\mathrm{d}x + \frac{1}{2}\int \eta^\pm\cdot\delta E_\gamma(\eta^\pm)\psi_R^\pm(x)\,\mathrm{d}x\\
&\qquad - \int \eta^\pm\cdot\partial_x w_*^\pm\partial_x\psi_R^\pm(x)\,\mathrm{d}x - \frac{1}{2}\int \eta^\pm\cdot\partial_x\eta^\pm\partial_x\psi_R^\pm(x)\,\mathrm{d}x\\
&\qquad - \gamma\int \eta^\pm\cdot(e_1\wedge w_*^\pm)\partial_x\psi_R^\pm(x)\,\mathrm{d}x.
\end{aligned}$$

Using both (4.1) and (4.2) along with Lemma 3.8, we get

$$\begin{aligned}
E^\pm(m) - \tilde{E}^\pm(w_*^\pm) = O(\|\eta^\pm\|_{H^1(\operatorname{supp}\psi_R^\pm)}^3)\\
&- \frac{1}{2}\int \beta_*|\eta^\pm|^2\psi_R^\pm(x)\,\mathrm{d}x + \frac{1}{2}\int\Big((-\partial_{xx}^2\nu^\pm + \Gamma^2\nu^\pm)\nu^\pm + (-\partial_{xx}^2\rho^\pm + \Gamma^2\rho^\pm)\rho^\pm\Big)\psi_R^\pm(x)\,\mathrm{d}x\\
&- \int \eta^\pm\cdot\partial_x w_*^\pm\partial_x\psi_R^\pm(x)\,\mathrm{d}x - \gamma\int \eta^\pm\cdot(e_1\wedge w_*^\pm)\partial_x\psi_R^\pm(x)\,\mathrm{d}x\\
&+ \frac{1}{4}\int |\eta^\pm|^2\partial_{xx}^2\psi_R^\pm(x)\,\mathrm{d}x.
\end{aligned}$$

We now use the fact that

$$|\eta^\pm|^2 = (\mu^\pm)^2 + (\nu^\pm)^2 + (\rho^\pm)^2 = (\nu^\pm)^2 + (\rho^\pm)^2 + \frac{1}{4}|\eta^\pm|^4.$$

Then we also use the fact that

$$\Gamma^2 - \beta_* = \Gamma^2(\cos^2\theta_* - \sin^2\theta_*), \tag{4.3}$$

so that

$$-\frac{1}{2}\int \beta_*|\eta^\pm|^2\psi_R^\pm(x)\,\mathrm{d}x + \frac{1}{2}\int\Big((-\partial_{xx}^2\nu^\pm + \Gamma^2\nu^\pm)\nu^\pm + (-\partial_{xx}^2\rho^\pm + \Gamma^2\rho^\pm)\rho^\pm\Big)\psi_R^\pm(x)\,\mathrm{d}x$$

$$= \frac{1}{2}\Big((L_\Gamma \nu^\pm, \nu^\pm)_{\psi_R^\pm} + (L_\Gamma \rho^\pm, \rho^\pm)_{\psi_R^\pm} \Big) - \frac{1}{8}\int \beta_*^\pm |\eta^\pm|^4 \psi_R^\pm(x)\,\mathrm{d}x.$$

Moreover, using (3.1), we get

$$\left| \int \eta^\pm \cdot \partial_x w_*^\pm \partial_x \psi_R^\pm(x)\,\mathrm{d}x \right| \leq \frac{C}{R} \|\eta^\pm\|_{L^2(\operatorname{supp}\partial_x\psi_R^\pm)} \|\partial_x w_*^\pm\|_{L^2(\operatorname{supp}\partial_x\psi_R^\pm)},$$

$$\left| \int \eta^\pm \cdot (e_1 \wedge w_*^\pm) \partial_x \psi_R^\pm(x)\,\mathrm{d}x \right| \leq \frac{C}{R} \|\eta^\pm\|_{L^2(\operatorname{supp}\partial_x\psi_R^\pm)} \|e_1 + w_*^\pm\|_{L^2(\operatorname{supp}\partial_x\psi_R^\pm)},$$

$$\left| \int |\eta^\pm|^2 \partial_{xx}^2 \psi_R^\pm(x)\,\mathrm{d}x \right| \leq \frac{C}{R^2} \|\eta^\pm\|^2_{L^2(\operatorname{supp}\partial_x\psi_R^\pm)}.$$

Therefore, there holds

$$\begin{aligned} &\left| E^\pm - \Big[\tilde{E}^\pm(w^\pm) + \frac{1}{2}\Big((L_\Gamma \nu^\pm, \nu^\pm)_{\psi_R^\pm} + (L_\Gamma \rho^\pm, \rho^\pm)_{\psi_R^\pm} \Big) \Big] \right| \\ &\qquad \leq C \Big[\|\eta^\pm\|^3_{H^1(\operatorname{supp}\psi_R^\pm)} + \frac{1}{R} \|\eta^\pm\|_{L^2(\operatorname{supp}\partial_x\psi_R^\pm)} \|\partial_x w_*^\pm\|_{L^2(\operatorname{supp}\partial_x\psi_R^\pm)} \\ &\qquad\quad + \frac{1}{R} \|\eta^\pm\|_{L^2(\operatorname{supp}\partial_x\psi_R^\pm)} \|e_1 + w_*^\pm\|_{L^2(\operatorname{supp}\partial_x\psi_R^\pm)} \\ &\qquad\quad + \frac{1}{R^2} \|\eta^\pm\|^2_{L^2(\operatorname{supp}\partial_x\psi_R^\pm)} \Big], \end{aligned}$$

and the conclusion comes from (3.10) and Lemma 3.1. □

4.2. Localised energy of the domain wall

However, in the previous lemma, $\tilde{E}^\pm(w_*^\pm)$ is not a constant : it still depends on the localisation $\psi_R^\pm$, and therefore on $y^\pm$. The following lemma estimates how far this quantity is from the constant $E(w_*) \coloneqq E_\gamma(w_*^\pm)$.

Lemma 4.2. *There exists $C > 0$ such that, for any $y^\pm$ such that $y^\pm - R > 0$ for $i = 1$ and 2, there holds*

$$\left| \tilde{E}^\pm(w_*^\pm) - E(w_*) \right| \leq C\, e^{2\Gamma(R - y^\pm)}.$$

Proof. By the properties of $w^\pm$ and $\psi_R^\pm$, we have

$$\begin{aligned} \left| \tilde{E}^\pm(w^\pm) - E(w_*) \right| &= \left| \frac{1}{2}\int \Big(|\partial_x w_*^\pm|^2 + 2\gamma \partial_x w_*^\pm \cdot (e_1 \wedge w_*^\pm) + \sin^2\theta_* \Big)(1 - \psi_R^\pm(x))\,\mathrm{d}x \right| \\ &\leq C \|w_*^\pm + e_1\|^2_{H^1(I_\pm)}, \end{aligned}$$

where $I_+ \coloneqq (-\infty, R - y^+)$ and $I_- \coloneqq (-R - y^-, \infty)$, and the conclusion follows from Lemma 3.1. □

4.3. Estimates on the quadratic terms

As for the quadratic terms in Proposition 4.1, we can estimate them by applying Lemma 3.10 to $\nu^\pm$ and $\rho^\pm$. Applying also Lemma 3.12, the following estimates hold.

Corollary 4.3. *Under the assumptions of Proposition 4.1, there holds*

$$\Big(L_\Gamma \nu^\pm, \nu^\pm \Big)_{\psi_R^\pm} \geq 4\lambda_0 \|\nu^\pm\|^2_{H^1(\psi_R^\pm(x)\,\mathrm{d}x)} - \frac{1}{\lambda_0}\Big(\int \sqrt{\psi_R^\pm} \nu^\pm \sin(\theta_*)\,\mathrm{d}x \Big)^2 - \frac{C}{R^2}\Big(\|\varepsilon\|^2_{L^2} + e^{2\Gamma(R \pm y^\mp)} \Big),$$

$$\Big(L_\Gamma \rho^\pm, \rho^\pm \Big)_{\psi_R^\pm} \geq 4\lambda_0 \|\rho^\pm\|^2_{H^1(\psi_R^\pm(x)\,\mathrm{d}x)} - \frac{1}{\lambda_0}\Big(\int \sqrt{\psi_R^\pm} \rho^\pm \sin(\theta_*)\,\mathrm{d}x \Big)^2 - \frac{C}{R^2}\Big(\|\varepsilon\|^2_{L^2} + e^{2\Gamma(R \pm y^\mp)} \Big),$$

$$\Big(L_\Gamma \nu^\pm, \nu^\pm \Big)_{\psi_R^\pm} \leq 2 \|\nu^\pm\|^2_{H^1(\psi_R^\pm(x)\,\mathrm{d}x)} + \frac{C}{R^2}\Big(\|\varepsilon\|^2_{L^2} + e^{2\Gamma(R \pm y^\mp)} \Big),$$

$$\Big(L_\Gamma \rho^\pm, \rho^\pm \Big)_{\psi_R^\pm} \leq 2 \|\rho^\pm\|^2_{H^1(\psi_R^\pm(x)\,\mathrm{d}x)} + \frac{C}{R^2}\Big(\|\varepsilon\|^2_{L^2} + e^{2\Gamma(R \pm y^\mp)} \Big),$$

4.4. Equivalence between localised energy and H^1 norm

Putting everything together, we get bounds by below and above for $E^{\pm}(m)$.

Corollary 4.4. *Under the assumptions of Lemma 2.1 and assuming $L_1 > R$, there exists $C > 0$ such that the following holds. With same notations as in the conclusion of Lemma 2.1, for all $t \in [0, T]$,*

$$2C_2^2 \|\eta^{\pm}\|^2_{H^1(\psi_R^{\pm}\,\mathrm{d}x)} + C\Big(\|\varepsilon\|^2_{H^1} + e^{2\Gamma(R-y^+)} + e^{2\Gamma(R+y^-)}\Big) \geq E^{\pm}(m) - E(w_*)$$
$$\geq \frac{2\lambda_0}{C^2}\|\eta^{\pm}\|^2_{H^1(\psi_R^{\pm}\,\mathrm{d}x)} - C\Big(\|\varepsilon\|^3_{H^1} + \frac{1}{R^2}\|\varepsilon\|^2_{H^1} + e^{-2\Gamma(R-y^+)} + e^{2\Gamma(R+y^-)}\Big).$$

Proof. We use Proposition 4.1 whose assumptions are satisfied thanks to the conclusion of Lemma 2.1. Thus, we get for all $t \in [0, T]$

$$E^{\pm}(m) - \tilde{E}^{\pm}(w_*^{\pm})$$
$$\geq \frac{1}{2}\Big((L_\Gamma \nu^{\pm}, \nu^{\pm})_{\psi_R^{\pm}} + (L_\Gamma \rho^{\pm}, \rho^{\pm})_{\psi_R^{\pm}}\Big)\Big] - C\Big[\|\varepsilon\|^3_{H^1} + \frac{1}{R^2}\|\varepsilon\|^2_{L^2} + e^{2(R-y^+)} + e^{2(R+y^-)}\Big],$$

From this inequality, we can substitute $\tilde{E}^{\pm}(w_*^{\pm})$ into $E(w_*)$ thanks to Lemma 4.2. Then, both $(L_\Gamma \nu^{\pm}, \nu^{\pm})_{\psi_R^{\pm}}$ and $(L_\Gamma \rho^{\pm}, \rho^{\pm})_{\psi_R^{\pm}}$ can be estimated by below by using Corollary 4.3. Moreover, the terms

$$\Big(\int \sqrt{\psi_R^{\pm}}\nu^{\pm}\sin(\theta_*)\,\mathrm{d}x\Big)^2 \qquad \text{and} \qquad \Big(\int \sqrt{\psi_R^{\pm}}\rho^{\pm}\sin(\theta_*)\,\mathrm{d}x\Big)^2$$

are controlled by the "almost-orthogonality" estimates of Lemma 3.14. From this estimate, we have

$$q(y^+ - y^-)^2 \leq e^{-\Gamma(y^+ - y^-)} \leq e^{-2\Gamma y^+} + e^{2\Gamma y^-},$$

and thus the conclusion. The estimate by above is obtained with similar computations. □

Proposition 2.3 follows by taking the sum of the two estimates and applying (3.11) from Lemma (3.12).

5. Evolution of the energy

In this section, we prove Proposition 2.4. The evolution of the energy is already known from [4]. We recall it here.

Lemma 5.1 ([4, Theorem 4.1])**.** *There holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\gamma(m) = -\alpha\int(|\delta E_\gamma(m)|^2 - |m\cdot\delta E_\gamma(m)|^2)\,\mathrm{d}x + \alpha h(t)\int(m\wedge e_1)\cdot(m\wedge\delta E_\gamma(m))\,\mathrm{d}x.$$

From this result, we define the so-called *dissipation term*

$$D := \int(|\delta E_\gamma(m)|^2 - |m\cdot\delta E_\gamma(m)|^2)\,\mathrm{d}x,$$

and the *forcing term*

$$F := \int(m\wedge e_1)\cdot(m\wedge\delta E_\gamma(m))\,\mathrm{d}x.$$

In particular, there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\gamma(m) = -\alpha D + \alpha h(t)F.$$

5.1. Localisation

Like previously, we will localise each of these terms :

$$D^{\pm}(m) := \int(|\delta E_\gamma(m)|^2 - |m\cdot\delta E_\gamma(m)|^2)\psi_R(\pm x)\,\mathrm{d}x,$$
$$F^{\pm}(m) := \int(m\wedge e_1)\cdot(m\wedge\delta E_\gamma(m))\psi_R(\pm x)\,\mathrm{d}x,$$

Therefore, we have $F = F^+ + F^-$ and $D = D^+ + D^-$. We also define, in a similar way as previously,

$$\tilde{D}^\pm(m) \coloneqq D^\pm(\tau_{y^\pm} m) = \int (|\delta E_\gamma(m)|^2 - |m \cdot \delta E_\gamma(m)|^2) \psi_R^\pm(x) \,\mathrm{d}x,$$
$$\tilde{F}^\pm(m) \coloneqq F^\pm(\tau_{y^+} m) = \int (m \wedge e_1) \cdot (m \wedge \delta E_\gamma(m)) \psi_R^\pm(x) \,\mathrm{d}x,$$

5.2. Estimates on the localised terms

5.2.1. *Dissipation term.* First, we show that the dissipation term is positive and can be estimated up to some negligible terms.

Lemma 5.2. *Under the assumptions of Proposition 4.1, there exists $C > 0$ such that, if $\|\varepsilon\|_{H^1} \leq 1$,*

$$\Big| D^\pm(m) - \Big(\big\| L_\Gamma \nu^\pm \big\|^2_{L^2(\psi_R^\pm \,\mathrm{d}x)} + \big\| L_\Gamma \rho^\pm \big\|^2_{L^2(\psi_R^\pm \,\mathrm{d}x)} \Big) \Big| \leq C\Big(\|\varepsilon\|^2_{H^2} (\|\varepsilon\|_{H^1} + e^{\Gamma(R \pm y^\mp)}) + e^{3\Gamma(R\pm y^\mp)} \Big). \tag{5.1}$$

Proof. First, we recall that

$$m = g^+.(\eta^+ + w_*^+) = g^-.(\eta^- + w_*^-),$$

so that

$$D^\pm(m) = \tilde{D}^\pm(\eta^\pm + w_*^\pm)$$

Once again, the pointwise estimate of the steps 3 and 5 of the proof of [4, Proposition 4.16] still holds here, so that we get:

$$\big|\delta E_\gamma(\eta^\pm + w_*^\pm)\big|^2 - \big|\delta E_\gamma(w_*^\pm)\big|^2 = 2\beta_* w_*^\pm \cdot \delta E_\gamma(\eta^\pm) + \big|\delta E_\gamma(\eta^\pm)\big|^2,$$

$$\begin{aligned}\big|(\eta^\pm + w_*^\pm) \cdot \delta E_\gamma(\eta^\pm + w_*^\pm)\big|^2 - \big|w_*^\pm \cdot \delta E_\gamma(w_*^\pm)\big|^2 &= 2\beta_*^2 \mu^\pm + 2\beta_* w_*^\pm \cdot \delta E_\gamma(\eta^\pm) + 2\beta_* \eta^\pm \cdot \delta E_\gamma(\eta^\pm) \\ &\quad + \big|w_*^\pm \cdot \delta E_\gamma(\eta^\pm)\big|^2 + O_4^3(\eta^\pm) + O_4^4(\eta^\pm),\end{aligned}$$

$$\big|w_*^\pm \cdot \delta E_\gamma(\eta^\pm)\big|^2 = (2\partial^2_{xx}\theta_* \nu^\pm + 2\partial_x \theta_* \partial_x \nu^\pm)^2 + O_4^3(\eta^\pm) + O_4^4(\eta^\pm),$$
$$\big|\delta E_\gamma(\eta^\pm)\big|^2 = (2\partial^2_{xx}\theta_* \nu^\pm + 2\partial_x \theta_* \partial_x \nu^\pm)^2 + (-\partial^2_{xx}\nu^\pm + \Gamma^2 \nu^\pm)^2 + (-\partial^2_{xx}\rho^\pm + \Gamma^2\rho^\pm)^2 + O_4^3(\eta^\pm) + O_4^4(\eta^\pm).$$

We also recall that $\mu^\pm = -\frac{1}{2}((\nu^\pm)^2 + (\rho^\pm)^2) - \frac{1}{8}|\eta^\pm|^4$ and that $\big|\delta E_{\gamma^\pm}(w_*^\pm)\big| = w_*^\pm \cdot \delta E_{\gamma^\pm}(w_*^\pm) = \beta_*$. Therefore:

$$D^\pm(m) = \int \Big[(-\partial_{xx}\nu^\pm + \Gamma^2 \nu^\pm - \beta_* \nu^\pm)^2 + (-\partial_{xx}\nu^\pm + \Gamma^2\nu^\pm - \beta_*\nu^\pm)^2 + O_4^3(\eta^\pm) + O_4^4(\eta^\pm) \Big] \psi_R^\pm(x) \,\mathrm{d}x.$$

Using (4.3) and Lemma 3.8, we get

$$\begin{aligned} D^\pm(m) - \int \Big[\big|L_\Gamma \nu^\pm\big|^2 + \big|L_\Gamma \rho^\pm\big|^2 \Big] \psi_R^\pm(x) \,\mathrm{d}x \\ = O\Big(\big\|\eta^\pm\big\|_{H^1(\operatorname{supp}\psi_R^\pm)} \big\|\eta^\pm\big\|^2_{H^2(\operatorname{supp}\psi_R^\pm)} \Big) + O\Big(\big\|\eta^\pm\big\|^2_{H^1(\operatorname{supp}\psi_R^\pm)} \big\|\eta^\pm\big\|^2_{H^2(\operatorname{supp}\psi_R^\pm)} \Big).\end{aligned}$$

The conclusion follows from (3.10) and the fact that both $\|\varepsilon\|_{H^1} \leq 1$ and $e^{\Gamma(R - y^\pm)} \leq 1$. □

Once again, the localisation remains in the quadratic terms of the left-hand side. Applying Lemma 3.10 to both $\nu^\pm$ and $\rho^\pm$, along with Lemma 3.12, we obtain the following estimates:

Corollary 5.3. *Under the assumptions of Proposition 4.1, there exists $C, \lambda_5 > 0$ such that, for $i = 1, 2$ and $j = 3 - i$ and $R \geq 1$,*

$$\begin{aligned} D^\pm \geq \lambda_5 \big\|\eta^\pm\big\|^2_{H^2(\psi_R^\pm \,\mathrm{d}x)} - C\Big(\|\varepsilon\|^2_{H^2} (\|\varepsilon\|_{H^1} + e^{\Gamma(R\pm y^\mp)} + \frac{1}{R}) + e^{2\Gamma(R \pm y^\mp)} \Big) \\ - \frac{1}{\lambda_0} \Big(\int \sqrt{\psi_R^\pm} \nu^\pm \sin(\theta_*) \,\mathrm{d}x \Big)^2 - \frac{1}{\lambda_0} \Big(\int \sqrt{\psi_R^\pm} \rho^\pm \sin(\theta_*) \,\mathrm{d}x \Big)^2, \end{aligned}$$

$$\begin{aligned} D \geq \lambda_5 \|\varepsilon\|^2_{H^2} - C\Big(\|\varepsilon\|^2_{H^2} (\|\varepsilon\|_{H^1} + e^{\Gamma(R - y^+)} + e^{\Gamma(R + y^-)} + \frac{1}{R}) + e^{2\Gamma(R - y^+)} + e^{2\Gamma(R+y^-)} \Big) \\ - \frac{1}{\lambda_0} \sum_\iota \Big(\int \sqrt{\psi_R^\iota} \nu^\iota \sin(\theta_*) \,\mathrm{d}x \Big)^2 + \Big(\int \sqrt{\psi_R^\iota} \rho^\iota \sin(\theta_*) \,\mathrm{d}x \Big)^2. \end{aligned} \tag{5.2}$$

5.2.2. *Forcing term.* In a second step, we show that the forcing term is negligible enough with respect to the dissipation term.

Proposition 5.4. *Under the assumptions of Proposition 4.1, there exists $C > 0$ such that, for $i = 1, 2$ and $j = 3 - i$, and if $\|\varepsilon\|_{H^1} \leq 1$,*

$$\left|F^{\pm}(m)\right| \leq C(\|\varepsilon\|_{H^1}^2 + e^{2\Gamma(R \pm y^{\mp})})$$

Proof. Similarly, we have

$$F^{\pm}(m) = \tilde{F}^{\pm}(\eta^{\pm} + w_*^{\pm})$$

Again, we will take advantage of the pointwise computations of [4].

$$\begin{aligned}((\eta^{\pm} + w_*^{\pm}) \wedge e_1) \cdot ((\eta^{\pm} + w_*^{\pm}) \wedge \delta E_\gamma(\eta^{\pm} + w_*^{\pm})) &= (w_*^{\pm} \wedge e_1) \cdot (w_*^{\pm} \wedge \delta E_\gamma(w_*^{\pm})) + (\eta^{\pm} \wedge e_1) \cdot (w_*^{\pm} \wedge \delta E_\gamma(w_*^{\pm})) \\ &+ (w_*^{\pm} \wedge e_1) \cdot (\eta^{\pm} \wedge \delta E_\gamma(w_*^{\pm}) + w_*^{\pm} \wedge \delta E_\gamma(\eta^{\pm})) + O_2^2(\eta^{\pm}) + O_2^3(\eta^{\pm}).\end{aligned}$$

Since $w_*^{\pm} \wedge \delta E_{\gamma^{\pm}}(w_*^{\pm}) = 0$, the first two terms are 0. Then, observe that $w_*^{\pm} \wedge e_1 = \sin(\theta_*) p_*^{\pm}$ and $n_*^{\pm} \wedge w_*^{\pm} = -p_*^{\pm}$, thus

$$\begin{aligned}(w_*^{\pm} \wedge e_1) \cdot (\eta^{\pm} \wedge \delta E_{\gamma^{\pm}}(w_*^{\pm})) &= \sin(\theta_*) p_*^{\pm} \cdot (\eta^{\pm} \wedge \beta_* w_*^{\pm}) = -\beta_* \sin\theta_* \nu^{\pm}, \\ (w_*^{\pm} \wedge e_1) \cdot (w_*^{\pm} \wedge \delta E_{\gamma^{\pm}}(\eta^{\pm})) &= \sin\theta_*(-\partial_{xx}^2 \nu^{\pm} - \gamma^2 \nu^{\pm}) + O_2^2(\eta^{\pm}).\end{aligned}$$

Using (4.3), we obtain

$$((\eta^{\pm} + w_*^{\pm}) \wedge e_1) \cdot ((\eta^{\pm} + w_*^{\pm}) \wedge \delta E_\gamma(\eta^{\pm} + w_*^{\pm})) = \sin\theta_*(L_\Gamma \nu^{\pm}) + O_2^2(\eta^{\pm}) + O_2^3(\eta^{\pm}).$$

Moreover, we know that $L_\Gamma(\sin\theta_*) = 0$, therefore

$$\int \sin\theta_* L_\Gamma \nu^{\pm} \psi_R^{\pm}(x)\,\mathrm{d}x = 2\int \partial_x(\sin\theta_*)\nu^{\pm}\partial_x\psi_R^{\pm}(x)\,\mathrm{d}x + \int \sin\theta_* \nu^{\pm}\partial_{xx}^2\psi_R^{\pm}(x)\,\mathrm{d}x.$$

Estimating these terms by using the fact that $\operatorname{supp}\partial_x\psi_R^{\pm} \subset [-y^{\pm} - R, -y^{\pm} + R]$ (and the same for $\partial_{xx}^2\psi_R^{\pm}$), we get

$$\begin{aligned}\left|\int \partial_x(\sin\theta_*)\nu^{\pm}\partial_x\psi_R^{\pm}(x)\,\mathrm{d}x\right| &\leq \frac{\|\partial_x\psi\|_{L^\infty}}{R}\|\partial_x\sin\theta_*\|_{L^2((-y^{\pm}-R,-y^{\pm}+R))}\left\|\nu^{\pm}\right\|_{L^2((-y^{\pm}-R,-y^{\pm}+R))} \\ &\leq C\frac{\|\partial_x\psi\|_{L^\infty}}{R}e^{\Gamma(R\pm y^{\mp})}\left\|\nu^{\pm}\right\|_{L^2((-y^{\pm}-R,-y^{\pm}+R))}, \\ \left|\int \sin\theta_*\nu^{\pm}\partial_{xx}^2\psi_R^{\pm}(x)\,\mathrm{d}x\right| &\leq \frac{\left\|\partial_{xx}^2\psi\right\|_{L^\infty}}{R^2}\|\sin\theta_*\|_{L^2((-y^{\pm}-R,-y^{\pm}+R))}\left\|\nu^{\pm}\right\|_{H^1((-y^{\pm}-R,-y^{\pm}+R))} \\ &\leq C\frac{\left\|\partial_{xx}^2\psi\right\|_{L^\infty}}{R^2}e^{\Gamma(R\pm y^{\mp})}\left\|\nu^{\pm}\right\|_{H^1((-y^{\pm}-R,-y^{\pm}+R))}.\end{aligned}$$

Thus, we obtain

$$\left|\int \sin\theta_* L_\Gamma\nu^{\pm}\psi_R^{\pm}(x)\,\mathrm{d}x\right| \leq \frac{C}{R^2}\Big(\left\|\nu^{\pm}\right\|_{H^1((-y^{\pm}-R,-y^{\pm}+R))}^2 + e^{2\Gamma(R\pm y^{\mp})}\Big),$$

and the conclusion follows in the same way, using again Lemma 3.8. ☐

By summing for $\iota = \pm 1$, we get an estimate for F.

Corollary 5.5. *Under the assumptions of Proposition 4.1, there exists $C > 0$ such that, if $\|\varepsilon\|_{H^1} \leq 1$,*

$$|F(m)| \leq C(\|\varepsilon\|_{H^1}^2 + e^{2\Gamma(R-y^+)} + e^{2\Gamma(R+y^-)}) \tag{5.3}$$

5.3. Dissipation estimate

Now, we prove Proposition 2.4 thanks to the previous lemmas.

By definition, we have $\frac{\mathrm{d}}{\mathrm{d}t}E_\gamma(m) = -\alpha D + \alpha h(t) F$. From the estimates of Corollaries 5.3 and 5.5 on D and F respectively, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_\gamma(m) + \lambda_5\|\varepsilon\|_{H^2}^2 \leq C\alpha\|\varepsilon\|_{H^2}^2\Big(\|\varepsilon\|_{H^1} + e^{\Gamma(R-y^+)} + e^{\Gamma(R+y^-)} + \frac{1}{R}\Big) + |h(t)|\|\varepsilon\|_{H^1}^2 \\ + C\alpha|h(t)|\Big(e^{2\Gamma(R-y^+)} + e^{2\Gamma(R+y^-)}\Big) \\ + \frac{\alpha}{\lambda_0}\sum_{i=1}^{2}\Big(\int\sqrt{\psi_R^{\pm}}\nu^{\pm}\sin(\theta_*)\,\mathrm{d}x\Big)^2 + \Big(\int\sqrt{\psi_R^{\pm}}\rho^{\pm}\sin(\theta_*)\,\mathrm{d}x\Big)^2.\end{aligned}$$

The conclusion follows by applying Lemma 3.14 to the last terms of the right-hand side.

6. Proof of the decomposition of the magnetization

In this section, we prove Lemma 2.1 which decomposes the magnetization with two nice gauges. For $g^{\pm} \in G$ such that $\pm y^{\pm} \geq 0$, we define the profile $P_{g^+,g^-} := g^+.w_*^+ + g^-.w_*^- + e_1$. We also call $P_{0,g^-} := P_{(0,0),g^-}$. First, we prove an intermediate result for P_{g^+,g^-} in the same context as Lemma 3.4.

Lemma 6.1. *For all $g^{[1]}, g^{[2]}, g^{[3]}, g^{[4]} \in G$, there holds*

$$\left\| P_{g^{[1]},g^{[2]}} - P_{g^{[3]},g^{[4]}} \right\|_{H^1} \leq C\Big(\left| g^{[1]} - g^{[3]} \right| + \left| g^{[2]} - g^{[4]} \right| \Big).$$

Moreover, there exists $y_{\min} > 0$ such that, if $(-1)^i y^{[i]} \geq y_0 \geq y_{\min}$ for all i for some y_0, there also holds

$$\left\| g^{[1]}.w_*^+ - g^{[3]}.w_*^+ \right\|_{H^1}^2 + \left\| g^{[2]}.w_*^- - g^{[4]}.w_*^- \right\|_{H^1}^2 \leq \left\| P_{g^{[1]},g^{[2]}} - P_{g^{[3]},g^{[4]}} \right\|_{H^1}^2 + C y_0 e^{-2\Gamma y_0}.$$

Proof. By using Lemma 3.4 and the fact that

$$P_{g^{[1]},g^{[2]}} - P_{g^{[3]},g^{[4]}} = (g^{[1]}.w_*^+ - g^{[3]}.w_*^+) + (g^{[2]}.w_*^- - g^{[4]}.w_*^-),$$

we get the first estimate. As for the second one, we expand

$$\begin{aligned} \left\| P_{g^{[1]},g^{[2]}} - P_{g^{[3]},g^{[4]}} \right\|_{H^1}^2 = \left\| g^{[1]}.w_*^+ - g^{[3]}.w_*^+ \right\|_{H^1}^2 + \left\| g^{[2]}.w_*^- - g^{[4]}.w_*^- \right\|_{H^1}^2 \\ + 2\langle g^{[1]}.w_*^+ - g^{[3]}.w_*^+, g^{[2]}.w_*^- - g^{[4]}.w_*^- \rangle_{H^1}. \end{aligned}$$

From Lemma 3.4, assuming for instance $y^{[1]} \leq y^{[3]}$ and $y^{[4]} \leq y^{[2]}$, we have for $j \in \{0, 1\}$

$$\left| \partial_x^j g^{[1]}.w_*^+(x) - \partial_x^j g^{[3]} w_*^+(x) \right| \leq C e^{-\Gamma \max(0, x - y^{[3]}, y^{[1]} - x)},$$

$$\left| \partial_x^j g^{[2]}.w_*^-(x) - \partial_x^j g^{[4]} w_*^-(x) \right| \leq C e^{-\Gamma \max(0, x - y^{[2]}, y^{[4]} - x)}.$$

Then, we can show that

$$\begin{aligned} &\left| \partial_x^j g^{[1]}.w_*^+(x) - \partial_x^j g^{[3]} w_*^+(x) \right| \left| \partial_x^j g^{[2]}.w_*^-(-x) - \partial_x^j g^{[4]} w_*^-(-x) \right| \\ &\qquad \leq C e^{-\Gamma(y^{[1]} - y^{[2]})} \times \begin{cases} e^{-\Gamma(x - y^{[1]})} & \text{if } x \geq y^{[1]} \\ 1 & \text{if } x \in [-y^{[2]}, y^{[1]}] \\ e^{\Gamma(x - y^{[2]})} & \text{if } x \leq -y^{[2]} \end{cases}. \end{aligned}$$

Therefore, we are able to prove that

$$\left| \langle g^{[1]}.w_*^+ - g^{[3]}.w_*^+, (g^{[2]}.w_*^- - g^{[4]}.w_*^-)(-x) \rangle_{H^1} \right| \leq C(y^{[1]} - y^{[2]}) e^{-\Gamma(y^{[1]} - y^{[2]})} \leq C y_0 e^{-2\Gamma y_0},$$

and the conclusion follows. □

6.1. Definition and properties of the functional

Define

$$\begin{aligned} \overline{\mathcal{F}} : (L^2 + L^\infty) \times G^2 &\to \mathbb{R}^4 \\ (m, g^+, g^-) &\mapsto \begin{pmatrix} \int m \cdot (g^+.\partial_x w_*^+) \, \mathrm{d}x \\ \int m \cdot (e_1 \wedge (g^+.w_*^+)) \, \mathrm{d}x \\ \int m \cdot (g^-.\partial_x w_*^-) \, \mathrm{d}x \\ \int m \cdot (e_1 \wedge (g^-.w_*^-)) \, \mathrm{d}x \end{pmatrix}. \end{aligned}$$

and

$$\begin{aligned} \mathcal{F} : \quad \mathcal{H}^1 \times G^2 &\to \mathbb{R}^4 \\ (m, g^+, g^-) &\mapsto \begin{pmatrix} \int \varepsilon \cdot (g^+.\partial_x w_*^+) \, \mathrm{d}x \\ \int \varepsilon \cdot (e_1 \wedge (g^+.w_*^+)) \, \mathrm{d}x \\ \int \varepsilon \cdot (g^-.\partial_x w_*^-) \, \mathrm{d}x \\ \int \varepsilon \cdot (e_1 \wedge (g^-.w_*^-)) \, \mathrm{d}x \end{pmatrix}, \end{aligned}$$

where $\mathcal{H}^1 := H^1 + e_1$, $\varepsilon := m - P_{g^+,g^-} \in H^1$, so that $\mathcal{F}(m, g^+, g^-) = \overline{\mathcal{F}}(m - P_{g^+,g^-}, g^+, g^-)$. Remark also that $\overline{\mathcal{F}}(\cdot, g^+, g^-)$ is linear. Then we also define $\mathcal{M} := \mathcal{H}^1 \cap L^\infty(\mathbb{R}, \mathbb{S}^2)$.

Proposition 6.2. *There exists $C > 0$ such that there holds for all $m \in X$ and all $g^+, g^- \in G$*

$$\left|\overline{\mathcal{F}}(m, g^+, g^-)\right| \le C\|m\|_X,$$

for $X = L^2$ or L^∞.

Proof. The result easily follows the fact that both $\partial_x w_*^\pm$ and $e_1 \wedge w_*^\pm$ are bounded and decay exponentially at infinity (for $i = 1, 2$) thanks to Lemma 3.1. ☐

Corollary 6.3. *There exists $C > 0$ such that there holds for all $m, m' \in X$ and all $g^\pm \in G$*

$$\left|\mathcal{F}(m, g^+, g^-) - \mathcal{F}(m', g^+, g^-)\right| \le C\|m - m'\|_X,$$

$$\left|\mathcal{F}(m, g^+, g^-)\right| \le C\left\|m - P_{g^+,g^-}\right\|_X,$$

for $X = L^2$ or L^∞.

Proof. Those estimates can be easily deduced from Proposition 6.2 and the definition of $\mathcal{F}$, which gives in particular $\mathcal{F}(m, g^+, g^-) - \mathcal{F}(m', g^+, g^-) = \overline{\mathcal{F}}(m - m', g^+, g^-)$. ☐

Similarly, we prove a similar property for $D_{g^+,g^-}\overline{\mathcal{F}}$.

Lemma 6.4. *There exists $C > 0$ such that there holds for all $m, m' \in X$ and all $g^\pm \in G$*

$$\left\|D_{g^+,g^-}\overline{\mathcal{F}}(m, g^+, g^-)\right\| \le C\|m\|_X,$$

for $X = L^2$ or L^∞.

Proof. From the definition of $\overline{\mathcal{F}}$, we see that

$$\partial_{y^+}\overline{\mathcal{F}}(m, g^+, g^-) = \begin{pmatrix} -\int m \cdot (g^+.\partial_{xx}^2 w_*^+)\,\mathrm{d}x \\ -\int m \cdot (e_1 \wedge (g^+.\partial_x w_*^+))\,\mathrm{d}x \\ 0 \\ 0 \end{pmatrix},$$

$$\partial_{\phi^+}\overline{\mathcal{F}}(m, g^+, g^-) = \begin{pmatrix} \int m \cdot (e_1 \wedge (g^+.\partial_x w_*^+))\,\mathrm{d}x \\ \int m \cdot (e_1 \wedge (e_1 \wedge (g^+.w_*^+)))\,\mathrm{d}x \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, since $\partial_x^2 w_*^+$, $e_1 \wedge \partial_x w_*^+$ and $e_1 \wedge w_*^+$ decay exponentially at infinity thanks to Lemma 3.1, we get the conclusion for these two differentiates. As for $\partial_{y^-}\overline{\mathcal{F}}$ and $\partial_{\phi^-}\overline{\mathcal{F}}$, the same arguments give the conclusion. ☐

Let also define $\mathcal{F}_0(\zeta^+, \zeta^-, g^+, g^-) \coloneqq \overline{\mathcal{F}}(P_{\zeta^+,\zeta^-}, g^+, g^-)$ for any $\zeta^\iota, g^\iota \in G$, and

$$A \coloneqq \frac{2}{\Gamma}\begin{pmatrix} 1 & \gamma & 0 & 0 \\ -\gamma & -1 & 0 & 0 \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & -\gamma & -1 \end{pmatrix}$$

We point out that A is invertible as soon as $\gamma^2 < 1$.

Lemma 6.5. *For all $g^\pm \in G$ such that $y^+ - y^- \ge y_{\min}$, there holds*

$$\left\|D_{\zeta^+,\zeta^-}\mathcal{F}_0(g^+, g^-, g^+, g^-) + A\right\| \le C(y^+ - y^-)e^{-\Gamma(y^+ - y^-)}.$$

Proof. From the definition of $\mathcal{F}_0$ and by noting $\zeta^\iota = (z^\iota, \alpha^\iota)$, we know that

$$\partial_{z^+}\mathcal{F}_0(\zeta^+, \zeta^-, g^+, g^-) = \overline{\mathcal{F}}(\partial_{z^+}(P_{\zeta^+,\zeta^-}), g^+, g^-) = -\overline{\mathcal{F}}(\zeta^+.\partial_x w_*^+, g^+, g^-).$$

Therefore, taking $\zeta^+ = g^+$ and applying Lemma 3.3, we get

$$\partial_{z^+}\mathcal{F}_0(g^+, g^-, g^+, g^-) = -\frac{2}{\Gamma}\begin{pmatrix} 1 \\ -\gamma \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \int g^+.\partial_x w_*^+(-x) \cdot (g^-.\partial_x w_*^-)\,\mathrm{d}x \\ \int g^+.\partial_x w_*^+(-x) \cdot (e_1 \wedge (g^-.w_*^-))\,\mathrm{d}x \end{pmatrix}.$$

The last term can be estimated thanks to Corollary 3.2. Then, there also holds

$$\partial_{\alpha^+}\mathcal{F}_0(\zeta^+,\zeta^-,g^+,g^-) = \overline{\mathcal{F}}(\partial_{\phi^+}(P_{\zeta^+,\zeta^-}),g^+,g^-) = \overline{\mathcal{F}}(e_1\wedge\zeta^+.w_*^+,g^+,g^-).$$

Thus, applying Lemma 3.3 again,

$$\partial_{\alpha^+}\mathcal{F}_0(g^+,g^-,g^+,g^-) = -\frac{2}{\Gamma}\begin{pmatrix}\gamma\\-1\\0\\0\end{pmatrix} - \begin{pmatrix}0\\0\\ \int e_1\wedge g^+.w_*^+(-x)\cdot(g^-.\partial_x w_*^-)\,\mathrm{d}x\\ \int e_1\wedge g^+.w_*^+(-x)\cdot(e_1\wedge(g^-.w_*^-))\,\mathrm{d}x\end{pmatrix},$$

and the last term can be estimated again with Corollary 3.2. Similar computations for $\partial_{z^-}\mathcal{F}_0(g^+,g^-,g^+,g^-)$ and $\partial_{\phi^-}\mathcal{F}_0(g^+,g^-,g^+,g^-)$ give the conclusion. □

Lemma 6.6. *For any $m\in\mathcal{M}$ and any $g^\pm\in G$ such that $y^+-y^-\geq y_{\min}$, there holds*

$$\left\|D_{g^+,g^-}\mathcal{F}(m,g^+,g^-)-A\right\| \leq C\Big(\left\|m-P_{g^+,g^-}\right\|_{H^1}+(y^+-y^-)e^{-\Gamma(y^+-y^-)}\Big)$$

Proof. From the definition of $\mathcal{F}$ and , there holds

$$\mathcal{F}(m,g^+,g^-) = \overline{\mathcal{F}}(m,g^+,g^-) - \overline{\mathcal{F}}(P_{g^+,g^-},g^+,g^-).$$

Therefore, using the fact that $\overline{\mathcal{F}}(\cdot,g^+,g^-)$ is linear and then so is $D_{g^+,g^-}\overline{\mathcal{F}}(\cdot,g^+,g^-)$, we obtain

$$\begin{aligned}D_{g^+,g^-}\mathcal{F}(m,g^+,g^-) &= D_{g^+,g^-}\overline{\mathcal{F}}(m,g^+,g^-) - D_{g^+,g^-}\overline{\mathcal{F}}(P_{g^+,g^-},g^+,g^-) - D_{\zeta^+,\zeta^-}\mathcal{F}_0(g^+,g^-,g^+,g^-)\\ &= D_{g^+,g^-}\overline{\mathcal{F}}(m-P_{g^+,g^-},g^+,g^-) - D_{\zeta^+,\zeta^-}\mathcal{F}_0(g^+,g^-,g^+,g^-).\end{aligned}$$

The conclusion is reached by applying Lemma 6.4 to the first term of the right-hand side and Lemma 6.5 to the second term. □

6.2. Implicit function theorem

Lemma 6.7. *1. There exists $C>0$, $y_{\min}>0$ and $\delta_0>0$ such that, for all $m\in\mathcal{M}$ and $g_0\in G$ satisfying $y_0<-y_{\min}$ and*

$$\delta := \|m-P_{0,g_0}\|_{H^1} < \delta_0,$$

there exists unique $\overline{g}^\pm\in G$ such that

- $\left|\overline{g}^- - g_0\right|\leq C\delta$ *and* $\left|\overline{g}^+\right|\leq C\delta$,
- $\left\|m-P_{\overline{g}^+,\overline{g}^-}\right\|_{H^1}\leq C\delta$,
- $\mathcal{F}(m,\overline{g}^+,\overline{g}^-)=0$.

2. Moreover, up to taking $\delta_0>0$ smaller and $y_{\min}$ larger, $(\overline{g}^+,\overline{g}^-)$ does not depend on g_0.

3. The application

$$\begin{aligned}\{m\in e_1+H^1 | \inf_{y_0>y_{\min}}\|m-P_{0,g_0}\|_{H^1}<\delta_0\} &\to G^2\\ m &\mapsto (\overline{g}^+,\overline{g}^-) \text{ satisfying the previous properties}\end{aligned}$$

is $\mathscr{C}^\infty$ with respect to the H^1 topology

Proof. 1st step : Existence and uniqueness

First, remark that $\mathcal{F}(m,\overline{g}^+,\overline{g}^-)=0$ is equivalent to $\overline{p}=\overline{p}-A^{-1}\mathcal{F}(m,\overline{p})$ where $\overline{p}=(\overline{g}^+,\overline{g}^-)$. We define $\mathcal{G}(m,p)=p-A^{-1}\mathcal{F}(m,p)$ for any $p\in\mathbb{R}^4$ and we look for a fixed point for this function. Moreover, there holds $D_p\mathcal{G}(m,p)=I_4-A^{-1}D_p\mathcal{F}(m,p)$. Therefore, by applying Lemma 6.6, there holds

$$\begin{aligned}\|D_p\mathcal{G}(m,p)\| &\leq \left\|A^{-1}\right\|\|D_p\mathcal{F}(m,p)-A\|\\ &\leq C\left\|A^{-1}\right\|\Big(q(y^+-y^-)+\left\|m-P_{g^+,g^-}\right\|_{H^1}\Big)\end{aligned}$$

Therefore, if we take $p \in B_{p_0}(\xi)$ (where $p_0 := ((0,0), g_0)$) for some $\xi > 0$ to be defined later and assuming $\xi < 1 < y_{\min}$, we get thanks to Lemma 6.1:

$$\begin{aligned}\|D_p\mathcal{G}(m,p)\| &\leq C\|A^{-1}\|\Big(q(y_0) + \delta + |g^+| + |g^- - g_0|\Big)\\ &\leq C\|A^{-1}\|\Big(q(y_0) + \delta + \xi\Big).\end{aligned}$$

Hence, $\|D_p\mathcal{G}(m,p)\| \leq \frac{1}{2}$ as soon as

$$C\|A^{-1}\|\Big(q(y_0) + \delta + \xi\Big) \leq \frac{1}{2}. \tag{6.1}$$

On the other hand, we know that $|\mathcal{F}(m,p_0)| \leq C\|m - P_{p_0}\|_{H^1}$ thanks to Lemma 6.2. Thus,

$$|\mathcal{G}(m,p_0) - p_0| \leq \|A^{-1}\||\mathcal{F}(m,p_0)| \leq C\|A^{-1}\|\delta.$$

Moreover, by assuming (6.1) so that $\|D_p\mathcal{G}(m,.)\| \leq \frac{1}{2}$ on $B_{p_0}(\xi)$, we get for all $p \in B_{p_0}(\xi)$,

$$|\mathcal{G}(m,p) - \mathcal{G}(m,p_0)| \leq \frac{1}{2}|p - p_0| \leq \frac{\xi}{2},$$

which yields

$$|\mathcal{G}(m,p) - p_0| \leq C\|A^{-1}\|\delta + \frac{\xi}{2}.$$

This means that $\mathcal{G}(m,p) \in B_{p_0}(\xi)$ as soon as $C\|A^{-1}\|\delta + \frac{\xi}{2} \leq \xi$, i.e.

$$C\|A^{-1}\|\delta \leq \frac{\xi}{2}. \tag{6.2}$$

From the previous computations, we conclude that $\mathcal{G}(m,\cdot)$ is a contraction on $B_{p_0}(\xi)$ as soon as (6.1) and (6.2) hold. Taking $\xi = 2C\|A^{-1}\|\delta$, (6.2) is thus satisfied and (6.1) is then also satisfied as soon as y_0 is large enough and δ small enough: to be more precise, as soon as

$$C((y_0+1)e^{-\Gamma y_0} + \delta) \leq \frac{1}{2}.$$

The conclusion then follows from the fixed point theorem for a contraction. Remark that we could have also taken $\xi = 2C\|A^{-1}\|\delta_0$ with $\delta_0 > 0$ small enough, which also gives that the solution is unique in $B_{p_0}(2C\|A^{-1}\|\delta_0)$.

2nd step : Dependence on g_0

Let $\delta_0' > 0$ and $y_{\min}' > 0$ and assume that there exists g_0 and g_0' in G such that $y_0 > -y_{\min}'$ and $\|m - P_{0,g_0}\| < \delta_0'$ and the same for g_0'. Then, there holds

$$\|P_{0,g_0'} - P_{0,g_0}\|_{H^1} \leq \|m - P_{0,g_0'}\|_{H^1} + \|m - P_{0,g_0}\|_{H^1} < 2\delta_0'.$$

On the other hand,

$$P_{0,g_0'} - P_{0,g_0} = g_0'.w_*^- - g_0.w_*^-,$$

therefore, as soon as δ_0' is small enough, we can apply Lemma 3.4 and get

$$|g_0' - g_0| \leq C\delta_0'.$$

Moreover, using Step 1, we get

$$|\overline{g}^+| + |\overline{g}^- - g_0| \leq C\delta_0',$$

and the same for $\overline{g}^{\iota\prime}$ with g_0'. At the end, we get

$$\left|\overline{g}^{+\prime}\right| + \left|\overline{g}^{-\prime} - g_0\right| \leq C{\delta_0}'.$$

Thus, taking ${\delta_0}' > 0$ small enough, we get that $({g^+}', {g^-}') \in B_{p_0}(2C\|A^{-1}\|\delta_0)$. By uniqueness of (g^+, g^-) in this ball, we get $({g^+}', {g^-}') = (g^+, g^-)$.

3rd step : Regularity of the application

In Step 1, we only considered p such that

$$\|D_p\mathcal{F}(m,p) - A\| \leq \frac{1}{2\|A^{-1}\|},$$

and thus $D_p\mathcal{F}(m,p)$ is invertible since

$$A^{-1}D_p\mathcal{F}(m,p) = I_4 - (I_4 - A^{-1}D_p\mathcal{F}(m,p)),$$

with

$$\left\|I_4 - A^{-1}D_p\mathcal{F}(m,p)\right\|_{H^1} \le \left\|A^{-1}\right\|\|D_p\mathcal{F}(m,p) - A\| \le \frac{1}{2}.$$

This is in particular true for $(\overline{g}^+, \overline{g}^-)$. Therefore, the regularity of the application at m can be deduced from the implicit function theorem applied on $\mathcal{F}$ (which is a $\mathscr{C}^\infty$ function since $\partial_x w_*^\iota$ and $e_1 \wedge w_*^+$ are H^∞) at the point $(m, \overline{g}^+, \overline{g}^-)$. □

6.3. Decomposition near a 2-domain wall

For any $T > 0$ possibly infinite, define $I_T \coloneqq [0,T]$ if $T < \infty$ or $I_T \coloneqq [0,\infty)$ if $T = \infty$

Lemma 6.8. *There exist $\delta_2 > 0$ and $y_{\min} > 0$ such that for all $T > 0$ possibly infinite and all $m \in \mathscr{C}(I_T, \mathcal{H}^1)$ satisfying*

$$\delta \coloneqq \sup_{t\in I_T} \inf_{\substack{\zeta^\pm\in G\\ \pm y^\pm \ge y_0}} \left\|m(t) - P_{\zeta^+,\zeta^-}\right\|_{H^1} < \delta_2$$

for some $y_0 > y_{\min}$, there exists $\overline{g}^\iota = (\overline{y}^\iota, \overline{\phi}^\iota) : I_T \to G$ for $i = 1,2$ and $\varepsilon \in \mathscr{C}(I_T, H^2)$ such that, for all $t \in I_T$,

- $\iota\overline{y}^\iota(t) \ge y_0 - 1 \ge y_{\min} - 1$,
- $m(t) = \overline{g}^+(t).w_*^+ + \overline{g}^-(t).w_*^- + e_1 + \varepsilon(t)$,
- $\mathcal{F}(m(t), \overline{g}^+, \overline{g}^-) = \overline{\mathcal{F}}(\varepsilon(t), \overline{g}^+, \overline{g}^-) = 0$,
- $\|\varepsilon(t)\|_{H^1} \le C\Big(\delta + q(y^+ - y^-)\Big)$.

Moreover, if m is $\mathscr{C}^1(I_T, \mathcal{H}^1)$ (resp. $W^{1,\infty}_{\text{loc}}(I_T, \mathcal{H}^1)$), then both $\overline{g}^\iota$ are $\mathscr{C}^1(I_T)$ (resp. locally Lipschitz).

Proof. As $m \in \mathscr{C}(I_T, \mathcal{H}^1)$, m is uniformly continuous on I_T if $T < \infty$ or on every $[0,T']$ for $T' < \infty$ if $T = \infty$. We assume $T < \infty$, since the proof can be easily adapted for the case $T = \infty$. We can thus find $0 = t_0 < t_1 < \cdots < t_N = T$ such that, for all $0 \le 0 \le N-1$ and $t \in [t_k, t_{k+1}]$, there holds $\|m(t) - m(t_k)\|_{H^1} \le \delta$. Then, for any k, we can find $g_k^\iota \in G$ ($\iota = \pm 1$) such that $y_k^\iota \ge y_0$ and

$$\left\|m(t) - P_{g_k^+, g_k^-}\right\|_{H^1} < 2\delta.$$

In particular, for all k,

$$\left\|P_{g_{k+1}^+, g_{k+1}^-} - P_{g_k^+, g_k^-}\right\|_{H^1} < 5\delta.$$

From Lemma 6.1, we thus get

$$\left\|g_{k+1}^+.w_*^+ - g_k^+.w_*^+\right\|_{H^1}^2 + \left\|g_{k+1}^-.w_*^- - g_k^-.w_*^-\right\|_{H^1}^2 \le 25\delta^2 + Cy_0e^{-2\Gamma y_0}.$$

Therefore, if we assume that δ is small enough and y_0 large enough, we can apply [4, Claim 4.12] and get integers n_{k+1}^+ and n_{k+1}^- such that, by noting $\mu = \delta + \sqrt{y_0}e^{-\Gamma y_0}$

$$\left|g_{k+1}^\iota + (0, 2\pi n_{k+1}^\iota) - g_k^\iota\right| \le C\left\|g_{k+1}^\iota.w_*^\iota - g_k^\iota.w_*^\iota\right\|_{H^1} \le C\mu.$$

We can then change every ϕ_k^ι by adding $2\pi n_k^{\iota\prime}$ for some well chosen $n_k^{\iota\prime} \in \mathbb{Z}$ such that, for all k,

$$\left|g_{k+1}^\iota - g_k^\iota\right| \le C\mu.$$

Consider now $\tilde{g}_1^\iota$ affine on each segment $[t_k, t_{k+1}]$ such that $\tilde{g}_1^\iota(t_k) = g_k^\iota$ for $k = 0, \dots, N$, and then consider smooth functions $\tilde{g}_0^\iota$ such that $\|\tilde{g}_0^\iota - \tilde{g}_1^\iota\|_{\mathscr{C}(I_T)} \le \mu$. Thus we get for any $t \in [t_k, t_{k+1}]$ by applying Lemma 3.4

$$\|\tilde{g}_1^\iota(t).w_*^\iota - \tilde{g}_1^\iota(t_k).w_*^\iota\|_{H^1} \le C|\tilde{g}_1^\iota(t) - \tilde{g}_1^\iota(t_k)| \le C|\tilde{g}_1^\iota(t_{k+1}) - \tilde{g}_1^\iota(t_k)| \le C\mu,$$

and therefore, by using Lemma 6.1,

$$\begin{aligned}\left\|m(t) - P_{\tilde{g}_0^+(t),\tilde{g}_0^-(t)}\right\|_{H^1} &\le \|m(t) - m(t_k)\|_{H^1} + \left\|m(t_k) - P_{\tilde{g}_1^+(t_k),\tilde{g}_1^-(t_k)}\right\|_{H^1} \\ &\quad + \left\|P_{\tilde{g}_1^+(t_k),\tilde{g}_1^-(t_k)} - P_{\tilde{g}_1^+(t),\tilde{g}_1^-(t)}\right\|_{H^1} + \left\|P_{\tilde{g}_1^+(t),\tilde{g}_1^-(t)} - P_{\tilde{g}_0^+(t),\tilde{g}_0^-(t)}\right\|_{H^1}\end{aligned}$$

$$\leq \delta + 2\delta + C\mu + C\mu \leq C\mu.$$

By assuming μ small enough, i.e. δ small enough and y_0 large enough, we can assume

$$\left\|(-\tilde{g}_0^+(t)).m(t) - P_{0,(\tilde{g}_0^+)'(t)+\tilde{g}_0^-(t)}\right\|_{H^1} = \left\|m(t) - P_{\tilde{g}_0^+(t),\tilde{g}_0^-(t)}\right\|_{H^1} \leq \delta_0$$

where δ_0 comes from Lemma 6.7 and $\tilde{g}_0^+ = (\tilde{y}_0^+, \tilde{\phi}_0^+)$ gives $(\tilde{g}_0^+)' = (\tilde{y}_0^+, -\tilde{\phi}_0^+)$. Moreover, for any $t \in [t_k, t_{k+1}]$, we have $|\tilde{g}_0^\iota(t) - g_k^\iota| \leq C\mu$, which gives

$$\pm\tilde{y}_0^\pm(t) \geq \pm y_k^\pm - C\mu \geq y_0 - C\mu \qquad \text{and} \qquad \tilde{y}_0^+(t) - \tilde{y}_0^-(t) \geq 2y_0 - C\mu \geq 2y_{\min} - 1,$$

for μ small enough. Therefore, we can apply Lemma 6.7 to $(-\tilde{g}_0^+(t)).m(t)$, which gives some $\overline{g}_0^+(t), \overline{g}_0^-(t) \in G$ for any $t \in I$ such that

- $\left|\overline{g}_0^-(t) - \left((\tilde{g}_0^+)'(t) + \tilde{g}_0^-(t)\right)\right| \leq C\mu$ and $\left|\overline{g}_0^+(t)\right| \leq C\mu$,
- $\left\|(-\tilde{g}_0^+(t)).m(t) - P_{\overline{g}_0^+,\overline{g}_0^-}\right\|_{H^1} \leq C\mu$,
- $\mathcal{F}((-\tilde{g}_0^+(t)).m(t), \overline{g}_0^+, \overline{g}_0^-) = 0$.

Then, define

$$\overline{g}^+ \coloneqq \tilde{g}_0^+ + \overline{g}_0^+, \qquad \overline{g}^- \coloneqq \overline{g}_0^- - (\tilde{g}_0^+)'.$$

These gauges satisfy $\mathcal{F}(m(t), \overline{g}^+(t), \overline{g}^-(t)) = 0$, and also $|\overline{g}^\iota(t) - \tilde{g}_0^\iota(t)| \leq C\mu$, which means that

$$\pm\overline{y}^\pm \geq \tilde{y}_0^\pm - C\mu \geq y_0 - C\mu \geq y_0 - 1,$$

for μ small enough and where $\overline{g}^\iota = (\overline{y}^\iota, \overline{\phi}^\iota)$. Moreover, since $\tilde{g}_0^+$ is smooth, $(-\tilde{g}_0^+).m$ has the same regularity as m. Therefore, if m is $\mathscr{C}^1(I_T, \mathcal{H}^1)$, then the regularity result of Lemma 6.7 gives that $\overline{g}_0^\iota \in \mathscr{C}^1(I_T)$ and so are $\overline{g}^\iota$. Similar arguments when $m \in W^{1,\infty}(I_T, \mathcal{H}^1)$ give the conclusion. □

6.4. Decomposition under LLG flow

Lemma 6.9. *If $m \in \mathscr{C}(I_T, \mathcal{H}^2)$ is a solution of* (LLG)*, then both $\overline{g}^\iota$ given by Lemma 6.8 are Lipschitz and satisfy, for a.e. $t \in I_T$,*

$$\left|\dot{\overline{g}}^\iota(t) - \dot{g}_*^\iota(t)\right| \leq C\Big(\|\varepsilon(t)\|_{H^1} + (\overline{y}^+(t) - \overline{y}^-(t))e^{-\Gamma(\overline{y}^+(t) - \overline{y}^-(t))}\Big),$$

Proof. Let assume first that $m \in \mathscr{C}(I_T, \mathcal{H}^3)$. From (LLG), we get $\partial_t m \in \mathscr{C}(I_T, H^1)$. Therefore, both $\overline{g}^\iota$ given by Lemma 6.8 are $\mathscr{C}^1(I_T)$. Then, we can compute using the fact that $\varepsilon \in \mathscr{C}^1(I_T, H^1)$:

$$\partial_t m = \partial_t \varepsilon - \dot{\overline{y}}^+ \overline{g}^+.\partial_x w_*^+ + \dot{\overline{\phi}}^+ e_1 \wedge \overline{g}^+.w_*^+ - \dot{\overline{y}}^- \overline{g}^-.\partial_x w_*^- + \dot{\overline{\phi}}^- e_1 \wedge \overline{g}^-.w_*^-$$

Then, δE_γ is linear and $\delta E_\gamma(w_*^\pm) = \beta_* w_*^\pm$, so

$$H(m) = -\overline{g}^+.(\beta_* w^+) - \overline{g}^-.(\beta_* w_*^-) - \delta E_\gamma(\varepsilon) + h(t)e_1,$$

with $\delta E_\gamma(\varepsilon) = O_2^1(\varepsilon)$. Then, we also have

$$\begin{aligned} m \wedge H(m) &= h(t)\Big[\overline{g}^+.w_*^+ \wedge e_1 + \overline{g}^-.w_*^- \wedge e_1\Big] \\ &\quad - \Big[\Big(\overline{g}^-.w_*^- + e_1\Big) \wedge \overline{g}^+.(\beta_* w_*^+) + (\overline{g}^+.w_*^+ + e_1) \wedge \Big(g^-.(\beta_* w_*^-)\Big)\Big] \\ &\quad + h(t)\varepsilon \wedge e_1 - \varepsilon \wedge \Big[\overline{g}^+.(\beta_* w_*^+) + \overline{g}^-.(\beta_* w_*^-)\Big] - P_{\overline{g}^+,\overline{g}^-} \wedge \delta E_\gamma(\varepsilon) - \varepsilon \wedge \delta E_\gamma(\varepsilon) \\ &= h(t)\Big[\overline{g}^+.w_*^+ \wedge e_1 + \overline{g}^-.w_*^- \wedge e_1\Big] + O(f_{\overline{y}^+,\overline{y}^-}(x)) + O_2^1(\varepsilon) + O_2^2(\varepsilon), \end{aligned}$$

by using Corollary 3.2 and where

$$f_{\overline{y}^+,\overline{y}^-}(x) = e^{-\Gamma(y^+ - y^-)} \times \begin{cases} e^{-\Gamma(x - y^+)} & \text{if } x \geq y^+ \\ 1 & \text{if } y \in [y^-, y^+] \\ e^{\Gamma(x + y^-)} & \text{if } x \leq y^- \end{cases}.$$

Last, there also holds

$$m \wedge (m \wedge H(m)) = h(t)\Big[\overline{g}^+.(w_*^+ \wedge (w_*^+ \wedge e_1) + g^-.(w_*^- \wedge (w_*^- \wedge e_1))\Big] + O(f_{\overline{y}^+,\overline{y}^-}(x)) + O_2^1(\varepsilon) + O_2^2(\varepsilon) + O_2^3(\varepsilon).$$

Hence,

$$\begin{aligned}\partial_t\varepsilon = \dot{\overline{y}}^+\overline{g}^+.\partial_x w_*^+ - \dot{\overline{\phi}}^+(e_1 \wedge \overline{g}^+.w_*^+) + \dot{\overline{y}}^-(\overline{g}^-.\partial_x w_*^-) - \dot{\overline{\phi}}^- e_1 \wedge (\overline{g}^-.w_*^-)& \\ + h(t)\Big[\overline{g}^+.w_*^+ \wedge e_1 + \overline{g}^-.w_*^- \wedge e_1\Big]& \\ - \alpha h(t)\Big[\overline{g}^+.(w_*^+ \wedge (w_*^+ \wedge e_1)) + \overline{g}^-.(w_*^- \wedge (w_*^- \wedge e_1))\Big]& \\ + O(f_{\overline{y}^+,\overline{y}^-}(x)) + O_2^1(\varepsilon) + O_2^2(\varepsilon) + O_2^3(\varepsilon).&\end{aligned} \tag{6.3}$$

From this equation, we can derive the first order of $\dot{\overline{g}}^\iota$. First, recall that $\overline{\mathcal{F}}(\varepsilon, \overline{g}^+, \overline{g}^-) = 0$. By differentiating with respect to t, we thus get for example

$$\begin{aligned}\int \partial_t\varepsilon \cdot (g^+.\partial_x w_*^+)\,\mathrm{d}x &= -\int \varepsilon \cdot \partial_t(g^+.\partial_x w_*^+)\,\mathrm{d}x \\ &= \dot{\overline{y}}^+ \int \varepsilon \cdot (g^+.\partial_{xx}^2 w_*^+)\,\mathrm{d}x - \dot{\overline{\phi}}^+ \int \varepsilon \cdot (e_1 \wedge g^+.\partial_x w_*^+)\,\mathrm{d}x \\ &= M_1(\varepsilon)\begin{pmatrix}\dot{\overline{y}}^+ \\ \dot{\overline{\phi}}^+\end{pmatrix},\end{aligned}$$

where the 2×2 matrix $M_1(\varepsilon)$ satisfies $\|M_1(\varepsilon)\| \leq C\|\varepsilon\|_{H^1}$. More generally, we obtain

$$\overline{\mathcal{F}}(\partial_t\varepsilon, \overline{g}^+, \overline{g}^-) = M_0(\varepsilon)\begin{pmatrix}\dot{\overline{y}}^+ \\ \dot{\overline{\phi}}^+ \\ \dot{\overline{y}}^- \\ \dot{\overline{\phi}}^-\end{pmatrix}, \tag{6.4}$$

with a 4×4 matrix $M_0(\varepsilon)$ such that $\|M_0(\varepsilon)\| \leq C\|\varepsilon\|_{H^1}$. On the other hand, we can also compute $\overline{\mathcal{F}}(\partial_t\varepsilon, \overline{g}^+, \overline{g}^-)$ with the relation (6.3), and use Lemma 3.3 for the zeroth order terms, Corollary 3.2 for the terms involving both w_*^+ and w_*^-, Proposition 6.2 for the terms in $O(f_{\overline{y}^+,\overline{y}^-}(x))$ and [4, Claim 4.9] for $O_2^\ell(\varepsilon)$. For example, one of the terms involving both w_*^+ and w_*^- is

$$\int (\overline{g}^-.\partial_x w_*^-) \cdot (\overline{g}^+.\partial_x w_*^+)\,\mathrm{d}x,$$

and from Corollary 3.2, we can estimate

$$\left|\int (\overline{g}^-.\partial_x w_*^-) \cdot (\overline{g}^+.\partial_x w_*^+)\,\mathrm{d}x\right| \leq Cq(\overline{y}^+ - \overline{y}^-).$$

From this, we obtain

$$\begin{aligned}\overline{\mathcal{F}}(\partial_t\varepsilon, \overline{g}^+, \overline{g}^-) = \frac{2}{\Gamma}(B + B_0)\begin{pmatrix}\dot{\overline{y}}^+ \\ \dot{\overline{\phi}}^+ \\ \dot{\overline{y}}^- \\ \dot{\overline{\phi}}^-\end{pmatrix} + \frac{2h(t)}{\Gamma}\begin{pmatrix}-\gamma + \alpha\Gamma \\ 1 \\ -\gamma - \alpha\Gamma \\ 1\end{pmatrix}& \\ + O\Big((\overline{y}^+ - \overline{y}^-)e^{-\Gamma(\overline{y}^+ - \overline{y}^-)}\Big) + O(\|\varepsilon\|_{H^1}) + O(\|\varepsilon\|_{H^1}^3),&\end{aligned} \tag{6.5}$$

where

$$B = \begin{pmatrix}1 & \gamma & 0 & 0 \\ -\gamma & -1 & 0 & 0 \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & -\gamma & -1\end{pmatrix} = \begin{pmatrix}B_\gamma & 0_2 \\ 0_2 & B_\gamma\end{pmatrix}, \qquad B_\gamma = \begin{pmatrix}1 & \gamma \\ -\gamma & -1\end{pmatrix}$$

and B_0 is a 4×4 matrix which satisfies $\|B_0\| \le C(\overline{y}^+ - \overline{y}^-)e^{-\Gamma(\overline{y}^+ - \overline{y}^-)}$. Hence, (6.4) and (6.5) imply

$$(B + \tilde{B}_0) \begin{pmatrix} \dot{\overline{y}}^+ \\ \dot{\overline{\phi}}^+ \\ \dot{\overline{y}}^- \\ \dot{\overline{\phi}}^- \end{pmatrix} = -h(t) \begin{pmatrix} -\gamma + \alpha\Gamma \\ 1 \\ -\gamma - \alpha\Gamma \\ 1 \end{pmatrix} + O\Big((\overline{y}^+ - \overline{y}^-)e^{-\Gamma(\overline{y}^+ - \overline{y}^-)}\Big) + O(\|\varepsilon\|_{H^1}) + O(\|\varepsilon\|^3_{H^1}),$$

with

$$\Big\|\tilde{B}_0\Big\| \le C\Big((\overline{y}^+ - \overline{y}^-)e^{-\Gamma(\overline{y}^+ - \overline{y}^-)} + \|\varepsilon\|_{H^1}\Big) \le C\Big(y_0 e^{-2\Gamma y_0} + \delta\Big).$$

We know that B_γ is invertible because $\gamma^2 < 1$, with inverse $\Gamma^{-2}B_\gamma$, and thus B is also invertible. Therefore, as soon as $y_0 e^{-2\Gamma y_0} + \delta$ is small enough, $B + \tilde{B}_0$ is invertible with inverse

$$(B + \tilde{B}_0)^{-1} = \frac{1}{\Gamma^2} \begin{pmatrix} B_\gamma & 0_2 \\ 0_2 & B_{-\gamma} \end{pmatrix} + O\Big((\overline{y}^+ - \overline{y}^-)e^{-\Gamma(\overline{y}^+ - \overline{y}^-)}\Big) + O(\|\varepsilon\|_{H^1}),$$

which leads to

$$(\dot{\overline{g}}^+, \dot{\overline{g}}^-) - (\dot{g}^+_*, \dot{g}^-_*) = O\Big((\overline{y}^+ - \overline{y}^-)e^{-\Gamma(\overline{y}^+ - \overline{y}^-)}\Big) + O(\|\varepsilon\|_{H^1}) + O(\|\varepsilon\|^3_{H^1}),$$

hence the conclusion for the $\mathcal{H}^3$ case. We point out that this estimate only depends on the H^1 norm of ε. Therefore, for the general case $m \in \mathscr{C}(I_T, \mathcal{H}^2)$, we can use a limiting argument. See [4] for further details. □

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