DESCRIPTION AND CLASSIFICATION OF 2-SOLITARY WAVES FOR NONLINEAR DAMPED KLEIN-GORDON EQUATIONS

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Abstract. We describe completely 2-solitary waves related to the ground state of the nonlinear damped Klein-Gordon equation

$$\partial_{tt}u + 2\alpha\partial_t u - \Delta u + u - |u|^{p-1}u = 0$$

on $\mathbb{R}^N$, for $1 \leq N \leq 5$ and energy subcritical exponents $p > 2$. The description is twofold.

First, we prove that 2-solitary waves with same sign do not exist. Second, we construct and classify the full family of 2-solitary waves in the case of opposite signs. Close to the sum of two remote solitary waves, it turns out that only the components of the initial data in the unstable direction of each ground state are relevant in the large time asymptotic behavior of the solution. In particular, we show that 2-solitary waves have a universal behavior: the distance between the solitary waves is asymptotic to $\log t$ as $t \to \infty$. This behavior is due to damping of the initial data combined with strong interactions between the solitary waves.

1. Introduction

1.1. **Setting of the problem.** We consider the nonlinear focusing damped Klein-Gordon equation

$$\partial_{tt}u + 2\alpha\partial_t u - \Delta u + u - f(u) = 0 \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1.1}$$

where $f(u) = |u|^{p-1}u$, $\alpha > 0$, $1 \leq N \leq 5$, and the exponent p corresponds to the energy sub-critical case, *i.e.*

$$2 < p < \infty \text{ for } N = 1,2 \text{ and } 2 < p < \frac{N+2}{N-2} \text{ for } N = 3,4,5. \tag{1.2}$$

The restriction $p > 2$ is discussed in Remark 1.6.
This equation also rewrites as a first order system for $\vec{u} = (u, \partial_t u) = (u, v)$

$$\begin{cases} \partial_t u = v \\ \partial_t v = \Delta u - u + f(u) - 2\alpha v. \end{cases}$$

It follows from [3, Theorem 2.3] that the Cauchy problem for (1.1) is locally well-posed in the energy space: for any initial data $(u_0, v_0) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, there exists a unique (in some class) maximal solution $u \in C([0,T_{\max}), H^1(\mathbb{R}^N)) \cap C^1([0,T_{\max}), L^2(\mathbb{R}^N))$ of (1.1). Moreover, if the maximal time of existence $T_{\max}$ is finite, then $\lim_{t\uparrow T_{\max}} \|\vec{u}(t)\|_{H^1\times L^2} = \infty$.
Setting $F(u) = \frac{1}{p+1}|u|^{p+1}$ and

$$E(\vec{u}) = \frac{1}{2}\int_{\mathbb{R}^N} \left\{|\nabla u|^2 + u^2 + (\partial_t u)^2 - 2F(u)\right\} \mathrm{d}x,$$

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for any $H^1 \times L^2$ solution $\vec{u}$ of (1.1), it holds

$$E(\vec{u}(t_2)) - E(\vec{u}(t_1)) = -2\alpha \int_{t_1}^{t_2} \|\partial_t u(t)\|_{L^2}^2 \, \mathrm{d}t. \tag{1.3}$$

In this paper, we are interested in the dynamics of 2-solitary waves related to the ground state Q, which is the unique positive, radial H^1 solution of

$$-\Delta Q + Q - f(Q) = 0, \quad x \in \mathbb{R}^N. \tag{1.4}$$

(See [2, 19].) The ground state generates the stationary solution $\vec{Q} = \binom{Q}{0}$ of (1.1). The function $-\vec{Q}$ as well as any translate $\vec{Q}(\cdot - z_0)$ are also solutions of (1.1).

The question of the existence of multi-solitary waves for (1.1) was first addressed by Feireisl in [13, Theorem 1.1], under suitable conditions on N and p, for an even number of solitary waves with specific geometric and sign configurations. His construction is based on variational and symmetry arguments to treat the instability direction of the solitary waves. The goal of the present paper is to fully understand 2-solitary waves by proving non-existence, existence and classification results using dynamical arguments.

1.2. **Main results.** First let us introduce a few basic notation. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ denote the canonical basis of $\mathbb{R}^N$. We denote by $\mathcal{B}_{\mathbb{R}^N}(\rho)$ (respectively, $\mathcal{S}_{\mathbb{R}^N}(\rho)$) the open ball (respectively, the sphere) of $\mathbb{R}^N$ of center the origin and of radius $\rho > 0$, for the usual norm $|\xi| = (\sum_{j=1}^N \xi_j^2)^{1/2}$. We denote by $\mathcal{B}_{H^1\times L^2}(\rho)$ the ball of $H^1 \times L^2$ of center the origin and of radius $\rho > 0$ for the norm $\|\binom{\varepsilon}{\eta}\|_{H^1\times L^2} = \left(\|\varepsilon\|_{H^1}^2 + \|\eta\|_{L^2}^2\right)^{1/2}$. We denote $\langle \cdot, \cdot \rangle$ the L^2 scalar product for real valued functions u_i or vector-valued functions $\vec{u}_i = (u_i, v_i)$ $(i = 1, 2)$:

$$\langle u_1, u_2 \rangle := \int u_1(x) u_2(x) \, \mathrm{d}x, \quad \langle \vec{u}_1, \vec{u}_2 \rangle := \int u_1(x) u_2(x) \, \mathrm{d}x + \int v_1(x) v_2(x) \, \mathrm{d}x.$$

It is well-known that the operator

$$\mathcal{L} = -\Delta + 1 - pQ^{p-1}$$

appearing after linearization of equation (1.1) around $\vec{Q}$, has a unique negative eigenvalue $-\nu_0^2$ $(\nu_0 > 0)$. We denote by Y the corresponding normalized eigenfunction (see Lemma 1.8 for details and references). In particular, it follows from explicit computations that setting

$$\nu^\pm = -\alpha \pm \sqrt{\alpha^2 + \nu_0^2} \quad \text{and} \quad \vec{Y}^\pm = \begin{pmatrix} Y \\ \nu^\pm Y \end{pmatrix},$$

the function $\vec{\varepsilon}^{\,\pm}(t, x) = \exp(\nu^\pm t) \vec{Y}^\pm(x)$ is solution of the linearized problem

$$\begin{cases} \partial_t \varepsilon = \eta \\ \partial_t \eta = -\mathcal{L}\varepsilon - 2\alpha\eta. \end{cases} \tag{1.5}$$

Since $\nu^+ > 0$, the solution $\vec{\varepsilon}^{\,+}$ illustrates the exponential instability of the solitary wave in positive time. In particular, we see that the presence of the damping $\alpha > 0$ in the equation does not remove the exponential instability of the Klein-Gordon solitary wave. An equivalent formulation of instability is obtained by setting

$$\zeta^\pm = \alpha \pm \sqrt{\alpha^2 + \nu_0^2} \quad \text{and} \quad \vec{Z}^\pm = \begin{pmatrix} \zeta^\pm Y \\ Y \end{pmatrix}$$

and observing that for any solution $\vec{\varepsilon}$ of (1.5),

$$a^\pm = \langle \vec{\varepsilon}, \vec{Z}^\pm \rangle \quad \text{satisfies} \quad \frac{\mathrm{d}a^\pm}{\mathrm{d}t} = \nu^\pm a^\pm.$$

We start with the definition of 2-solitary waves.

Definition 1.1. A solution $\vec{u} \in \mathcal{C}([T,\infty), H^1 \times L^2)$ of (1.1), for some $T \in \mathbb{R}$, is called a 2-*solitary wave* if there exist $\sigma_1, \sigma_2 = \pm 1$, a sequence $t_n \to \infty$ and a sequence $(\xi_{1,n}, \xi_{2,n}) \in \mathbb{R}^{2N}$ such that

$$\lim_{n\to\infty} \left\{ \left\| u(t_n) - \sum_{k=1,2} \sigma_k Q(\cdot - \xi_{k,n}) \right\|_{H^1} + \|\partial_t u(t_n)\|_{L^2} \right\} = 0$$

and $\lim_{n\to\infty} |\xi_{1,n} - \xi_{2,n}| \to \infty$.

Remark 1.2. We observe that if u is a global solution of (1.1) satisfying

$$\lim_{t\to\infty} \left\| u(t) - \sum_{k=1,2} \sigma_k Q(\cdot - \xi_k(t)) \right\|_{H^1} = 0 \qquad (1.6)$$

where $\lim_{t\to\infty} |\xi_1(t) - \xi_2(t)| \to \infty$, then u is a 2-solitary wave. Indeed, it follows from (1.3) and (1.6) that $t \mapsto E(\vec{u}(t))$ is lower bounded. Thus, from (1.3), it holds $\int_0^\infty \|\partial_t u(t)\|_{L^2}^2 \,\mathrm{d}t < \infty$ and so $\lim_{n\to\infty} \|\partial_t u(t_n)\|_{L^2} = 0$ for some sequence $t_n \to \infty$.

Our first result concerns the non-existence of 2-solitary waves with same signs.

Theorem 1.3. *There exists no* 2*-solitary wave of* (1.1) *with* $\sigma_1 = \sigma_2$*.*

In the next result, we show that 2-solitary waves with opposite sign satisfy a universal asymptotic behavior.

Theorem 1.4 (Description of 2-solitary waves)**.** *For any* 2*-solitary wave* $\vec{u} = (u, \partial_t u)$ *of* (1.1)*, there exist* $\sigma_1 = \pm 1$*,* $\sigma_2 = -\sigma_1$*,* $z_\infty \in \mathbb{R}^N$*,* $\omega_\infty \in \mathcal{S}_{\mathbb{R}^N}(1)$*,* $T > 0$ *and a function* $t \in [T,\infty) \mapsto (z_1(t), z_2(t)) \in \mathbb{R}^N \times \mathbb{R}^N$ *such that for all* $t \in [T,\infty)$*,*

$$\left\| u(t) - \sum_{k=1,2} \sigma_k Q(\cdot - z_k(t)) \right\|_{H^1} + \|\partial_t u(t)\|_{L^2} \lesssim t^{-1}, \qquad (1.7)$$

and for $k = 1, 2$*,*

$$z_k(t) = z_\infty + \frac{(-1)^k}{2} \left(\log t - \frac{N-1}{2} \log\log t + c_0 \right) \omega_\infty + O\left(\frac{\log\log t}{\log t} \right), \qquad (1.8)$$

where c_0 *is a constant depending on* N *and* α*.*

Finally, we describe the full family of 2-solitary waves for initial data close to the sum of two remote solitary waves.

Theorem 1.5 (Classification of 2-solitary waves)**.** *There exist* $C, \delta > 0$ *and a Lipschitz map*

$$H : (\mathbb{R}^N \setminus \bar{\mathcal{B}}_{\mathbb{R}^N}(10|\log\delta|)) \times \mathcal{B}_{H^1\times L^2}(\delta) \to \mathbb{R}^2, \quad (L, \vec{\phi}) \mapsto H(L, \vec{\phi})$$

such that

$$|H(L, \vec{\phi})| < C \left(e^{-\frac{L}{2}} + \|\vec{\phi}\|_{H^1\times L^2} \right),$$

with the following property. Given any $L, \vec{\phi}, h_1, h_2$ *such that*

$$|L| > 10|\log\delta|, \quad \|\vec{\phi}\|_{H^1\times L^2} < \delta, \quad |h_1| + |h_2| < \delta,$$

the solution $\vec{u}$ *of* (1.1) *with initial data*

$$\vec{u}(0) = \left(\vec{Q} + h_1 \vec{Y}^+ \right) \left(\cdot - \frac{L}{2} \right) - \left(\vec{Q} + h_2 \vec{Y}^+ \right) \left(\cdot + \frac{L}{2} \right) + \vec{\phi}$$

is a 2*-solitary wave if and only if* $(h_1, h_2) = H(L, \vec{\phi})$*.*

This result essentially means that locally around the sum of two sufficiently separated solitons with opposite signs, the initial data of 2-solitary waves form a codimension-2 Lipschitz manifold (the unstable directions being directed by $\vec{Y}^+$ translated around each soliton).

We refer to §2.4 for a formal discussion on the dynamics of 2-solitary waves of (1.1) justifying the main results of Theorems 1.3, 1.4 and 1.5.

Remark 1.6. We discuss the condition on p in (1.2). The energy sub-criticality condition is necessary for the existence of solitary waves and allows to work in the framework of finite energy solutions. The condition $p > 2$ could be waived for some of the above results, but it would complicate the analysis and weaken the results. Keeping in mind that the most relevant case is $p = 3$, we will not pursue further here the question of lowering p.

Remark 1.7. In a succeeding paper, the first three authors have complemented this work by a complete description of the asymptotic behavior of *all global solutions* of (1.1) in the one-dimensional case. Indeed, in the 1D case, it is possible to take advantage of the uniqueness of the ground state as solution of the stationary problem (1.4) (up to invariance), and of the fact that any global solution of (1.1) is bounded (see references in (1.1)). For space dimensions greater than 1, the description of the dynamics of general global solutions of (1.1) remains a challenging open problem because of the existence of bound states solutions of (1.4) other than the ground state (excited states) and of the possibility of involved geometric configurations of solitary waves. We refer to partial results in this direction by the first and third authors in [9].

1.3. **Previous results.** The question of the long time asymptotic behavior of solutions of the damped Klein-Gordon equation in relation with the bound states was addressed in several articles; see *e.g.* [3, 12, 13, 14, 22]. Notably, under some conditions on N and p, results in [13, 22] state that for any sequence of time, any global bounded solution of (1.1) converges to a sum of decoupled bound states after extraction of a subsequence of times. (Note that such result would allow us to weaken the definition of 2-solitary wave given in Definition 1.1; however, we have preferred a stronger definition valid in any case.) In [3], for radial solutions in dimension $N \geq 2$, the convergence of any global solution to one equilibrium is proved to hold for the whole sequence of time. As discussed in [3], such results are closely related to the general soliton resolution conjecture for global bounded solutions of dispersive problems; see [10, 11] for details and results related to this conjecture for the undamped energy critical wave equation.

The existence and properties of multi-solitary waves is a classical question for integrable models (see for instance [27] for the Korteweg-de Vries equation and [32] for the 1D cubic Schrödinger equation). As mentioned above, [13] gave the first construction of such solutions for (1.1). Since then the same question has been addressed for various non-integrable and undamped nonlinear dispersive and wave equations. We refer to [4, 6, 8, 23] for the generalized Korteweg-de Vries equation, the nonlinear Schrödinger equation, the Klein-Gordon equation and the wave equation, in situations where unstable ground states are involved. See also references therein for previous works related to stable ground states. In those works, the distance between two traveling waves is asymptotic to Ct for $C > 0$, as $t \to \infty$. The more delicate case of multi-solitary waves with logarithmic distance is treated in [18, 25, 26, 30, 31] for Korteweg-de Vries and Schrödinger type equations and systems, both in stable and unstable cases. Note that the logarithmic distance in the latter works is non-generic while it is the universal behavior for the damped equation (1.1). See also [15, 16, 17] for works on the non-existence, existence and

classification of radial two-bubble solutions for the energy critical wave equation in large dimensions.
The construction of (center-) stable manifolds in the neighborhood of unstable ground state was addressed in several situations, see *e.g.* [1, 20, 21, 24, 28, 29].
While the initial motivation and several technical tools originate from some of the above mentioned papers, we point out that the present article is self-contained except for the local Cauchy theory for (1.1) (see [3]) and elliptic theory for (1.4) and its linearization (we refer to [2, 8, 19]).

This paper is organized as follows. Section 2 introduces all the technical tools involved in a dynamical approach to the 2-solitary wave problem for (1.1): computation of the nonlinear interaction, modulation, parameter estimates and energy estimates. Theorems 1.3 and 1.4 are proved in Section 3. Finally, Theorem 1.5 is proved in Section 4.

1.4. **Recollection on the ground state.** The ground state Q rewrites $Q(x) = q(|x|)$ where $q > 0$ satisfies

$$q'' + \frac{N-1}{r}q' - q + q^p = 0, \quad q'(0) = 0, \quad \lim_{r\to\infty} q(r) = 0. \tag{1.9}$$

It is well-known and easily checked that for a constant $\kappa > 0$, for all $r > 1$,

$$\left|q(r) - \kappa r^{-\frac{N-1}{2}} e^{-r}\right| + \left|q'(r) + \kappa r^{-\frac{N-1}{2}} e^{-r}\right| \lesssim r^{-\frac{N+1}{2}} e^{-r}. \tag{1.10}$$

Due to the radial symmetry, there hold the following cancellation (which we will use repetitively):

$$\forall i \neq j, \quad \int \partial_{x_i} Q(x) \partial_{x_j} Q(x)\, \mathrm{d}x = 0. \tag{1.11}$$

Let

$$\mathcal{L} = -\Delta + 1 - pQ^{p-1}, \quad \langle \mathcal{L}\varepsilon, \varepsilon\rangle = \int \left\{|\nabla\varepsilon|^2 + \varepsilon^2 - pQ^{p-1}\varepsilon^2\right\} \mathrm{d}x.$$

We recall standard properties of the operator $\mathcal{L}$ (see *e.g.* [8, Lemma 1]).

Lemma 1.8. (i) Spectral properties. *The unbounded operator $\mathcal{L}$ on L^2 with domain H^2 is self-adjoint, its continuous spectrum is $[1, \infty)$, its kernel is* $\operatorname{span}\{\partial_{x_j} Q : j = 1, \ldots, N\}$ *and it has a unique negative eigenvalue $-\nu_0^2$, with corresponding smooth normalized radial eigenfunction Y ($\|Y\|_{L^2} = 1$) Moreover, on $\mathbb{R}^N$,*

$$\left|\partial_x^\beta Y(x)\right| \lesssim e^{-\sqrt{1+\nu_0^2}|x|} \quad \textit{for any } \beta = (\beta_1, \ldots, \beta_N) \in \mathbb{N}^N.$$

(ii) Coercivity property. *There exists $c > 0$ such that, for all $\varepsilon \in H^1$,*

$$\langle \mathcal{L}\varepsilon, \varepsilon\rangle \geq c\|\varepsilon\|_{H^1}^2 - c^{-1}\left(\langle\varepsilon, Y\rangle^2 + \sum_{j=1}^N \langle\varepsilon, \partial_{x_j} Q\rangle^2\right).$$

2. Dynamics of two solitary waves

We prove in this section a general decomposition result close to the sum of two decoupled solitary waves. Let any $\sigma_1 = \pm 1$, $\sigma_2 = \pm 1$ and denote $\sigma = \sigma_1\sigma_2$. Consider time dependent $\mathcal{C}^1$ parameters $(z_1, z_2, \ell_1, \ell_2) \in \mathbb{R}^{4N}$ with $|\ell_1| \ll 1$, $|\ell_2| \ll 1$ and $|z| \gg 1$ where

$$z = z_1 - z_2 \quad \text{and} \quad \ell = \ell_1 - \ell_2.$$

Define the modulated ground state solitary waves, for $k = 1, 2$,

$$Q_k = \sigma_k Q(\cdot - z_k) \quad \text{and} \quad \vec{Q}_k = \begin{pmatrix} Q_k \\ -(\ell_k \cdot \nabla) Q_k \end{pmatrix}. \tag{2.1}$$

Set

$$R = Q_1 + Q_2, \qquad \vec{R} = \vec{Q}_1 + \vec{Q}_2,$$

and the nonlinear interaction term

$$G = f(Q_1 + Q_2) - f(Q_1) - f(Q_2). \tag{2.2}$$

The following functions are related to the exponential instabilities around each solitary wave:

$$Y_k = \sigma_k Y(\cdot - z_k), \quad \vec{Y}_k^{\pm} = \sigma_k \vec{Y}^{\pm}(\cdot - z_k), \quad \vec{Z}_k^{\pm} = \sigma_k \vec{Z}^{\pm}(\cdot - z_k).$$

2.1. **Nonlinear interactions.** A key of the understanding of the dynamics of 2-solitary waves is the computation of the first order of the projections of the nonlinear interaction term G on the directions ∇Q_1 and ∇Q_2 (see *e.g.* [31, Lemma 7]).

Lemma 2.1. *The following estimates hold for* $|z| \gg 1$.

(i) Bounds. *For any* $0 < m' < m$,

$$\int |Q_1 Q_2|^m \lesssim e^{-m'|z|}, \tag{2.3}$$

$$\int |F(R) - F(Q_1) - F(Q_2) - f(Q_1)Q_2 - f(Q_2)Q_1| \lesssim e^{-\frac{5}{4}|z|}. \tag{2.4}$$

(ii) Sharp bounds. *For any* $m > 0$,

$$\int |Q_1||Q_2|^{1+m} \lesssim q(|z|), \tag{2.5}$$

$$\|G\|_{L^2} \lesssim \|Q_1^{p-1} Q_2\|_{L^2} + \|Q_1 Q_2^{p-1}\|_{L^2} \lesssim q(|z|). \tag{2.6}$$

(iii) Asymptotics. *It holds*

$$|\langle f(Q_2), Q_1\rangle - \sigma c_1 g_0 q(|z|)| \lesssim |z|^{-1} q(|z|) \tag{2.7}$$

where

$$g_0 = \frac{1}{c_1}\int Q^p(x) e^{-x_1}\,\mathrm{d}x > 0, \quad c_1 = \|\partial_{x_1} Q\|_{L^2}^2. \tag{2.8}$$

(iv) Sharp asymptotics. *There exists a smooth function* $g : [0,\infty) \to \mathbb{R}$ *such that, for any* $0 < \theta < \min(p-1, 2)$ *and* $r, r' > 1$

$$|g(r) - g_0 q(r)| \lesssim r^{-1} q(r), \quad |g(r) - g(r')| \lesssim |r - r'| \max(q(r), q(r')), \tag{2.9}$$

and

$$\left|\langle G, \nabla Q_1\rangle - \sigma c_1 \frac{z}{|z|} g(|z|)\right| \lesssim e^{-\theta|z|}, \tag{2.10}$$

$$\left|\langle G, \nabla Q_2\rangle + \sigma c_1 \frac{z}{|z|} g(|z|)\right| \lesssim e^{-\theta|z|}. \tag{2.11}$$

Proof. (i) By (1.10), $|Q(y)| \lesssim e^{-|y|}$, and thus

$$\int |Q_1 Q_2|^m \,\mathrm{d}y \lesssim \int e^{-m|y|} e^{-m'|y+z|}\,\mathrm{d}y \lesssim e^{-m'|z|} \int e^{-(m-m')|y|}\,\mathrm{d}y \lesssim e^{-m'|z|}.$$

Then, we observe that using $p \geq 2$, by Taylor expansion:

$$|F(Q_1 + Q_2) - F(Q_1) - F(Q_2) - f(Q_1)Q_2 - f(Q_2)Q_1| \lesssim |Q_1 Q_2|^{\frac{3}{2}},$$

which reduces the proof of (2.4) to applying (2.3) with $m = \frac{3}{2}$ and $m' = \frac{5}{4}$.

(ii) We estimate

$$\begin{aligned}
&\int |Q_1||Q_2|^{1+m}\,\mathrm{d}x = \int Q(y-z)Q^{1+m}(y)\,\mathrm{d}y\\
&\lesssim q(|z|)\int_{|y|<\frac34|z|} e^{|y|}Q^{1+m}(y)\,\mathrm{d}y + e^{-|z|}\int_{|y|>\frac34|z|} e^{|y|}Q^{1+m}(y)\,\mathrm{d}y \lesssim q(|z|).
\end{aligned}$$

Next, using

$$|G| \lesssim |Q_1|^{p-1}|Q_2| + |Q_1||Q_2|^{p-1},$$

and (using $p > 2$)

$$\begin{aligned}
&\int Q_1^2|Q_2|^{2(p-1)}\,\mathrm{d}x = \int Q^2(y-z)Q^{2(p-1)}(y)\,\mathrm{d}y\\
&\lesssim [q(|z|)]^2\int_{|y|<\frac34|z|} e^{2|y|}Q^{2(p-1)}(y)\,\mathrm{d}y + e^{-2|z|}\int_{|y|>\frac34|z|} e^{2|y|}Q^{2(p-1)}(y)\,\mathrm{d}y\\
&\lesssim [q(|z|)]^2.
\end{aligned}$$

(iii) We claim the following estimate

$$\left|\int Q^p(y)Q(y+z)\,\mathrm{d}y - c_1 g_0\kappa|z|^{-\frac{N-1}{2}}e^{-|z|}\right| \lesssim |z|^{-1}q(|z|). \tag{2.12}$$

Observe that (2.7) follows directly from (2.12) and (1.10).
Proof of (2.12). First, for $|y| < \frac34|z|$ (and so $|y+z| \ge |z|-|y| \ge \frac14|z| \gg 1$), we have using (1.10),

$$\left|Q(y+z) - \kappa|y+z|^{-\frac{N-1}{2}}e^{-|y+z|}\right| \lesssim |y+z|^{-\frac{N+1}{2}}e^{-|y+z|} \lesssim |z|^{-\frac{N+1}{2}}e^{-|z|}e^{|y|}.$$

In particular,

$$\left|\int_{|y|<\frac34|z|} Q^p(y)\Big[Q(y+z) - \kappa|y+z|^{-\frac{N-1}{2}}e^{-|y+z|}\Big]\,\mathrm{d}y\right| \lesssim |z|^{-\frac{N+1}{2}}e^{-|z|}. \tag{2.13}$$

Moreover, for $|y| < \frac34|z|$, we have the expansions

$$\begin{aligned}
\left||y+z|^{-\frac{N-1}{2}} - |z|^{-\frac{N-1}{2}}\right| &\lesssim |z|^{-\frac{N+1}{2}}|y|,\\
\left||y+z| - |z| - \frac{y\cdot z}{|z|}\right| &\lesssim |z|^{-1}|y|^2,
\end{aligned}$$

and so

$$\left||y+z|^{-\frac{N-1}{2}}e^{-|y+z|} - |z|^{-\frac{N-1}{2}}e^{-|z|-\frac{y\cdot z}{|z|}}\right| \lesssim |z|^{-\frac{N+1}{2}}e^{-|z|}(1+|y|^2)e^{|y|}.$$

Inserted into (2.13), this yields

$$\left|\int_{|y|<\frac34|z|} Q^p(y)\Big[Q(y+z) - \kappa|z|^{-\frac{N-1}{2}}e^{-|z|-\frac{y\cdot z}{|z|}}\Big]\,\mathrm{d}y\right| \lesssim |z|^{-\frac{N+1}{2}}e^{-|z|}.$$

Next, using (1.10), we observe

$$\int_{|y|>\frac34|z|} Q^p(y)Q(y+z)\,\mathrm{d}y \lesssim e^{-\frac34 p|z|},$$

and

$$\int_{|y|>\frac34|z|} Q^p(y)e^{-|z|-\frac{y\cdot z}{|z|}}\,\mathrm{d}y \lesssim e^{-|z|}\int_{|y|>\frac34|z|} Q^p(y)e^{|y|}\,\mathrm{d}y \lesssim e^{-\frac34 p|z|}.$$

Gathering these estimates, we have proved

$$\left|\int Q^p(y)Q(y+z)\,\mathrm{d}y - \kappa|z|^{-\frac{N-1}{2}}e^{-|z|}\int Q^p(y)e^{-\frac{y\cdot z}{|z|}}\,\mathrm{d}y\right| \lesssim |z|^{-\frac{N+1}{2}}e^{-|z|}.$$

Last, the identity $\int Q^p(y)e^{-\frac{y\cdot z}{|z|}}\,\mathrm{d}y = \int Q^p(y)e^{-y_1}\,\mathrm{d}y$ (recall that Q is radially symmetric) and the definition of g_0 imply (2.12).
(iv) First, using the Taylor formula, it holds

$$|G - p|Q_1|^{p-1}Q_2| \lesssim |Q_1|^{p-2}Q_2^2 + |Q_2|^{p-1}|Q_1|.$$

Thus, using (2.3), we obtain for any $1 < \theta < \min(p-1, 2)$,

$$|\langle G, \nabla Q_1\rangle - \sigma_1\sigma_2 H(z)| \lesssim \int Q^2(y)Q^{p-1}(y+z)\,\mathrm{d}y \lesssim e^{-\theta|z|} \tag{2.14}$$

where we set $H(z) = \int \nabla(Q^p)(y)Q(y+z)\,\mathrm{d}y$. Second, we claim that there exists a function $g : [0,\infty) \to \mathbb{R}$ such that $H(z) = c_1\frac{z}{|z|}g(|z|)$. Indeed, remark by using the change of variable $y = 2x_1\mathbf{e}_1 - x$ that

$$\begin{aligned} H(r\mathbf{e}_1) &= p\int \frac{y}{|y|}q'(|y|)q^{p-1}(|y|)q(|y+r\mathbf{e}_1|)\,\mathrm{d}y \\ &= p\int \frac{y_1\mathbf{e}_1}{|y|}q'(|y|)q^{p-1}(|y|)q(|y+r\mathbf{e}_1|)\,\mathrm{d}y. \end{aligned}$$

Thus, we set

$$g(r) = \frac{\mathbf{e}_1\cdot H(r\mathbf{e}_1)}{c_1} \quad \text{so that} \quad H(r\mathbf{e}_1) = c_1\mathbf{e}_1 g(r).$$

Let $\omega \in \mathcal{S}_{\mathbb{R}^N}(1)$ be such that $z = |z|\omega$ and let U be an orthogonal matrix of size N such that $U\mathbf{e}_1 = \omega$. Then, using the change of variable $y = Ux$,

$$\begin{aligned} H(z) &= p\int \frac{y}{|y|}q'(|y|)q^{p-1}(|y|)q(|y+z|)\,\mathrm{d}y \\ &= p\int \frac{Ux}{|x|}q'(|x|)q^{p-1}(|x|)q(|x+|z|\mathbf{e}_1|)\,\mathrm{d}x = UH(|z|\mathbf{e}_1) = c_1\frac{z}{|z|}g(z). \end{aligned}$$

Together with (2.14), this proves (2.10). The proof of (2.11) is the same but it is important to notice the change of sign due to $H(-z) = -H(z)$.
Last, we observe that proceeding as in the proof of (2.12), it holds

$$\left|\mathbf{e}_1\cdot H(r\mathbf{e}_1) - \kappa r^{-\frac{N-1}{2}}e^{-r}\int \partial_{x_1}(Q^p)(y)e^{-y_1}\,\mathrm{d}y\right| \lesssim r^{-\frac{N+1}{2}}e^{-r}.$$

Moreover, by integration by parts, $\int \partial_{x_1}(Q^p)(y)e^{-y_1}\,\mathrm{d}y = \int Q^p(y)e^{-y_1}\,\mathrm{d}y$, which proves the first estimate of (2.9).
For the difference estimate of (2.9), it suffices to show that for z, z' with modulus greater than 1, then

$$|H(z) - H(z')| \lesssim |z - z'|\max(q(|z|), q(|z'|)).$$

We write the difference

$$\begin{aligned} H(z) - H(z') &= \int \nabla(Q^p)(y)(Q(y+z) - Q(y+z'))\,\mathrm{d}y \\ &= \int \nabla(Q^p)(y)\int_0^1 \nabla Q(y + (1-\tau)z + \tau z')\cdot(z-z')d\tau dy, \end{aligned}$$

so that

$$|H(z) - H(z')| \le \int_0^1 (z-z')\cdot\left(\int \nabla(Q^p)(y)\nabla Q(y+(1-\tau)z+\tau z')dy\right)d\tau.$$

The conclusion comes that, arguing as for the proof of (2.12), there holds

$$\begin{aligned} \left|\int \nabla(Q^p)(y)\nabla Q(y+(1-\tau)z+\tau z')dy\right| &\lesssim q(|(1-\tau)z + \tau z'|) \\ &\lesssim \max(q(|z|), q(|z'|)). \end{aligned}$$

□

2.2. **Decomposition around the sum of two solitary waves.** The following quantity measures the proximity of a function $\vec{u} = (u, v)$ to the sum of two distant solitary waves, for $\gamma > 0$,

$$d(\vec{u}\,;\gamma) = \inf_{|\xi_1 - \xi_2| > |\log \gamma|} \Big\| u - \sum_{k=1,2} \sigma_k Q(\cdot - \xi_k) \Big\|_{H^1} + \|v\|_{L^2}.$$

We state a decomposition result for solutions of (1.1).

Lemma 2.2. *There exists $\gamma_0 > 0$ such that for any $0 < \gamma < \gamma_0$, $T_1 \le T_2$, and any solution $\vec{u} = (u, \partial_t u)$ of (1.1) on $[T_1, T_2]$ satisfying*

$$\sup_{t \in [T_1, T_2]} d(\vec{u}(t); \gamma) < \gamma, \tag{2.15}$$

there exist unique $\mathcal{C}^1$ functions

$$t \in [T_1, T_2] \mapsto (z_1, z_2, \ell_1, \ell_2)(t) \in \mathbb{R}^{4N},$$

such that the solution $\vec{u}$ decomposes on $[T_1, T_2]$ as

$$\vec{u} = \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \vec{Q}_1 + \vec{Q}_2 + \vec{\varepsilon}, \quad \vec{\varepsilon} = \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix} \tag{2.16}$$

with the following properties on $[T_1, T_2]$.

(i) Orthogonality and smallness. *For any $k = 1, 2$, $j = 1, \dots, N$,*

$$\langle \varepsilon, \partial_{x_j} Q_k \rangle = \langle \eta, \partial_{x_j} Q_k \rangle = 0 \tag{2.17}$$

and

$$\|\vec{\varepsilon}\|_{H^1 \times L^2} + \sum_{k=1,2} |\ell_k| + e^{-2|z|} \lesssim \gamma. \tag{2.18}$$

(ii) Equation of $\vec{\varepsilon}$.

$$\begin{cases} \partial_t \varepsilon = \eta + \mathrm{Mod}_\varepsilon \\ \partial_t \eta = \Delta \varepsilon - \varepsilon + f(R + \varepsilon) - f(R) - 2\alpha\eta + \mathrm{Mod}_\eta + G \end{cases} \tag{2.19}$$

where

$$\mathrm{Mod}_\varepsilon = \sum_{k=1,2} (\dot{z}_k - \ell_k) \cdot \nabla Q_k,$$

$$\mathrm{Mod}_\eta = \sum_{k=1,2} \big(\dot{\ell}_k + 2\alpha \ell_k\big) \cdot \nabla Q_k - \sum_{k=1,2} (\ell_k \cdot \nabla)(\dot{z}_k \cdot \nabla) Q_k.$$

(iii) Equations of the geometric parameters. *For $k = 1, 2$,*

$$|\dot{z}_k - \ell_k| \lesssim \|\vec{\varepsilon}\|_{H^1 \times L^2}^2 + \sum_{k=1,2} |\ell_k|^2, \tag{2.20}$$

$$|\dot{\ell}_k + 2\alpha \ell_k| \lesssim \|\vec{\varepsilon}\|_{H^1 \times L^2}^2 + \sum_{k=1,2} |\ell_k|^2 + q(|z|). \tag{2.21}$$

(iv) Refined equation for ℓ_k. *For any $1 < \theta < \min(p-1, 2)$, $k = 1, 2$,*

$$\Big| \dot{\ell}_k + 2\alpha \ell_k - (-1)^k \sigma \frac{z}{|z|} g(|z|) \Big| \lesssim \|\vec{\varepsilon}\|_{H^1 \times L^2}^2 + \sum_{k=1,2} |\ell_k|^2 + e^{-\theta |z|}. \tag{2.22}$$

(v) Equations of the exponential directions. *Let*

$$a_k^\pm = \langle \vec{\varepsilon}, \vec{Z}_k^\pm \rangle. \tag{2.23}$$

Then,

$$\Big| \frac{\mathrm{d}}{\mathrm{d}t} a_k^\pm - \nu^\pm a_k^\pm \Big| \lesssim \|\vec{\varepsilon}\|_{H^1 \times L^2}^2 + \sum_{k=1,2} |\ell_k|^2 + q(|z|). \tag{2.24}$$

Remark 2.3. We see on estimate (2.21) the damping of the Lorentz parameters ℓ_k. The more precise estimate (2.22) involves the nonlinear interactions which becomes preponderant for large time.

Remark 2.4. After completion of this work, the authors were kindly informed of an alternate choice of decomposition and orthogonality conditions. Indeed, a suitable extension of the natural symplectic spectral decomposition introduced in [29, Section 2.1] to the case of two solitons with damping would simplify the equations of z_1 and z_2, at the cost of a slightly more complicated initial formulation. We have chosen to keep our original setting mainly to ensure compatibility with the notation of the succeeding published article [7].

Proof. Proof of (i). The existence and uniqueness of the geometric parameters is proved for fixed time. Let $0 < \gamma \ll 1$. First, for any $u \in H^1$ such that

$$\inf_{|\xi_1-\xi_2|>|\log\gamma|} \Big\| u - \sum_{k=1,2} \sigma_k Q(\cdot - \xi_k) \Big\|_{H^1} \le \gamma, \tag{2.25}$$

we consider $z_1(u)$ and $z_2(u)$ achieving the infimum

$$\Big\| u - \sum_{k=1,2} \sigma_k Q(\cdot - z_k(u)) \Big\|_{L^2} = \inf_{\xi_1,\xi_2\in\mathbb{R}^N} \Big\| u - \sum_{k=1,2} \sigma_k Q(\cdot - \xi_k) \Big\|_{L^2}.$$

Then, for $\gamma > 0$ small enough, the minimum is attained for $z_1(u)$ and $z_2(u)$ such that $|z_1(u) - z_2(u)| > |\log\gamma| - C$, for some $C > 0$. Let

$$\varepsilon(x) = u(x) - \sigma_1 Q(x - z_1(u)) - \sigma_2 Q(x - z_2(u)), \quad \|\varepsilon\|_{L^2} \le \gamma.$$

By standard arguments since $|z_1(u) - z_2(u)| > |\log\gamma| - C$, it holds

$$\langle \varepsilon, \partial_{x_j} Q(\cdot - z_1(u)) \rangle = \langle \varepsilon, \partial_{x_j} Q(\cdot - z_2(u)) \rangle = 0. \tag{2.26}$$

For u and $\tilde u$ as in (2.25), we compare the corresponding z_k, $\tilde z_k$ and ε, $\tilde\varepsilon$. First, for ζ, $\tilde\zeta \in \mathbb{R}^N$, setting $\check\zeta = \zeta - \tilde\zeta$, we observe the following estimates

$$Q(\cdot - \zeta) - Q(\cdot - \tilde\zeta) = -(\check\zeta \cdot \nabla) Q(\cdot - \zeta) + O_{H^1}(|\check\zeta|^2), \tag{2.27}$$

$$\nabla Q(\cdot - \zeta) - \nabla Q(\cdot - \tilde\zeta) = -(\check\zeta \cdot \nabla^2) Q(\cdot - \zeta) + O_{H^1}(|\check\zeta|^2). \tag{2.28}$$

Thus, denoting $\check u = u - \tilde u$, $\check z_k = z_k - \tilde z_k$, $\check\varepsilon = \varepsilon - \tilde\varepsilon$, we obtain

$$\check u = -\sum_{k=1,2} \sigma_k (\check z_k \cdot \nabla) Q(\cdot - z_k) + \check\varepsilon + O_{H^1}(|\check z_1|^2 + |\check z_2|^2).$$

(In the O_{H^1}, there is no dependence on $\check u$ or $\check\varepsilon$). Projecting on each $\nabla Q(\cdot - z_k)$, using (2.26) and the above estimates, we obtain

$$|\check z_1| + |\check z_2| \lesssim \|\check u\|_{L^2} + (|\check z_1| + |\check z_2|)(e^{-\frac12 |z|} + \|\check\varepsilon\|_{L^2} + |\check z_1| + |\check z_2|)$$

and thus, for γ, $\check z_1$ and $\check z_2$ small,

$$\|\check\varepsilon\|_{H^1} \lesssim \|\check u\|_{H^1}, \quad |\check z_1| + |\check z_2| \lesssim \|\check u\|_{L^2}. \tag{2.29}$$

Therefore, for γ small enough, this proves uniqueness and Lipschitz continuity of z_1 and z_2 with respect to u in L^2.
Now, let $v \in L^2$ and z_1, z_2 be such that $\|v\|_{L^2} < \gamma$ and $|z_1 - z_2| \gg 1$. Set

$$\eta(x) = v(x) + \sigma_1(\ell_1 \cdot \nabla) Q(x - z_1) + \sigma_2(\ell_2 \cdot \nabla) Q(x - z_2).$$

Then, it is easy to check that the $2N$ conditions

$$\langle \eta, \partial_{x_j} Q(\cdot - z_k) \rangle = 0 \tag{2.30}$$

for $j = 1, \ldots, N$, $k = 1, 2$, are equivalent to a linear system in the components of ℓ_1 and ℓ_2 whose matrix is a perturbation of the identity up to a multiplicative constant. In particular, it is invertible and the existence and uniqueness of parameters $\ell_1(v,z_1,z_2),\ell_2(v,z_1,z_2)\in\mathbb{R}^N$ satisfying (2.30) and $|\ell_1|+|\ell_2|\lesssim\|v\|_{L^2}$ is clear. Moroever, with similar notation as before, it holds

$$\|\check{\eta}\|_{L^2}+|\check{\ell}_1|+|\check{\ell}_2|\lesssim\|\check{v}\|_{L^2}+|\check{z}_1|+|\check{z}_2|. \tag{2.31}$$

Estimate (2.18) is now proved. In the rest of this proof, we formally derive the equations of $\vec{\varepsilon}$ and the geometric parameters from the equation of u. This derivation can be justified rigorously and used to prove by the Cauchy-Lipschitz theorem that the parameters are $\mathcal{C}^1$ functions of time (see for instance [5, Proof of Lemma 2.7]).
Proof of (ii). First, by the definition of ε and η,

$$\partial_t\varepsilon=\partial_t u-\sum_{k=1,2}\partial_t Q_k=\eta+\sum_{k=1,2}(\dot{z}_k-\ell_k)\cdot\nabla Q_k.$$

Second,

$$\begin{aligned}\partial_t\eta&=\partial_{tt}u+\sum_{k=1,2}\partial_t(\ell_k\cdot\nabla Q_k)\\&=\Delta u-u+f(u)-2\alpha\partial_t u+\sum_{k=1,2}\dot{\ell}_k\cdot\nabla Q_k-\sum_{k=1,2}(\ell_k\cdot\nabla)(\dot{z}_k\cdot\nabla)Q_k.\end{aligned}$$

By (2.16), $\Delta Q_k-Q_k+f(Q_k)=0$ (from (1.4)) and the definition of G,

$$\begin{aligned}\Delta u-u+f(u)-2\alpha\partial_t u&=\Delta\varepsilon-\varepsilon+f(R+\varepsilon)-f(R)-2\alpha\eta\\&\quad+2\alpha\sum_{k=1,2}\ell_k\cdot\nabla Q_k+G.\end{aligned}$$

Therefore,

$$\begin{aligned}\partial_t\eta&=\Delta\varepsilon-\varepsilon+f(R+\varepsilon)-f(R)-2\alpha\eta\\&\quad+\sum_{k=1,2}(\dot{\ell}_k+2\alpha\ell_k)\cdot\nabla Q_k-\sum_{k=1,2}(\ell_k\cdot\nabla)(\dot{z}_k\cdot\nabla)Q_k+G.\end{aligned}$$

Proof of (iii)-(iv). We derive (2.20) from (2.17). For any $j=1,\dots,N$, we have

$$0=\frac{\mathrm{d}}{\mathrm{d}t}\langle\varepsilon,\partial_{x_j}Q_1\rangle=\langle\partial_t\varepsilon,\partial_{x_j}Q_1\rangle+\langle\varepsilon,\partial_t(\partial_{x_j}Q_1)\rangle.$$

Thus (2.19) gives

$$\langle\eta,\partial_{x_j}Q_1\rangle+\langle\mathrm{Mod}_\varepsilon,\partial_{x_j}Q_1\rangle-\langle\varepsilon,\dot{z}_1\cdot\nabla\partial_{x_j}Q_1\rangle=0.$$

The first term is zero due to the orthogonality (2.17). Hence, using the expression of Mod_ε

$$(\dot{z}_{1,j}-\ell_{1,j})\|\partial_{x_j}Q\|_{L^2}^2=-\int(\dot{z}_2-\ell_2)\cdot\nabla Q_2(\partial_{x_j}Q_1)\,\mathrm{d}x+\langle\varepsilon,\dot{z}_1\cdot\nabla\partial_{x_j}Q_1\rangle. \tag{2.32}$$

From this formula, we deduce

$$|\dot{z}_{1,j}-\ell_{1,j}|\lesssim|\dot{z}_2-\ell_2|\int|\nabla Q_2(x)||\nabla Q_1(x)|\,\mathrm{d}x+|\dot{z}_1|\,\|\varepsilon\|_{L^2}.$$

Thus, also using (2.3) with $m=1$ and $m'=\frac{1}{2}$, we obtain

$$|\dot{z}_1-\ell_1|\lesssim|\dot{z}_2-\ell_2|e^{-\frac{1}{2}|z|}+|\dot{z}_1-\ell_1|\,\|\vec{\varepsilon}\|_{H^1\times L^2}+|\ell_1|\,\|\vec{\varepsilon}\|_{H^1\times L^2}.$$

Since $\|\vec{\varepsilon}\|_{H^1\times L^2}\lesssim\gamma$, this yields

$$|\dot{z}_1-\ell_1|\lesssim|\dot{z}_2-\ell_2|e^{-\frac{1}{2}|z|}+|\ell_1|\,\|\vec{\varepsilon}\|_{H^1\times L^2}.$$

Similarly, it holds

$$|\dot{z}_2-\ell_2|\lesssim|\dot{z}_1-\ell_1|e^{-\frac{1}{2}|z|}+|\ell_2|\,\|\vec{\varepsilon}\|_{H^1\times L^2},$$

and thus, for large $|z|$,

$$\sum_{k=1,2} |\dot{z}_k - \ell_k| \lesssim (|\ell_1| + |\ell_2|) \|\vec{\varepsilon}\|_{H^1\times L^2},$$

which implies (2.20).
Next, we derive (2.21)-(2.22). From (2.17), it holds

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\langle \eta, \partial_{x_j} Q_1\rangle = \langle \partial_t \eta, \partial_{x_j} Q_1\rangle + \langle \eta, \partial_t(\partial_{x_j} Q_1)\rangle.$$

Thus, by (2.17) and (2.19), we have

$$\begin{aligned} 0 &= \langle \Delta\varepsilon - \varepsilon + f'(Q_1)\varepsilon, \partial_{x_j} Q_1\rangle + \langle f(R+\varepsilon) - f(R) - f'(R)\varepsilon, \partial_{x_j} Q_1\rangle \\ &\quad + \langle (f'(R) - f'(Q_1))\varepsilon, \partial_{x_j} Q_1\rangle \\ &\quad + \langle \mathrm{Mod}_\eta, \partial_{x_j} Q_1\rangle + \langle G, \partial_{x_j} Q_1\rangle - \langle \eta, (\dot{z}_1\cdot\nabla)\partial_{x_j} Q_1\rangle. \end{aligned}$$

Since $\partial_{x_j} Q_1$ satisfies $\Delta\partial_{x_j} Q_1 - \partial_{x_j} Q_1 + f'(Q_1)\partial_{x_j} Q_1 = 0$, the first term is zero. Hence, using the expression of Mod_η

$$\begin{aligned} \big(\dot{\ell}_{1,j} + 2\alpha\ell_{1,j}\big)\|\partial_{x_j} Q\|_{L^2}^2 &= \sum_{k=1,2} \langle (\ell_k\cdot\nabla)(\dot{z}_k\cdot\nabla) Q_k, \partial_{x_j} Q_1\rangle \\ &- \int (\dot{\ell}_2 + 2\alpha\ell_2)\cdot\nabla Q_2(\partial_{x_j} Q_1) - \langle f(R+\varepsilon) - f(R) - f'(R)\varepsilon, \partial_{x_j} Q_1\rangle \\ &- \langle (f'(R) - f'(Q_1))\varepsilon, \partial_{x_j} Q_1\rangle - \langle G, \partial_{x_j} Q_1\rangle + \langle \eta, (\dot{z}_1\cdot\nabla)\partial_{x_j} Q_1\rangle. \end{aligned} \tag{2.33}$$

By Taylor expansion (as f is $\mathcal{C}^2$), we have

$$f(R+\varepsilon) - f(R) - f'(R)\varepsilon = \varepsilon^2 \int_0^1 (1-\theta) f''(R+\theta\varepsilon)\,\mathrm{d}\theta,$$

and by the H^1 sub-criticality of the exponent $p > 2$, we infer

$$\big|\langle f(R+\varepsilon) - f(R) - f'(R)\varepsilon, \partial_{x_j} Q_1\rangle\big| \lesssim \|\varepsilon\|_{H^1}^2. \tag{2.34}$$

Then, again by Taylor expansion and $p > 2$,

$$|f'(R) - f'(Q_1)||\partial_{x_j} Q_1| \lesssim |Q_2||Q_1|^{p-1} + |Q_1||Q_2|^{p-1}.$$

Thus, using also the Cauchy-Schwarz inequality and (2.6),

$$|\langle (f'(R) - f'(Q_1))\varepsilon, \partial_{x_j} Q_1\rangle| \lesssim \|\varepsilon\|_{L^2}^2 + [q(|z|)]^2. \tag{2.35}$$

By (2.10) and the definition of c_1 in (2.8), we have, for any $1 < \theta < \min(p-1, 2)$,

$$\left|\frac{\langle G, \partial_{x_j} Q_1\rangle}{\|\partial_{x_1} Q\|_{L^2}^2} - \sigma\frac{z_j}{|z|} g(|z|)\right| \lesssim e^{-\theta|z|}.$$

Last, by (2.20), we have

$$|\langle \eta, (\dot{z}_1\cdot\nabla)\partial_{x_j} Q_1\rangle| \lesssim (|\dot{z}_1 - \ell_1| + |\ell_1|)\|\vec{\varepsilon}\|_{H^1\times L^2} \lesssim \|\vec{\varepsilon}\|_{H^1\times L^2}^2 + \sum_{k=1,2} |\ell_k|^2.$$

Combining the above estimates, we obtain

$$\left|\big(\dot{\ell}_1 + 2\alpha\ell_1\big) + \sigma\frac{z}{|z|} g(|z|)\right| \lesssim \big|\dot{\ell}_2 + 2\alpha\ell_2\big| e^{-m|z|} + \sum_{k=1,2} |\ell_k|^2 + \|\vec{\varepsilon}\|_{H^1\times L^2}^2 + e^{-\theta|z|}.$$

Similarly, from $(\eta, \partial_{x_j} Q_2) = 0$, we check

$$\left|\big(\dot{\ell}_2 + 2\alpha\ell_2\big) - \sigma\frac{z}{|z|} g(|z|)\right| \lesssim \big|\dot{\ell}_1 + 2\alpha\ell_1\big| e^{-m|z|} + \sum_{k=1,2} |\ell_k|^2 + \|\vec{\varepsilon}\|_{H^1\times L^2}^2 + e^{-\theta|z|}.$$

These estimates imply (2.22); (2.21) follows readily using (2.9).

Proof of (v). By (2.19), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}a_1^{\pm} &= \langle \partial_t \vec{\varepsilon}, \vec{Z}_1^{\pm}\rangle + \langle \vec{\varepsilon}, \partial_t \vec{Z}_1^{\pm}\rangle \\ &= (\zeta^{\pm} - 2\alpha)\langle \eta, Y_1\rangle + \langle \Delta\varepsilon - \varepsilon + f'(Q_1)\varepsilon, Y_1\rangle \\ &\quad + \langle f(R+\varepsilon) - f(R) - f'(R)\varepsilon, Y_1\rangle + \langle (f'(R) - f'(Q_1))\varepsilon, Y_1\rangle \\ &\quad + \langle G, Y_1\rangle + \zeta^{\pm}\langle \mathrm{Mod}_{\varepsilon}, Y_1\rangle + \langle \mathrm{Mod}_{\eta}, Y_1\rangle - \langle \vec{\varepsilon}, \dot{z}_1 \cdot \nabla \vec{Z}_1^{\pm}\rangle.\end{aligned}$$

Using $\zeta^{\pm} - 2\alpha = \nu^{\pm}$ and (2.23), $\mathcal{L}Y = -\nu_0^2 Y$ and $\nu_0^2 = \nu^{\pm}\zeta^{\pm}$, we observe that

$$(\zeta^{\pm} - 2\alpha)\langle \eta, Y_1\rangle + \langle \Delta\varepsilon - \varepsilon + f'(Q_1)\varepsilon, Y_1\rangle = \nu^{\pm}a_1^{\pm}.$$

Using the decay properties of Y in Lemma 1.8 and proceeding as before for (2.34) and (2.35),

$$|\langle f(R+\varepsilon) - f(R) - f'(R)\varepsilon, Y_1\rangle| + |\langle (f'(R) - f'(Q_1))\varepsilon, Y_1\rangle| \lesssim \|\varepsilon\|_{L^2}^2 + e^{-\frac{3}{2}|z|}.$$

Next, by (2.6), $|\langle G, Y_1\rangle| \lesssim q(|z|)$. Last, by (2.20) and (2.21),

$$|\langle \mathrm{Mod}_{\varepsilon}, Y_1\rangle| + |\langle \mathrm{Mod}_{\eta}, Y_1\rangle| + |\langle \vec{\varepsilon}, \dot{z}_1 \cdot \nabla \vec{Z}_1^{\pm}\rangle| \lesssim \|\vec{\varepsilon}\|_{H^1\times L^2}^2 + \sum_{k=1,2} |\ell_k|^2 + q(|z|).$$

Gathering these estimates, and proceeding similarly for $a_2^{\pm}$, (2.24) is proved. □

2.3. **Energy estimates.** For $\mu > 0$ small to be chosen, we denote $\rho = 2\alpha - \mu$. Consider the nonlinear energy functional

$$\mathcal{E} = \int \left\{ |\nabla\varepsilon|^2 + (1 - \rho\mu)\varepsilon^2 + (\eta + \mu\varepsilon)^2 - 2[F(R+\varepsilon) - F(R) - f(R)\varepsilon] \right\}. \quad (2.36)$$

Lemma 2.5. *There exists $\mu > 0$ such that in the context of Lemma 2.2, the following hold.*

(i) Coercivity and bound.

$$\mu\|\vec{\varepsilon}\|_{H^1\times L^2}^2 - \frac{1}{2\mu}\sum_{k=1,2}\left((a_k^+)^2 + (a_k^-)^2\right) \le \mathcal{E} \le \frac{1}{\mu}\|\vec{\varepsilon}\|_{H^1\times L^2}^2. \quad (2.37)$$

(ii) Time variation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \le -2\mu\mathcal{E} + \frac{1}{\mu}\|\vec{\varepsilon}\|_{H^1\times L^2}\left[\|\vec{\varepsilon}\|_{H^1\times L^2}^2 + \sum_{k=1,2}|\ell_k|^2 + q(|z|)\right]. \quad (2.38)$$

Remark 2.6. The above lemma is valid for any small enough $\mu > 0$. For future needs, we assume further

$$\mu \le \min\left(1, \alpha, |\nu_-|\right). \quad (2.39)$$

One checks that a usual linearized energy, corresponding to $\mu = 0$ in the definition of $\mathcal{E}$, would only gives damping for the component η. This is the reason why we introduce the modified energy $\mathcal{E}$. For the simplicity of notation, the same small constant $\mu > 0$ is used in (2.37) and (2.38) though in the former estimate the small constant is related to the coercivity constant c of Lemma 1.8, while in the latter it is related to the damping α.

Proof. Proof of (i). The upper bound on $\mathcal{E}$ in (2.37) easily follows from the energy subcriticality of p. The coercivity is proved for fixed time and so we omit the time dependency. By translation invariance, we assume without loss of generality that $z_1 = -z_2 = \frac{z}{2}$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying the following properties

$$\chi = 1 \text{ on } [0,1], \quad \chi = 0 \text{ on } [2, +\infty), \quad \chi' \le 0 \text{ on } \mathbb{R}.$$

For $\lambda = \frac{|z|}{4} \gg 1$, let

$$\chi_1(t,x) = \chi\left(\frac{|x - z_1(t)|}{\lambda(t)}\right), \quad \chi_2(t,x) = (1 - \chi_1^2(x))^{\frac{1}{2}},$$

so that $\chi_1^2 + \chi_2^2 = 1$. We define $\varepsilon_k = \varepsilon(\cdot + z_k)\chi_k(\cdot + z_k)$ for $k = 1, 2$ and we decompose $\mathcal{E}$

$$\mathcal{E} = \langle \mathcal{L}\varepsilon_1, \varepsilon_1\rangle + \langle \mathcal{L}\varepsilon_2, \varepsilon_2\rangle + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 + \mathcal{E}_5, \tag{2.40}$$

where

$$\begin{aligned}
\mathcal{E}_1 &= \int |\nabla\varepsilon|^2 - \int |\nabla\varepsilon_1|^2 - \int |\nabla\varepsilon_2|^2, \\
\mathcal{E}_2 &= -\rho\mu \int \varepsilon^2 + \int (\eta + \mu\varepsilon)^2, \\
\mathcal{E}_3 &= -2\int \left[F(R+\varepsilon) - F(R) - f(R)\varepsilon - \frac{\varepsilon^2}{2}f'(R)\right], \\
\mathcal{E}_4 &= -\int \varepsilon^2\left[f'(R) - f'(Q_1) - f'(Q_2)\right], \\
\mathcal{E}_5 &= -\int \varepsilon^2 f'(Q_1)(1-\chi_1^2) - \int \varepsilon^2 f'(Q_2)(1-\chi_2^2).
\end{aligned}$$

First, using (2.17), we have

$$\langle \varepsilon_k, \partial_{x_j} Q\rangle = \sigma_k\langle \varepsilon\chi_k, \partial_{x_j} Q_k\rangle = \sigma_k\langle \varepsilon, (\chi_k - 1)\partial_{x_j} Q_k\rangle = O(e^{-\frac{|z|}{4}}\|\varepsilon\|_{L^2}),$$

and

$$\langle \varepsilon_k, Y\rangle = \sigma_k\langle \varepsilon\chi_k, Y_k\rangle = \sigma_k\langle \varepsilon, Y_k\rangle + O(e^{-\frac{|z|}{4}}\|\varepsilon\|_{L^2}).$$

Thus, applying (ii) of Lemma 1.8 to ε_k, one obtains

$$\langle \mathcal{L}\varepsilon_k, \varepsilon_k\rangle \geq c\|\varepsilon_k\|_{H^1}^2 - C\langle \varepsilon, Y_k\rangle^2 - C(e^{-\frac{|z|}{4}} + \lambda^{-2})\|\varepsilon\|_{L^2}^2.$$

Second,

$$\int |\nabla\varepsilon_k|^2 = \int |\nabla\varepsilon|^2\chi_k^2 - \int \varepsilon^2(\Delta\chi_k)\chi_k,$$

and thus

$$|\mathcal{E}_1| \lesssim \lambda^{-2}\|\varepsilon\|_{L^2}^2. \tag{2.41}$$

Then, using $(a+b)^2 \geq \frac{1}{2}a^2 - b^2$, we see that

$$\mathcal{E}_2 \geq \frac{1}{2}\int \eta^2 - \mu(\rho+\mu)\int \varepsilon^2.$$

Next, using

$$\left|F(R+\varepsilon) - F(R) - f(R)\varepsilon - \frac{1}{2}f'(R)\varepsilon^2\right| \lesssim |\varepsilon|^3 + |\varepsilon|^{p+1},$$

and $\int |\varepsilon|^3 + |\varepsilon|^{p+1} \lesssim \|\varepsilon\|_{H^1}^3 + \|\varepsilon\|_{H^1}^{p+1}$ by energy subcriticality of p, so that

$$|\mathcal{E}_3| \lesssim \|\varepsilon\|_{H^1}^3 + \|\varepsilon\|_{H^1}^{p+1}.$$

Then, as in the proof of Lemma 2.1, using $p > 2$, for some $m > 0$, it holds

$$|f'(R) - f'(Q_1) - f'(Q_2)| \lesssim |Q_1|^{p-2}|Q_2| + |Q_1||Q_2|^{p-2} \lesssim e^{-m|z|},$$

and so $|\mathcal{E}_4| \lesssim e^{-m|z|}\|\varepsilon\|_{L^2}^2$. Last, since $|f'(Q_k)(1-\chi_k^2)| \lesssim Q(\lambda)$ for $k = 1, 2$, we have $|\mathcal{E}_5| \lesssim Q(\lambda)\|\varepsilon\|_{L^2}^2$.

Gathering these estimates, for γ, μ small and $|z|$ large, we obtain

$$\mathcal{E} \geq \frac{c}{2}\|\varepsilon\|_{H^1}^2 + \frac{1}{2}\|\eta\|_{L^2}^2 - C\sum_{k=1,2}\langle \varepsilon, Y_k\rangle^2.$$

Using

$$\langle \varepsilon, Y_k\rangle = \frac{\langle \vec{\varepsilon}, \vec{Z}_k^+ - \vec{Z}_k^-\rangle}{\zeta^+ - \zeta^-} = \frac{a_k^+ - a_k^-}{\zeta^+ - \zeta^-},$$

we have proved (2.37).

Proof of (ii). From direct computations and integration by parts, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} &= \int \partial_t\varepsilon\left[-\Delta\varepsilon + (1-\rho\mu)\varepsilon - f(R+\varepsilon) + f(R)\right] + (\partial_t\eta + \mu\partial_t\varepsilon)(\eta+\mu\varepsilon)\\ &\quad + \int \sum_{k=1,2} (\dot{z}_k\cdot\nabla Q_k)\left[f(R+\varepsilon) - f(R) - f'(R)\varepsilon\right] = \mathbf{g_1} + \mathbf{g_2}.\end{aligned}$$

Using (2.19), integration by parts and $2\alpha = \rho+\mu$, we compute

$$\begin{aligned}\mathbf{g_1} &= -\mu\int \left\{|\nabla\varepsilon|^2 + (1-\rho\mu)\varepsilon^2 - \varepsilon[f(R+\varepsilon) - f(R)]\right\} - \rho\int(\eta+\mu\varepsilon)^2\\ &\quad + \int \mathrm{Mod}_\varepsilon\big\{-\Delta\varepsilon + (1-\rho\mu)\varepsilon - \big[f(R+\varepsilon) - f(R)\big]\big\}\\ &\quad + \int(\eta+\mu\varepsilon)[\mathrm{Mod}_\eta + \mu\mathrm{Mod}_\varepsilon] + \int(\eta+\mu\varepsilon)G\\ &= \mathbf{g_{1,1}} + \mathbf{g_{1,2}} + \mathbf{g_{1,3}} + \mathbf{g_{1,4}}.\end{aligned}$$

Note that by $0<\mu<\alpha$, one has $\rho-\mu = 2(\alpha-\mu)>0$ and so

$$\begin{aligned}\mathbf{g_{1,1}} &= -\mu\mathcal{E} - 2\mu\int\left[F(R+\varepsilon) - F(R) - f(R)\varepsilon - \tfrac{1}{2}f'(R)\varepsilon^2\right]\\ &\quad + \mu\int\varepsilon[f(R+\varepsilon) - f(R) - f'(R)\varepsilon] - (\rho-\mu)\int(\eta+\mu\varepsilon)^2 \le -\mu\mathcal{E} + C\|\varepsilon\|_{H^1}^3,\end{aligned}$$

where we have estimated, using $p>2$, Hölder inequality, the sub-criticality of p and Sobolev embedding,

$$\begin{aligned}&\int\left|F(R+\varepsilon) - F(R) - f(R)\varepsilon - \tfrac{1}{2}f'(R)\varepsilon^2\right|\\ &\quad + \int|\varepsilon[f(R+\varepsilon) - f(R) - f'(R)\varepsilon]| \lesssim \int|\varepsilon|^3|R|^{p-2} + |\varepsilon|^{p+1} \lesssim \|\varepsilon\|_{H^1}^3.\end{aligned}$$

Using the Cauchy-Schwarz inequality and (2.20)-(2.21), we also derive the following estimates

$$|\mathbf{g_{1,2}}| \lesssim (|\dot{z}_1 - \ell_1| + |\dot{z}_2 - \ell_2|)\,\|\varepsilon\|_{H^1} \lesssim \|\vec{\varepsilon}\|_{H^1\times L^2}\left[\|\vec{\varepsilon}\|_{H^1\times L^2}^2 + \sum_{k=1,2}|\ell_k|^2\right],$$

and (also using the orthogonality conditions (2.17))

$$\begin{aligned}|\mathbf{g_{1,3}}| &= \left|\int(\eta+\mu\varepsilon)\Big(\sum_{k=1,2}(\ell_k\cdot\nabla)(\dot{z}_k\cdot\nabla)Q_k\Big)\right|\\ &\lesssim (|\ell_1|+|\ell_2|)(|\dot{z}_1-\ell_1| + |\dot{z}_2-\ell_2| + |\ell_1|+|\ell_2|)(\|\varepsilon\|_{L^2} + \|\eta\|_{L^2})\\ &\lesssim \|\vec{\varepsilon}\|_{H^1\times L^2}\left[\|\vec{\varepsilon}\|_{H^1\times L^2}^2 + \sum_{k=1,2}|\ell_k|^2\right].\end{aligned}$$

Next, from (2.6) and the Cauchy-Schwarz inequality,

$$|\mathbf{g_{1,4}}| \lesssim \|G\|_{L^2}(\|\varepsilon\|_{L^2} + \|\eta\|_{L^2}) \lesssim q(|z|)\|\vec{\varepsilon}\|_{H^1\times L^2}.$$

Last, by (2.20), proceeding as before, we see that

$$|\mathbf{g_2}| \lesssim (|\dot z_1| + |\dot z_2|)\|\varepsilon\|_{H^1}^2 \lesssim (|\dot z_1 - \ell_1| + |\dot z_2 - \ell_2| + |\ell_1| + |\ell_2|)\|\vec\varepsilon\|_{H^1\times L^2}^2$$
$$\lesssim \|\vec\varepsilon\|_{H^1\times L^2}^2\left[\|\vec\varepsilon\|_{H^1\times L^2}^2 + \sum_{k=1,2}|\ell_k|\right].$$

Gathering the above estimates, (2.38) is proved, taking μ small enough. □

2.4. **Trichotomy of evolution.** Estimates (2.20), (2.21), (2.22), (2.24) and (2.38) give basic information on the evolution of the various components of a solution in the framework of the decomposition introduced in Lemma 2.2.
We introduce notation related to modified parameters that allow us to justify the following trichotomy in the evolution of the solution.

ODE behavior for $|z|$: The distance $z = z_1 - z_2$ formally satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{q(|z|)}\right] = -\frac{\sigma g_0}{\alpha}.$$

Note that for $\sigma = -1$, $\log t$ is an approximate solution of this ODE. This justifies the rigidity results in Theorem 1.4. In contrast, when $\sigma = 1$, there are no solution such that $|z| \to \infty$, which explains the non-existence result in Theorem 1.3.

Exponential growth: The parameters a_k^+ are related to the forward exponential instability of the solitary wave. They will require a specific approach, involving backward in time arguments. The existence of exactly one direction of instability for each solitary wave justifies Theorem 1.5.

Damped evolution: The parameters ℓ_k, a_k^- and the remainder $\vec\varepsilon$ (without its unstable components a_k^+) enjoy exponential damping; they will be easily estimated provided that the other parameters are locked, see Proposition 3.1.

First, we set

$$y = z + \frac{\ell}{2\alpha}, \quad r = |y|. \tag{2.42}$$

Second, we define

$$b = \sum_{k=1,2}(a_k^+)^2. \tag{2.43}$$

Third, we introduce notation for the damped components:

$$\mathcal{F} = \mathcal{E} + \mathcal{G}, \quad \mathcal{G} = \sum_{k=1,2}|\ell_k|^2 + \frac{1}{2\mu}\sum_{k=1,2}(a_k^-)^2, \tag{2.44}$$

and for all the components of the solution

$$\mathcal{N} = \left[\|\vec\varepsilon\|_{H^1\times L^2}^2 + \sum_{k=1,2}|\ell_k|^2\right]^{\frac12}. \tag{2.45}$$

Last, we define

$$\mathcal{M} = \frac{1}{\mu^2}\left(\mathcal{F} - \frac{b}{2\nu^+}\right). \tag{2.46}$$

In the following lemma, we rewrite the estimates of Lemmas 2.2 and 2.5 in terms of these new parameters.

Lemma 2.7. *In the context of Lemma 2.2, the following hold.*

(i) Comparison with original variables.

$$\big|r - |z|\big| \le |y - z| \lesssim \mathcal{N}, \quad |g(|z|) - g_0 q(r)| \lesssim q(r)(\mathcal{N} + r^{-1}), \tag{2.47}$$

$$\mu \mathcal{N}^2 \le \mu \|\vec{\varepsilon}\|_{H^1\times L^2}^2 + \sum_{k=1,2} |\ell_k|^2 \le \mathcal{F} + \frac{b}{2\mu} \lesssim \mathcal{N}^2. \tag{2.48}$$

(ii) ODE behavior for the distance. *For some* $K > 0$,

$$\left| \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{q(r)}\right] + \frac{\sigma g_0}{\alpha} \right| \le \frac{K}{q(r)} \left(\mathcal{N}^2 + r^{-1} q(r)\right). \tag{2.49}$$

(iii) Exponential instability.

$$|\dot{b} - 2\nu^+ b| \lesssim \mathcal{N}^3 + q(r)\mathcal{N}. \tag{2.50}$$

(iv) Damped components.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} + 2\mu\mathcal{F} \lesssim \mathcal{N}^3 + q(r)\mathcal{N}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G} + 2\mu\mathcal{G} \lesssim \mathcal{N}^3 + q(r)\mathcal{N}. \tag{2.51}$$

(v) Liapunov type functional.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M} \le -\mathcal{N}^2 + C[q(r)]^2. \tag{2.52}$$

(vi) Refined estimates for the distance. *Setting*

$$R^+ = \frac{1}{q(r)} \exp(K\mathcal{M}) \quad \textit{and} \quad R^- = \frac{1}{q(r)} \exp(-K\mathcal{M}),$$

(K is given in (2.49)*), it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t} R^+ \le \left(-\frac{\sigma g_0}{\alpha} + 2K r^{-1}\right) \exp(K\mathcal{M}), \tag{2.53}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} R^- \ge \left(-\frac{\sigma g_0}{\alpha} - 2K r^{-1}\right) \exp(-K\mathcal{M}). \tag{2.54}$$

Proof. Proof of (2.47). It follows from the triangle inequality that

$$\big|r - |z|\big| \le |y - z| \le \frac{|\ell|}{2\alpha} \lesssim \mathcal{N}.$$

The second part of (2.47) then follows from (2.9).
Proof of (2.48). It follows readily from (2.37).
Proof of (2.49). First, from (2.20), (2.22) and (2.47), we note (θ is as in (2.22))

$$\begin{aligned} \dot{y} = \dot{z} + \frac{\dot{\ell}}{2\alpha} &= -\frac{\sigma}{\alpha}\frac{z}{|z|} g(|z|) + O(\mathcal{N}^2 + e^{-\theta|z|}) \\ &= -\frac{\sigma g_0}{\alpha}\frac{y}{r} q(r) + O(\mathcal{N}^2 + r^{-1} q(r)). \end{aligned} \tag{2.55}$$

Hence,

$$\dot{r} = \frac{\dot{y}\cdot y}{r} = -\frac{\sigma g_0}{\alpha} q(r) + O(\mathcal{N}^2 + r^{-1} q(r)).$$

Using also $|q'(r) + q(r)| \lesssim r^{-1} q(r)$ (from (1.10)), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{q(r)}\right] = -\frac{\dot{r} q'(r)}{[q(r)]^2} = -\frac{\sigma g_0}{\alpha} + \frac{1}{q(r)} O(\mathcal{N}^2 + r^{-1} q(r)).$$

Proof of (2.50). It follows from (2.24) and $|a_k^+| \lesssim \|\vec{\varepsilon}\|_{H^1\times L^2} \le \mathcal{N}$.

Proof of (2.51). From the expression of $\mathcal{F}$ and then (2.38), (2.21) and (2.24)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} &= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + 2\sum_{k=1,2} \dot{\ell}_k \cdot \ell_k + \frac{1}{\mu}\sum_{k=1,2} \dot{a}_k^- a_k^- \\ &\leq -2\mu\mathcal{E} - 4\alpha \sum_{k=1,2} |\ell_k|^2 + \frac{\nu^-}{\mu}\sum_{k=1,2}(a_k^-)^2 + O(\mathcal{N}^3 + q(r)\mathcal{N}).\end{aligned}$$

Since $0 < \mu < \alpha$ and $0 < \mu < |\nu^-|$ (see (2.39)) we obtain (2.51) for $\mathcal{F}$. The proof for $\mathcal{G}$ is the same.

Proof of (2.52). First, it follows from combining (2.50) and (2.51) that

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M} \leq -2\mu\left(\mathcal{F} + \frac{b}{2\mu}\right) + O(\mathcal{N}^3 + q(r)\mathcal{N}),$$

Thus, from (2.48), we observe that

$$\mathcal{N}^2 \leq \frac{1}{\mu}\left(\mathcal{F} + \frac{b}{2\mu}\right) \leq -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M} + O(\mathcal{N}^3 + q(r)\mathcal{N}).$$

and (2.52) follows for $\mathcal{N}$ small enough.

Proof of (2.53)-(2.54). It follows from direct computation and (2.49) and (2.52) that (for r large enough)

$$\exp(-K\mathcal{M})\frac{\mathrm{d}}{\mathrm{d}t}R^+ = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{q(r)}\right] + \frac{K}{q(r)}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M} \leq -\frac{\sigma g_0}{\alpha} + 2Kr^{-1}.$$

The computation for $\frac{d}{dt}R^-$ is similar. □

2.5. **Energy of a 2-solitary wave.** We observe from the definition and the energy property (1.3) that a 2-solitary wave u of (1.1) satisfies

$$\lim_{t\to\infty} E(\vec{u}(t)) = \int |\nabla Q|^2 + Q^2 - 2F(Q) = 2E(Q,0). \tag{2.56}$$

More precisely, we expand the energy for a solution close to a 2-solitary wave.

Lemma 2.8. *In the context of Lemma 2.2, the following holds*

$$E(\vec{u}) = 2E(Q,0) - \sigma c_1 g_0 q(r) + O(r^{-1}q(r)) + O(q(r)\mathcal{N}) + O(\mathcal{N}^2). \tag{2.57}$$

Proof. Expanding $E(u, \partial_t u)$ using the decomposition (2.16), integration by parts, the equation $-\Delta Q_k + Q_k - f(Q_k) = 0$ and the definition of G in (2.2), we find

$$\begin{aligned}2E(u,\partial_t u) &= \int |\partial_t u|^2 + 2E(R,0) - 2\int G\varepsilon \\ &\quad + \int \left(|\nabla\varepsilon|^2 + \varepsilon^2 - 2F(R+\varepsilon) + 2F(R) + 2f(R)\varepsilon\right).\end{aligned}$$

Thus, using (2.6), the subcriticality of p and Sobolev embedding, there hold

$$2E(u,\partial_t u) = \int |\partial_t u|^2 + 2E(R,0) + O\big(q(r)\|\vec{\varepsilon}\|_{H^1\times L^2} + \|\vec{\varepsilon}\|_{H^1\times L^2}^2\big).$$

Note that $\partial_t u = \eta - \sum_{k=1,2}(\ell_k \cdot \nabla)Q_k$ implies $\|\partial_t u\|_{L^2} \lesssim \mathcal{N}$. Next, by direct computation, $-\Delta Q_1 + Q_1 - f(Q_1) = 0$ and then (2.7), (2.9) and (2.4)

$$\begin{aligned}E(Q_1+Q_2,0) &= 2E(Q,0) + \int (\nabla Q_1 \cdot \nabla Q_2 + Q_1Q_2 - f(Q_1)Q_2 - f(Q_2)Q_1) \\ &\quad - \int (F(R) - F(Q_1) - F(Q_2) - f(Q_1)Q_2 - f(Q_2)Q_1) \\ &= 2E(Q,0) - \int f(Q_2)Q_1 + O(e^{-\frac{5}{4}|z|}) \\ &= 2E(Q,0) - \sigma c_1 g(|z|) + O(|z|^{-1}g(|z|)).\end{aligned}$$

Using (2.47), $g(|z|) = g_0 q(r) + O(r^{-1}q(r)) + O(q(r)\mathcal{N})$ and the proof is complete. □

As a consequence of (1.3), (2.56) and (2.57), we obtain the following estimate.

Corollary 2.9. *Let $\vec{u}$ be a 2-solitary wave solution of (1.1) satisfying the decomposition of Lemma 2.2 on $[t,\infty)$, for some $t \in \mathbb{R}$. Then,*

$$\int_t^{+\infty} \|\partial_t u(s)\|_{L^2}^2 \, \mathrm{d}s \lesssim q(r(t)) + \mathcal{N}^2(t). \tag{2.58}$$

3. Proofs of Theorems 1.3 and 1.4

3.1. **General estimates.**

Proposition 3.1. *There exists $C > 0$ and $\delta_1 > 0$ such that the following hold. For any $0 < \delta < \delta_1$ and any 2-solitary wave $\vec{u}$ of (1.1), there exists $T_\delta \in \mathbb{R}$ such that $\vec{u}(t)$ admits a decomposition as in Lemma 2.2 in a neighborhood of T_δ with*

$$\mathcal{N}(T_\delta) \le \delta \quad \text{and} \quad q(r(T_\delta)) \le \delta^2. \tag{3.1}$$

Moreover, for any such T_δ, it holds:

Same sign case: *if $\sigma = 1$ then there exists $T_* > T_\delta$ such that $\vec{u}$ admits a decomposition as in Lemma 2.2 on $[T_\delta, T_*]$ with*

$$\forall t \in [T_\delta, T_*], \quad \mathcal{N}(t) \le C\delta \quad \text{and} \quad q(r(T_*)) = \delta^{\frac{3}{2}}. \tag{3.2}$$

Opposite sign case: *if $\sigma = -1$ then $\vec{u}$ admits a decomposition as in Lemma 2.2 for all $t \ge T_\delta$ and satisfies*

$$\forall t \in [T_\delta, \infty), \quad \mathcal{N}(t) \le C\delta \quad \text{and} \quad \left| \frac{1}{q(r(t))} - \frac{g_0}{\alpha} t \right| \lesssim \frac{t}{|\log \delta|} + \tilde{C} \tag{3.3}$$

for some $\tilde{C} > 0$.

Proof. Let $\vec{u}$ be a 2-solitary wave of (1.1). For δ to be taken small enough, the existence of T_δ satisfying (3.1) is a consequence of Definition 1.1 and Lemma 2.2. For a constant $C > 1$ to be taken large enough, we introduce the following bootstrap estimate

$$\mathcal{N} \le C\delta, \quad q(r) \le \delta^{\frac{3}{2}}, \quad b \le C\delta^2. \tag{3.4}$$

Set

$$T_* = \sup\left\{ t \in [T_\delta, \infty) \text{ such that (3.4) holds on } [T_\delta, t] \right\} > T_\delta.$$

In the remainder of the proof, the implied constants in $\lesssim$ or O *do not depend* on the constant C appearing in the bootstrap assumption (3.4). We start by improving the bootstrap assumption on $\mathcal{N}$ and b.

Estimate on $\mathcal{N}$. We now improve the bootstrap assumption (3.4) on $\mathcal{N}$.
From (2.51) and (3.4), it holds on $[T_\delta, T_*)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{2\mu t} \mathcal{F} \right] \lesssim C^3 \delta^3 e^{2\mu t} + C\delta^{\frac{5}{2}} e^{2\mu t} \lesssim \delta^2 e^{2\mu t},$$

for $\delta > 0$ small enough. From (2.48) and (3.1), we have $\mathcal{F}(T_\delta) \lesssim \delta^2$. Thus, integrating the above estimate on $[T_\delta, t]$, for any $t \in [T_\delta, T_*)$, it holds $\mathcal{F} \lesssim \delta^2$. In particular, by (2.48), we obtain

$$\|\vec{\varepsilon}\|_{H^1 \times L^2}^2 \lesssim \mathcal{F} + \frac{b}{2\mu} \lesssim C\delta^2. \tag{3.5}$$

Arguing similarly for the quantity $\mathcal{G}$, we have

$$\sum_{k=1,2} |\ell_k|^2 + \sum_{k=1,2} (a_k^-)^2 \lesssim \mathcal{G} \lesssim \delta^2. \tag{3.6}$$

Hence we obtain

$$\forall t \in [T_\delta, T_*), \quad \mathcal{N}(t) \lesssim \sqrt{C}\delta.$$

For C large enough, this strictly improves the estimate (3.4) of $\mathcal{N}$ on $[T_\delta, T_*)$.

Estimate on b. We now prove that for C large enough, it holds

$$\forall t \in [T_\delta, T_*), \quad b(t) \le \frac{C}{2}\delta^2.$$

From (2.58) in Corollary 2.9 , we have

$$\int_{T_\delta}^{+\infty} \|\partial_t u(s)\|_{L^2}^2 \,\mathrm{d}s \lesssim q(r(T_\delta)) + \mathcal{N}^2(T_\delta) \lesssim \delta^2. \tag{3.7}$$

By (3.1), we have $b(T_\delta) \lesssim \delta^2$.
For the sake of contradiction, take C large and assume that there exists $t_2 \in [T_\delta, T_*)$ such that

$$b(t_2) = \frac{C}{2}\delta^2, \quad b(t) < \frac{C}{2}\delta^2 \quad \text{on } [T_\delta, t_2).$$

On the one hand, by continuity of b, there exists $t_1 \in [T_\delta, t_2)$ such that

$$b(t_1) = \frac{C}{4}\delta^2 \quad \text{and} \quad b(t) > \frac{C}{4}\delta^2 \quad \text{on } (t_1, t_2].$$

Using (2.50) and (3.4), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} b = 2\nu^+ b + O(C^3\delta^3 + C\delta^{\frac{5}{2}})$$

which implies (for δ small enough with respect to $1/C$)

$$t_2 - t_1 = \frac{\log 2}{2\nu^+} + O(\delta^{\frac{1}{2}}), \tag{3.8}$$

and thus

$$\int_{t_1}^{t_2} b \gtrsim C\delta^2. \tag{3.9}$$

On the other hand, by (2.1), (2.16), $\partial_t u = \eta - \sum_{k=1,2}(\ell_k \cdot \nabla) Q_k$ and (3.6) we have $\|\eta\|_{L^2}^2 \lesssim \|\partial_t u\|_{L^2}^2 + \delta^2$ on $[T_\delta, T_*]$. Thus, using (3.7) and (3.8),

$$\int_{t_1}^{t_2} \|\eta(t)\|_{L^2}^2 \,\mathrm{d}t \lesssim \delta^2. \tag{3.10}$$

By the definition of $a_k^\pm$, one has

$$a_k^+ = \zeta^+ \langle \varepsilon, Y_k \rangle + \langle \eta, Y_k \rangle, \quad a_k^- = \zeta^- \langle \varepsilon, Y_k \rangle + \langle \eta, Y_k \rangle$$

and thus

$$a_k^+ = \frac{\zeta^+}{\zeta^-} a_k^- + \frac{\zeta^- - \zeta^+}{\zeta^-} \langle \eta, Y_k \rangle.$$

Combining (3.6), (3.8) and (3.10), we find $\displaystyle\int_{t_1}^{t_2} b \lesssim \delta^2$, a contradiction with (3.9) for C large enough.

We can now prove the two statements of Proposition 3.1. First observe that (3.5) and (3.6) give on $[T_\delta, T_*)$

$$\exp(\pm K\mathcal{M}) = 1 + O(C\delta^2).$$

Same sign case. We use the quantity R^+ defined in Lemma 2.7. From (2.53), for r large,

$$\frac{\mathrm{d}}{\mathrm{d}t} R^+ \le \left(-\frac{g_0}{\alpha} + 2Kr^{-1}\right) \exp\left(K\mathcal{M}\right) \le -\frac{g_0}{2\alpha}.$$

Therefore, for all $t \in [T_\delta, T_*]$, it holds

$$R^+(t) \le R^+(T_\delta) - \frac{g_0}{2\alpha}(t - T_\delta).$$

Assuming $T_* = \infty$, $R^+(t)$ becomes negative for some time, which is absurd. Since all the estimates in (3.4) have been strictly improved on $[T_\delta, T_*]$ except the one for $q(r)$, it follows by a continuity argument that $q(r(T_*)) = \delta^{\frac{3}{2}}$.

Opposite sign case. Here we use both R^+ and R^-. First notice that on $[T_\delta, T_*]$,

$$R^-(1 + O(C\delta^2)) \le \frac{1}{q(r)} \le R^+(1 + O(C\delta^2)). \tag{3.11}$$

From (2.53),

$$\frac{\mathrm{d}}{\mathrm{d}t} R^+ \le \frac{g_0}{\alpha}\left(1 + \frac{O(K)}{|\log \delta|}\right)(1 + O(C\delta^2)) \le \frac{g_0}{\alpha}\left(1 + \frac{O(K)}{|\log \delta|}\right),$$

for $\delta > 0$ small enough with respect to C. Thus, for any $t \in [T_\delta, T_*)$, it holds by integration on $[T_\delta, t]$

$$R^+(t) \le R^+(T_\delta) + \frac{g_0}{\alpha}\left(1 + \frac{O(K)}{|\log \delta|}\right)(t - T_\delta). \tag{3.12}$$

Similarly, we check that

$$R^-(t) \ge R^-(T_\delta) + \frac{g_0}{\alpha}\left(1 - \frac{O(K)}{|\log \delta|}\right)(t - T_\delta). \tag{3.13}$$

Note that (3.1), (3.11) and (3.13) imply that

$$\forall t \in [T_\delta, T_*), \quad \frac{1}{q(r(t))} \ge \frac{1}{2}\left(\delta^{-2} + \frac{g_0}{\alpha}(t - T_\delta)\right).$$

In particular, the estimate on $q(r)$ in (3.4) is strictly improved for δ small enough, As a consequence $T_* = \infty$.
Last, we observe that (3.11), (3.12) and (3.13) imply (3.3) where $\tilde{C}$ can be written in terms of $R^\pm(T_\delta)$, T_δ and δ. □

3.2. **Proof of Theorem 1.3.** Let $\vec{u}$ be a 2-solitary wave of (1.1) with $\sigma_1 = \sigma_2$. Let $\delta > 0$ to be fixed later, and let T_δ and T_* be as Proposition 3.1. Using (2.57) at $t = T_*$ and (3.2), we have

$$E(u(T_*), \partial_t u(T_*)) = 2E(Q, 0) - c_1 g_0 \delta^{\frac{3}{2}} + O(|\log \delta|^{-1}\delta^{\frac{3}{2}}).$$

This allows to fix a $\delta > 0$ small enough so that

$$E(u(T_*), \partial_t u(T_*)) < 2E(Q, 0).$$

This contradicts the fact that the energy is decreasing and converges to $2E(Q, 0)$ as $t \to \infty$.

3.3. **Proof of Theorem 1.4.** Let $\vec{u}$ be a 2-solitary wave of (1.1) with $\sigma_1 = -\sigma_2$.

Proposition 3.2. *There exists $T > 0$ such that the decomposition of $\vec{u}$ satisfies*

$$\forall t \ge T, \quad q(r(t)) \lesssim t^{-1}, \quad \mathcal{N}(t) \lesssim t^{-1}. \tag{3.14}$$

Proof. Let $0 < \delta < \delta_1$ in the framework of Proposition 3.1. From (3.3), there exists $T > 0$ large enough such that

$$\forall t \ge T/2, \quad \mathcal{N}(t) \le C\delta, \quad q(r(t)) \le \frac{2g_0}{\alpha t}.$$

In particular, from (2.50) and (2.51)

$$\left|\frac{\mathrm{d}b}{\mathrm{d}t} - 2\nu^+ b\right| \lesssim \mathcal{N}^3 + t^{-1}\mathcal{N}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} + 2\mu\mathcal{F} \lesssim \mathcal{N}^3 + t^{-1}\mathcal{N}. \tag{3.15}$$

Our goal is to obtain the decay rate of $\mathcal{N}$. The above bounds are not quite enough for this, due to the cubic term in $\mathcal{N}$ for which we have no decay yet, only smallness. In order to get around this, we will work on a modification $\tilde{b}$ of b. Recall (2.48):

$$0 \leq \mathcal{N}^2 \lesssim \mathcal{F} + \frac{b}{2\mu}.$$

For $0 < \epsilon \ll 1$ to be fixed, we observe that taking $\delta < \epsilon^2$, it holds

$$\mathcal{N}^3 + t^{-1}\mathcal{N} \lesssim \epsilon^2 \mathcal{N}^2 + \epsilon^{-2} t^{-2} \lesssim \epsilon^2 \left(\mathcal{F} + \frac{b}{2\mu} \right) + \epsilon^{-2} t^{-2}. \tag{3.16}$$

(Here and below the implicit constants do not depend on ε). Set $\tilde{b} = b - \epsilon \mathcal{F}$ and observe that

$$\tilde{b} = b - \epsilon \mathcal{F} = \left(1 + \frac{\epsilon}{2\mu} \right) b - \epsilon \left(\mathcal{F} + \frac{1}{2\mu} b \right) \leq \left(1 + \frac{\epsilon}{2\mu} \right) b.$$

Therefore, using (3.15),

$$\begin{aligned} \frac{\mathrm{d}\tilde{b}}{\mathrm{d}t} &\geq 2\nu^+ b + 2\epsilon\mu \mathcal{F} - C\epsilon^2 \left(\mathcal{F} + \frac{b}{2\mu} \right) - C\epsilon^{-2} t^{-2} \\ &\geq (2\nu^+ - \epsilon) b + \left(2\epsilon\mu - C\epsilon^2 \right) \left(\mathcal{F} + \frac{b}{2\mu} \right) - C\epsilon^{-2} t^{-2} \\ &\geq \nu^+ \tilde{b} - C\epsilon^{-2} t^{-2}, \end{aligned}$$

where $\epsilon > 0$ is taken small enough so that

$$(2\nu^+ - \epsilon) \left(1 + \frac{\epsilon}{2\mu} \right)^{-1} \geq \nu^+, \quad 2\epsilon\mu - C\epsilon^2 > 0.$$

This bound is suitable for our purpose. Integrating on $[t, \tau] \subset [T/2, +\infty)$, we obtain

$$\tilde{b}(t) - e^{-\nu^+(\tau - t)} \tilde{b}(\tau) \lesssim \epsilon^{-2} \int_t^\tau e^{-\nu^+(s-t)} s^{-2} \, \mathrm{d}s \lesssim \epsilon^{-2} t^{-2}.$$

Passing to the limit as $\tau \to \infty$, one obtains for all $t \geq T/2$,

$$\tilde{b}(t) \lesssim \epsilon^{-2} t^{-2}, \quad b(t) \leq C\epsilon^{-2} t^{-2} + \epsilon \mathcal{F}(t).$$

Inserting this information into the equations (3.15) of $\mathcal{F}$ and b and using (3.16), it holds

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathcal{F} + \frac{b}{2\mu} \right) &\leq -2\mu \mathcal{F} + \frac{\nu^+}{\mu} b + C\epsilon^2 \left(\mathcal{F} + \frac{b}{2\mu} \right) + C\epsilon^{-2} t^{-2} \\ &\leq -(2\mu - C\epsilon^2) \left(\mathcal{F} + \frac{b}{2\mu} \right) + \left(1 + \frac{\nu^+}{\mu} \right) b + C\epsilon^{-2} t^{-2} \\ &\leq -\mu \left(\mathcal{F} + \frac{b}{2\mu} \right) + C\epsilon^{-2} t^{-2}, \end{aligned}$$

by possibly choosing a smaller $\epsilon > 0$. Integrating on $[T/2, t]$, we obtain

$$\left(\mathcal{F} + \frac{b}{2\mu} \right)(t) - e^{-\mu(t - \frac{T}{2})} \left(\mathcal{F} + \frac{b}{2\mu} \right)(T/2) \lesssim \epsilon^{-2} \int_{\frac{T}{2}}^t e^{-\mu(t-s)} s^{-2} \, \mathrm{d}s$$

and so

$$\mathcal{N}^2(t) \lesssim \left(\mathcal{F} + \frac{b}{2\mu} \right)(t) \lesssim \epsilon^{-2} t^{-2} + \epsilon^{-2} \delta^2 e^{\mu T/2} e^{-\mu t}$$

For $t \geq T > 0$, the last term is bounded by $e^{-\mu t/2} \lesssim t^{-2}$, and this proves (3.14). □

We complete the proof of Theorem 1.4. The estimate (1.7) is a consequence of (2.16) and (3.14). Since (3.3) holds for any $\delta > 0$, it means that

$$\lim_{t\to\infty} \frac{1}{tq(r(t))} = \frac{g_0}{\alpha},$$

and, using the expansion (1.10) of q, this is equivalent to

$$\lim_{t\to\infty} \frac{1}{t} r(t)^{\frac{N-1}{2}} e^{r(t)} = \frac{\kappa g_0}{\alpha}. \tag{3.17}$$

Setting $c_0 = \log \frac{\kappa g_0}{\alpha}$, this implies

$$r(t) = \log t - \frac{1}{2}(N-1)\log\log t + c_0 + s(t), \text{ where } s(t) \to 0 \text{ as } t \to \infty. \tag{3.18}$$

Finally, we prove the refined estimate (1.8) on $z(t)$. We observe from (2.9), (3.14), and $|y - z| \lesssim \mathcal{N} \lesssim t^{-1}$ that

$$\left| \frac{z}{|z|} g(|z|) - \frac{y}{|y|} g(|y|) \right| \lesssim |z - y| \max(g(|z|), g(y)|) \lesssim t^{-2}.$$

Let $1 < \theta < \min(p - 1, 2)$. From (2.55), we obtain

$$\dot{y} = \frac{1}{\alpha} \frac{z}{|z|} g(|z|) + O(\mathcal{N}^2 + e^{-\theta|z|}) = \frac{1}{\alpha} \frac{y}{|y|} g(|y|) + O(t^{-\theta}).$$

Set $y = r\omega$, where $r = |y|$ and $\omega \in \mathcal{S}_{\mathbb{R}^N}(1)$. Then, using also (1.10), (2.9) and (3.18), we have

$$\begin{aligned} \dot{r} &= \frac{g(r)}{\alpha} + O(t^{-\theta}) = r^{-\frac{N-1}{2}} e^{-r+c_0} + O(t^{-\theta}) + O(r^{-\frac{N+1}{2}} e^{-r}) \\ &= r^{-\frac{N-1}{2}} e^{-r+c_0} \left(1 + O\left(\frac{1}{\log t}\right)\right), \end{aligned}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^{\frac{N-1}{2}} e^{r-c_0}\right) = 1 + O\left(\frac{1}{\log t}\right).$$

By integration,

$$r(t) - c_0 + \frac{N-1}{2} \log r(t) = \log t + O\left(\frac{1}{\log t}\right). \tag{3.19}$$

From (3.18) and (3.19), we compute

$$\begin{aligned} s(t) &= -\frac{N-1}{2} \log\left(\frac{r(t)}{\log t}\right) + O\left(\frac{1}{\log t}\right) \\ &= \left(\frac{N-1}{2}\right)^2 \frac{\log\log t}{\log t} + O\left(\frac{1}{\log t}\right). \end{aligned}$$

which implies the following improvement of (3.18)

$$r(t) = \log t - \frac{1}{2}(N-1)\log\log t + c_0 + \left(\frac{N-1}{2}\right)^2 \frac{\log\log t}{\log t} + O\left(\frac{1}{\log t}\right).$$

Concerning ω, by the above expressions of $\dot{y}$ and $\dot{r}$, we have

$$r\dot{\omega} = \dot{y} - \dot{r}\omega = O(t^{-\theta}).$$

Thus, there exists $\omega^\infty \in \mathcal{S}_{\mathbb{R}^N}(1)$ such that $\omega = \omega^\infty + O(t^{1-\theta})$. Hence, we obtain

$$y = \omega^\infty \left(\log t - \frac{1}{2}(N-1)\log\log t + c_0 + \left(\frac{N-1}{2}\right)^2 \frac{\log\log t}{\log t}\right) + O\left(\frac{1}{\log t}\right).$$

Now, observe that setting $y_k = z_k + \frac{\ell_k}{2\alpha}$, by (2.20), (2.22) and (3.14), we find $|\dot y_1 + \dot y_2| \lesssim t^{-\theta}$. Hence, there exists z_∞ such that $y_1 + y_2 = 2z_\infty + O(t^{1-\theta})$. As $y_i = \frac{1}{2}((y_1+y_2) + (-1)^i y)$ and $|y_k - z_k| \lesssim t^{-1}$, we infer (1.8).

4. Proof of Theorem 1.5

4.1. **Preliminary result.** We use the notation from the beginning of Section 2. We also use the constant

$$\beta := \frac{1}{2\sqrt{\alpha^2+\nu_0^2}} = \langle \vec Y^+, \vec Z^+\rangle^{-1} > 0. \tag{4.1}$$

Lemma 4.1. *For any $(z_1, z_2, \ell_1, \ell_2) \in \mathbb{R}^{4N}$ with $|z|$ large enough, there exist linear maps*

$$B : \mathbb{R}^2 \to \mathbb{R}^2, \quad V_j : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{for } j = 1, \dots, N,$$

smooth in $(z_1, z_2, \ell_1, \ell_2)$, satisfying

$$\|B - \beta \,\mathrm{Id}\,\| \lesssim e^{-\frac{1}{2}|z|}, \quad \|V_j\| \lesssim e^{-\frac{1}{2}|z|},$$

and such that the function $W(a_1, a_2) : \mathbb{R}^N \to \mathbb{R}$ defined by

$$W(a_1,a_2) := \sum_{k=1,2} \left\{ B_k(a_1,a_2) Y_k + \sum_{j=1}^N V_{k,j}(a_1,a_2)\partial_{x_j} Q_k \right\},$$

satisfies, for all $k = 1, 2$, $j = 1, \dots, N$,

$$\langle W(a_1,a_2), \partial_{x_j} Q_k\rangle = 0, \quad \langle W(a_1,a_2), Y_k\rangle = \beta a_k.$$

In particular, setting

$$\vec W(a_1,a_2) = \begin{pmatrix} W(a_1,a_2) \\ \nu^+ W(a_1,a_2)\end{pmatrix} \quad \text{it holds} \quad \langle \vec W(a_1,a_2), \vec Z_k^+\rangle = a_k.$$

Proof. Define

$$W(a_1,a_2) = \sum_{k=1,2}\left\{ b_k Y_k + \sum_{j=1}^N v_{k,j}\partial_{x_j} Q_k\right\},$$

our goal is to solve for b_k, $v_{k,j}$ in function of a_1, a_2. Using the relations $\langle Y_1, \partial_{x_j} Q_1\rangle = 0$, $\langle \partial_{x_{j'}} Q_1, \partial_{x_j} Q_1\rangle = 0$ for $j \neq j'$, and the estimate $\langle \partial_{x_{j'}} Q_2, \partial_{x_j} Q_1\rangle = O(e^{-\frac{1}{2}|z|})$ for any j, j' (see (2.3)), we observe that the condition $\langle W(a_1,a_2), \partial_{x_j} Q_1\rangle = 0$ is equivalent to a linear relation between the coefficients in the definition of $W(a_1,a_2)$ of the form

$$\|\partial_{x_j} Q\|_{L^2}^2 v_{1,j} = O(e^{-\frac{1}{2}|z|}) b_2 + \sum_{j'=1}^N O(e^{-\frac{1}{2}|z|}) v_{2,j'}.$$

Similarly, the condition $\langle W(a_1,a_2), \partial_{x_j} Q_2\rangle = 0$ is equivalent to

$$\|\partial_{x_j} Q\|_{L^2}^2 v_{2,j} = O(e^{-\frac{1}{2}|z|}) b_1 + \sum_{j'=1}^N O(e^{-\frac{1}{2}|z|}) v_{1,j'}.$$

Moreover, since $\langle Y, Y\rangle = 1$ (see Lemma 1.8) and $\langle Y_1, Y_2\rangle = O(e^{-\frac{1}{2}|z|})$, the conditions $\langle W(a_1,a_2), Y_k\rangle = \beta a_k$ write

$$b_1 = \beta a_1 + O\left\{\left(|b_2| + \sum_{j'=1}^N |v_{2,j'}|\right) e^{-\frac{1}{2}|z|}\right\},$$

$$b_2 = \beta a_2 + O\left\{\left(|b_1| + \sum_{j'=1}^N |v_{1,j'}|\right) e^{-\frac{1}{2}|z|}\right\}.$$

The existence and desired properties of b_k and $v_{k,j}$ for $k = 1, 2$ and $j = 1, \dots, N$ thus follow from inverting a linear system for $|z|$ large enough. □

4.2. **Construction of a family of 2-solitary waves.**

Proposition 4.2. *Assume $\sigma = -1$. For $\delta > 0$ small enough, for any*

$$\begin{cases} (z_1(0), z_2(0)) \in \mathbb{R}^{2N} \text{ with } |z_1(0) - z_2(0)| > 5|\log\delta|, \\ (\ell_1(0), \ell_2(0)) \in \mathcal{B}_{\mathbb{R}^{2N}}(\delta), \\ \vec{\varepsilon}_\perp(0) \in \mathcal{B}_{H^1\times L^2}(\delta) \text{ with (2.17) and } \langle \vec{\varepsilon}_\perp(0), \vec{Z}_k^+(0)\rangle = 0 \text{ for } k = 1, 2, \end{cases} \tag{4.2}$$

there exists $(a_1^+(0), a_2^+(0)) \in \bar{\mathcal{B}}_{\mathbb{R}^2}(\delta^{\frac{5}{4}})$ such that the solution $\vec{u}$ of (1.1) with the initial data

$$\vec{u}(0) = \vec{Q}_1(0) + \vec{Q}_2(0) + \vec{W}(a_1^+(0), a_2^+(0)) + \vec{\varepsilon}_\perp(0)$$

is a 2-solitary wave.

Proof. Decomposition. For any $t \geq 0$ such that the solution $\vec{u}(t)$ satisfies (2.15), we decompose it according to Lemma 2.2. Note that by the properties of the function W in Lemma 4.1 and the orthogonality properties (2.17) of $\vec{\varepsilon}_\perp(0)$ assumed in (4.2), the initial data $\vec{u}(0)$ is modulated, in the sense that $(z_1(0), z_2(0), \ell_1(0), \ell_2(0))$ and

$$\vec{\varepsilon}(0) = \vec{W}(a_1^+(0), a_2^+(0)) + \vec{\varepsilon}_\perp(0),$$

are the parameters of the decomposition of $\vec{u}(0)$. In particular, it holds from (4.2)

$$\mathcal{N}(0) \lesssim \delta, \quad q(r(0)) \lesssim \delta^2. \tag{4.3}$$

Moreover, by Lemma 4.1, for $k = 1, 2$,

$$\langle \vec{\varepsilon}(0), \vec{Z}_k^+(0)\rangle = \langle \vec{W}(a_1^+(0), a_2^+(0)), Z_k^+\rangle = a_k^+(0),$$

which is consistent with the definition of a_k^+ in (v) of Lemma 2.2.

Bootstrap estimates. We introduce the following bootstrap estimates

$$\mathcal{N} \leq \delta^{\frac{3}{4}}, \quad q(r) \leq \delta^{\frac{3}{2}}, \quad b \leq \delta^{\frac{5}{2}}. \tag{4.4}$$

We set

$$T_* = \sup\{t \in [0, \infty) \text{ such that (4.4) holds on } [0, t]\} \geq 0.$$

Estimates on the damped components. The estimate on $\mathcal{N}$ is strictly improved on $[0, T_*]$ as in the proof of Proposition 3.1. In particular, $\mathcal{N} \lesssim \delta$ on $[0, T_*]$.

Estimate on the distance. Note that $r(t) \gtrsim \log\delta$ and thus, from (2.54),

$$\frac{\mathrm{d}R^-}{\mathrm{d}t} \geq \frac{g_0}{\alpha}\left(1 - \frac{C}{|\log\delta|}\right)(1 - C\delta^2) \geq \frac{g_0}{2\alpha}.$$

Thus, by integration on $[0, t]$, for any $t \in [0, T_*]$, it holds $R^-(t) \geq R^-(0) + \frac{g_0 t}{2\alpha}$. Using (3.11) and next (4.3), we obtain

$$\frac{1}{q(r(t))} \geq \left(\frac{1 - C\delta^2}{q(r(0))} + \frac{g_0 t}{2\alpha}\right)(1 - C\delta^2) \gtrsim \delta^{-2}.$$

This strictly improves the estimate of $q(r)$ in (3.4).

Transversality condition. From (2.50) and $\mathcal{N} \lesssim \delta$, we observe that for any time $t \in [0, T_*]$ where it holds $b(t) = \delta^{\frac{5}{2}}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} b(t) \geq 2\nu^+ b(t) - C\delta^3 \geq 2\nu^+\delta^{\frac{5}{2}} - C\delta^3 \geq \nu^+\delta^{\frac{5}{2}} > 0,$$

for $\delta > 0$ small enough. This transversality condition is enough to justify the existence of at least a couple $(a_1^+(0), a_2^+(0)) \in \bar{\mathcal{B}}_{\mathbb{R}^2}(\delta^{\frac{5}{4}})$ such that $T_* = \infty$.

Indeed, for the sake of contradiction assume that for all $(a_1^+(0), a_2^+(0)) \in \bar{\mathcal{B}}_{\mathbb{R}^2}(\delta^{\frac{5}{4}})$, it holds $T_* < \infty$. Then, a contradiction follows from the following observations (see for instance more details in [6] or in [8, Section 3.1]).

Continuity of T_*: the above transversality condition proves that the map

$$(a_1^+(0), a_2^+(0)) \in \bar{\mathcal{B}}_{\mathbb{R}^2}(\delta^{\frac{5}{4}}) \mapsto T_* \in [0, +\infty)$$

is continuous and that $T_* = 0$ when $(a_1^+(0), a_2^+(0)) \in \mathcal{S}_{\mathbb{R}^2}(\delta^{\frac{5}{4}})$.

Construction of a retraction: As a consequence, the map

$$(a_1^+(0), a_2^+(0)) \in \bar{\mathcal{B}}_{\mathbb{R}^2}(\delta^{\frac{5}{4}}) \mapsto (a_1^+(T_*), a_2^+(T_*)) \in \mathcal{S}_{\mathbb{R}^2}(\delta^{\frac{5}{4}})$$

is continuous and its restriction to the sphere $\mathcal{S}_{\mathbb{R}^2}(\delta^{\frac{5}{4}})$ is the identity.

This is a contradiction with the no retraction theorem for continuous maps from the ball to the sphere. □

Remark 4.3. The use of initial data similar to the ones in Proposition 4.2 allows to correct a flaw in the articles [6, 8, 23] dealing with the construction of multi-solitons in several contexts. For example, in [8], the initial data U_0 chosen page 18 is not exactly modulated, and this is why Lemma 6 provides the modulation keeping track of the free parameters necessary for the topological argument. However, this modulation involves a translation parameter denoted by $\tilde{\mathbf{y}}$ in [8] (similar to (z_1, z_2) in the present paper) on which the topological argument finally depends. To close the choice of the free parameters for initial data as chosen in [8], an extra argument would be needed (like a fixed point result). It is simpler though to define an initial data already modulated as in Proposition 4.2, or in [16, Lemma 3.1].

4.3. **Lipschitz estimate and uniqueness.** The heart of Theorem 1.5 is the following proposition.

Proposition 4.4. *For $\delta > 0$ small enough, let $(z_1(0), z_2(0), \ell_1(0), \ell_2(0), \vec{\varepsilon}_\perp(0))$ and $(\tilde{z}_1(0), \tilde{z}_2(0), \tilde{\ell}_1(0), \tilde{\ell}_2(0), \vec{\tilde{\varepsilon}}_\perp(0))$ be as in* (4.2)*, and let $(a_1^+(0), a_2^+(0)) \in \mathcal{B}_{\mathbb{R}^2}(\delta)$, $(\tilde{a}_1^+(0), \tilde{a}_2^+(0)) \in \mathcal{B}_{\mathbb{R}^2}(\delta)$ be such that the solutions $\vec{u}(t)$ and $\vec{\tilde{u}}(t)$ of* (1.1) *with initial data*

$$\vec{u}(0) = \vec{Q}_1(0) + \vec{Q}_2(0) + \vec{W}(a_1^+(0), a_2^+(0)) + \vec{\varepsilon}_\perp(0),$$
$$\vec{\tilde{u}}(0) = \vec{\tilde{Q}}_1(0) + \vec{\tilde{Q}}_2(0) + \vec{W}(\tilde{a}_1^+(0), \tilde{a}_2^+(0)) + \vec{\tilde{\varepsilon}}_\perp(0),$$

are 2-solitary waves. Then it holds

$$\sum_{k=1,2} |a_k^+(0) - \tilde{a}_k^+(0)| \lesssim \delta^{\frac{1}{4}} \left(\|\vec{\varepsilon}_\perp(0) - \vec{\tilde{\varepsilon}}_\perp(0)\|_{H^1 \times L^2} + \sum_{k=1,2} \Big\{ |z_k(0) - \tilde{z}_k(0)| + |\ell_k(0) - \tilde{\ell}_k(0)| \Big\} \right).$$

Proof. We will split the proof in several steps. For $\delta > 0$ small enough, from Proposition 3.1 and the assumption on their initial data $\vec{u}(0)$, $\vec{\tilde{u}}(0)$, the 2-solitary waves $\vec{u}$ and $\vec{\tilde{u}}$ decompose for any $t \geq 0$ as in Lemma 2.2 $\vec{u} = \vec{Q}_1 + \vec{Q}_2 + \vec{\varepsilon}$, $\vec{\tilde{u}} = \vec{\tilde{Q}}_1 + \vec{\tilde{Q}}_2 + \vec{\tilde{\varepsilon}}$ and the parameters of their decompositions $(z_k, \ell_k, \mathcal{N})_{k=1,2}$ and $(\tilde{z}_k, \tilde{\ell}_k, \tilde{\mathcal{N}})_{k=1,2}$ respectively satisfy (3.3) for all $t \geq 0$. Denote

$$\check{z}_k = z_k - \tilde{z}_k, \quad \check{\ell}_k = \ell_k - \tilde{\ell}_k, \quad \check{Q}_k = Q_k - \tilde{Q}_k, \quad \check{\varepsilon} = \varepsilon - \tilde{\varepsilon}, \quad \check{\eta} = \eta - \tilde{\eta}, \tag{4.5}$$

$$\check{\mathcal{N}} = \left[\|\vec{\check{\varepsilon}}\|^2_{H^1 \times L^2} + \sum_{k=1,2} |\check{z}_k|^2 + \sum_{k=1,2} |\check{\ell}_k|^2 \right]^{\frac{1}{2}}. \tag{4.6}$$

(Notice the extra term in $\check{\mathcal{N}}$).

If $\sum_{k=1,2}|\check{z}_k(0)| > \delta^{\frac{3}{4}}$, since $(a_1^+(0), a_2^+(0))$, $(\tilde{a}_1^+(0), \tilde{a}_2^+(0)) \in \bar{\mathcal{B}}_{\mathbb{R}^2}(\delta)$, the desired estimate is satisfied. Thus, we assume now $\sum_{k=1,2}|\check{z}_k(0)| \le \delta^{\frac{3}{4}}$, and we work only for time t such that $\sum_{k=1,2}|\check{z}_k(t)|$ is small (see (4.19)).

Equation of $\vec{\check{\varepsilon}}$. By direct computation from (2.19) for $\vec{\varepsilon}$ and $\vec{\tilde{\varepsilon}}$, we have

$$\begin{cases} \partial_t \check{\varepsilon} = \check{\eta} + \mathrm{Mod}_{\check{\varepsilon},1} + \mathrm{Mod}_{\check{\varepsilon},2} \\ \partial_t \check{\eta} = \Delta\check{\varepsilon} - \check{\varepsilon} + f'(R)\check{\varepsilon} - 2\alpha\check{\eta} + \mathrm{Mod}_{\check{\eta},1} + \mathrm{Mod}_{\check{\eta},2} + \check{G} \end{cases} \tag{4.7}$$

where

$$\mathrm{Mod}_{\check{\varepsilon},1} = \sum_{k=1,2}\left(\dot{\check{z}}_k - \check{\ell}_k\right)\cdot\nabla Q_k, \quad \mathrm{Mod}_{\check{\varepsilon},2} = \sum_{k=1,2}\left(\dot{\tilde{z}}_k - \tilde{\ell}_k\right)\cdot\nabla \check{Q}_k,$$

$$\begin{aligned}\mathrm{Mod}_{\check{\eta},1} &= \sum_{k=1,2}\left(\dot{\check{\ell}}_k + 2\alpha\check{\ell}_k\right)\cdot\nabla Q_k \\ &\quad - \sum_{k=1,2}(\check{\ell}_k\cdot\nabla)(\dot{z}_k\cdot\nabla)Q_k - \sum_{k=1,2}(\tilde{\ell}_k\cdot\nabla)(\dot{\check{z}}_k\cdot\nabla)Q_k, \\ \mathrm{Mod}_{\check{\eta},2} &= \sum_{k=1,2}\left(\dot{\tilde{\ell}}_k + 2\alpha\tilde{\ell}_k\right)\cdot\nabla\check{Q}_k - \sum_{k=1,2}(\tilde{\ell}_k\cdot\nabla)(\dot{\tilde{z}}_k\cdot\nabla)\check{Q}_k\end{aligned}$$

and

$$\check{G} = \left[f(R+\varepsilon) - f(R) - f'(R)\varepsilon\right] - \left[f(\tilde{R}+\tilde{\varepsilon}) - f(\tilde{R}) - f'(R)\tilde{\varepsilon}\right] + G - \tilde{G}.$$

Step 1. We claim the following estimate on the nonlinear term $\check{G}$.

Lemma 4.5.

$$\|\check{G}\|_{L^2} \lesssim \check{\mathcal{N}}(\mathcal{N} + \tilde{\mathcal{N}} + e^{-\frac{1}{2}|z|}). \tag{4.8}$$

Proof. We will in fact derive pointwise bounds. We decompose $\check{G} = \sum_{j=1}^{5}\check{G}_j$ where

$$\begin{aligned}\check{G}_1 &= [f'(R) - f'(\tilde{R})]\tilde{\varepsilon}, \\ \check{G}_2 &= [f(R+\varepsilon) - f(R) - f'(R)\varepsilon] - [f(R+\tilde{\varepsilon}) - f(R) - f'(R)\tilde{\varepsilon}], \\ \check{G}_3 &= [f(R+\tilde{\varepsilon}) - f(R) - f'(R)\tilde{\varepsilon}] - [f(\tilde{R}+\tilde{\varepsilon}) - f(\tilde{R}) - f'(\tilde{R})\tilde{\varepsilon}], \\ \check{G}_4 &= [f(Q_1+Q_2) - f(Q_2)] - [f(Q_1+\tilde{Q}_2) - f(\tilde{Q}_2)], \\ \check{G}_5 &= [f(Q_1+\tilde{Q}_2) - f(Q_1)] - [f(\tilde{Q}_1+\tilde{Q}_2) - f(\tilde{Q}_1)].\end{aligned}$$

For $\check{G}_1$, using $|f'(R) - f'(\tilde{R})| \lesssim \left(|Q_1|^{p-1} + |Q_2|^{p-1}\right)\sum_{k=1,2}|\check{z}_k|$, we have

$$|\check{G}_1| \lesssim |\tilde{\varepsilon}|\left(|Q_1|^{p-1} + |Q_2|^{p-1}\right)\sum_{k=1,2}|\check{z}_k|.$$

For $\check{G}_2$, by Taylor formular, we have

$$\begin{aligned}\check{G}_2 &= f(R+\tilde{\varepsilon}+\check{\varepsilon}) - f(R+\tilde{\varepsilon}) - f'(R)\check{\varepsilon} \\ &= \check{\varepsilon}\int_0^1 f'(R+\tilde{\varepsilon}+\theta\check{\varepsilon}) - f'(R)\,\mathrm{d}\theta = \check{\varepsilon}\int_0^1(\tilde{\varepsilon}+\theta\check{\varepsilon})\int_0^1 f''(R+\rho(\tilde{\varepsilon}+\theta\check{\varepsilon}))\,\mathrm{d}\rho\,\mathrm{d}\theta \\ &= \check{\varepsilon}\int_0^1\int_0^1(\theta\varepsilon + (1-\theta)\tilde{\varepsilon})f''(R+\rho(\tilde{\varepsilon}+\theta\check{\varepsilon}))\,\mathrm{d}\rho\,\mathrm{d}\theta.\end{aligned}$$

As $|f''(x+y)| = p(p-1)|x+y|^{p-2} \lesssim |x|^{p-2} + |y|^{p-2}$, we get

$$|\check{G}_2| \lesssim |\check{\varepsilon}|(|\varepsilon| + |\tilde{\varepsilon}|)\left(|R|^{p-2} + |\tilde{\varepsilon}|^{p-2} + |\varepsilon|^{p-2}\right).$$

For $\check G_3$: by Taylor formula,

$$\begin{aligned}\big|[f(R+\tilde\varepsilon)-f(R)]-[f(\tilde R+\tilde\varepsilon)-f(\tilde R)]\big| &= \Big|\tilde\varepsilon\int_0^1 [f'(R+\theta\tilde\varepsilon)-f'(\tilde R+\theta\tilde\varepsilon)]\,\mathrm{d}\theta\Big|\\ &\lesssim (|\check z_1|+|\check z_2|)|\tilde\varepsilon|\big(|R|^{p-2}+|\tilde R|^{p-2}+|\tilde\varepsilon|^{p-2}\big),\end{aligned}$$

and

$$\big|[f'(R)-f'(\tilde R)]\check\varepsilon\big| \lesssim (|\check z_1|+|\check z_2|)|\check\varepsilon|\big(|R|^{p-2}+|\tilde R|^{p-2}\big).$$

It follows that

$$|\check G_3| \lesssim (|\check z_1|+|\check z_2|)|\tilde\varepsilon|\big(|R|^{p-2}+|\tilde R|^{p-2}+|\tilde\varepsilon|^{p-2}\big).$$

For $\check G_4$: Taylor formula gives similarly

$$\begin{aligned}|\check G_4| &= \left|Q_1\int_0^1 \left[f'(Q_2+\theta Q_1)-f'(\tilde Q_2+\theta Q_1)\right]\mathrm{d}\theta\right|\\ &\lesssim |Q_1||Q_2-\tilde Q_2|\left(|Q_1|^{p-2}+|Q_2|^{p-2}\right) \lesssim |\check z_2|\left(|Q_1|^{p-1}|Q_2|+|Q_1||Q_2|^{p-1}\right).\end{aligned}$$

For $\check G_5$, a similar argument yields

$$|\check G_5| \lesssim |\check z_1|(|Q_1|^{p-1}|Q_2|+|Q_1||Q_2|^{p-1}).$$

Estimate (4.8) then follows from (2.6) and Sobolev embeddings. □

Step 2. Equations for the parameters.

Lemma 4.6.

(i) Equations of the geometric parameters. *For $k=1,2$,*

$$|\dot{\check z}_k-\check\ell_k| \lesssim \check{\mathcal N}(\mathcal N+\tilde{\mathcal N}), \tag{4.9}$$

$$|\dot{\check\ell}_k+2\alpha\check\ell_k| \lesssim \check{\mathcal N}(\mathcal N+\tilde{\mathcal N}+e^{-\frac12|z|}). \tag{4.10}$$

(ii) Equations of the exponential directions. *Let $\check a_k^\pm=\langle\vec{\check\varepsilon},\vec Z_k^\pm\rangle$. Then,*

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\check a_k^\pm-\nu^\pm\check a_k^\pm\right| \lesssim \check{\mathcal N}(\mathcal N+\tilde{\mathcal N}+e^{-\frac12|z|}). \tag{4.11}$$

Proof. (i) Using (2.32) with $j=1$ for u and $\tilde u$ and taking the difference, we have

$$\begin{aligned}0 &= (\dot{\check z}_{1,j}-\check\ell_{1,j})\|\partial_{x_j}Q\|_{L^2}^2+\langle(\dot{\check z}_2-\check\ell_2)\cdot\nabla Q_2,\partial_{x_j}Q_1\rangle\\ &\quad+\langle(\dot{\tilde z}_2-\tilde\ell_2)\cdot\nabla\check Q_2,\partial_{x_j}Q_1\rangle+\langle(\dot{\tilde z}_2-\tilde\ell_2)\cdot\nabla\tilde Q_2,\partial_{x_j}\check Q_1\rangle\\ &\quad-\langle\check\varepsilon,(\dot z_1\cdot\nabla)\,\partial_{x_j}Q_1\rangle-\langle\tilde\varepsilon,(\dot{\check z}_1\cdot\nabla)\partial_{x_j}Q_1\rangle-\langle\tilde\varepsilon,(\dot{\tilde z}_1\cdot\nabla)\partial_{x_j}\check Q_1\rangle.\end{aligned}$$

Using (2.3), (2.20) and the estimates

$$|\langle\partial_{x_{j'}}\check Q_1,\partial_{x_j}Q_1\rangle|+|\langle\partial_{x_{j'}}\check Q_2,\partial_{x_j}Q_1\rangle|+|\langle\partial_{x_{j'}}Q_2,\partial_{x_j}\check Q_1\rangle| \lesssim |\check z_1|+|\check z_2|,$$

it follows that

$$\begin{aligned}|\dot{\check z}_{1,j}-\check\ell_{1,j}| &\lesssim e^{-\frac12|z|}|\dot{\check z}_2-\check\ell_2|+(|\check z_1|+|\check z_2|)\tilde{\mathcal N}^2+(|\ell_1|+\mathcal N^2)\check{\mathcal N}\\ &\quad+\tilde{\mathcal N}(|\check\ell_1|+|\dot{\check z}_1-\check\ell_1|)+\tilde{\mathcal N}|\check z_1|(\mathcal N^2+|\ell_1|).\end{aligned}$$

By symmetry, an analogous estimate holds for $|\dot{\check z}_{2,j}-\check\ell_{2,j}|$. Summing up both estimate gives (4.9). Using (2.33) for u and $\tilde u$ and taking the difference, we obtain (4.10) similarly.

(ii) The proof of (4.11) is similar. See also the proof of (v) in Lemma 2.2. □

Step 3. Energy estimates. For $\mu>0$ small to be chosen, denote $\rho=2\alpha-\mu$, and consider the following energy functional

$$\check{\mathcal E}=\int\big\{|\nabla\check\varepsilon|^2+(1-\rho\mu)\check\varepsilon^2+(\check\eta+\mu\check\varepsilon)^2-f'(R)\check\varepsilon^2\big\}. \tag{4.12}$$

Lemma 4.7. *There exists $\mu > 0$ such that the following hold.*

(i) Coercivity and bound.

$$2\mu\|\vec{\check{\varepsilon}}\|^2_{H^1\times L^2} - \frac{1}{2\mu}\left[\tilde{\mathcal{N}}^2\check{\mathcal{N}}^2 + \sum_{k=1,2}\left((\check{a}_k^+)^2 + (\check{a}_k^-)^2\right)\right] \le \check{\mathcal{E}} \le \frac{1}{\mu}\|\vec{\check{\varepsilon}}\|^2_{H^1\times L^2}. \quad (4.13)$$

(ii) Time variation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\check{\mathcal{E}} \le -2\mu\check{\mathcal{E}} + \frac{1}{\mu}\check{\mathcal{N}}^2(\mathcal{N} + \tilde{\mathcal{N}} + e^{-\frac{1}{2}|z|}).$$

Proof. (i) The upper bound in (4.13) is clear. Next, adapting the proof of (i) of Lemma 2.5, one checks

$$\check{\mathcal{E}} \ge 2\mu\|\vec{\check{\varepsilon}}\|^2_{H^1\times L^2} - \frac{1}{2\mu}\left[\sum_{k,j}|\langle\check{\varepsilon}, \partial_{x_j}Q_k\rangle|^2 + \sum_{k=1,2}\left((\check{a}_k^+)^2 + (\check{a}_k^-)^2\right)\right].$$

By (2.17) for u and $\tilde{u}$, we have $|\langle\check{\varepsilon}, \partial_{x_j}Q_k\rangle| = |\langle\tilde{\varepsilon}, \partial_{x_j}\check{Q}_k\rangle| \lesssim \tilde{\mathcal{N}}\check{\mathcal{N}}$ and this implies the lower bound in (4.13).
(ii) We follow the computation of the proof of (ii) of Lemma 2.5. First,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\check{\mathcal{E}} &= \int \partial_t\check{\varepsilon}\left[-\Delta\check{\varepsilon} + (1-\rho\mu)\check{\varepsilon} - f'(R)\check{\varepsilon}\right] + (\partial_t\check{\eta} + \mu\partial_t\check{\varepsilon})\,(\check{\eta} + \mu\check{\varepsilon}) \\ &\quad + \int\sum_{k=1,2}(\dot{z}_k\cdot\nabla Q_k)f''(R)\check{\varepsilon}^2 = \mathbf{g_1} + \mathbf{g_2}.\end{aligned}$$

Second, using (4.7) and integration by parts,

$$\begin{aligned}\mathbf{g_1} &= -\mu\int\left[|\nabla\check{\varepsilon}|^2 + (1-\rho\mu)\check{\varepsilon}^2 - f'(R)\check{\varepsilon}^2\right] - \rho\int(\check{\eta} + \mu\check{\varepsilon})^2 \\ &\quad + \sum_{k=1,2}\int\check{\varepsilon}\left[-\Delta\mathrm{Mod}_{\check{\varepsilon},k} + (1-\rho\mu)\mathrm{Mod}_{\check{\varepsilon},k} - f'(R)\mathrm{Mod}_{\check{\varepsilon},k}\right] \\ &\quad + \sum_{k=1,2}\int(\check{\eta} + \mu\check{\varepsilon})\left[\mathrm{Mod}_{\check{\eta},k} + \mu\mathrm{Mod}_{\check{\varepsilon},k}\right] + \int(\check{\eta} + \mu\check{\varepsilon})\check{G}.\end{aligned}$$

The first line in the expression of $\mathbf{g_1}$ above is less than $-\mu\check{\mathcal{E}}$ (taking $\mu \le \rho$). Next, using (2.20), (2.21), (4.8), (4.9) and (4.10), we check that the remaining terms in $\mathbf{g_1}$ and $\mathbf{g_2}$ are estimated by $\check{\mathcal{N}}^2(\mathcal{N} + \tilde{\mathcal{N}} + e^{-\frac{1}{2}|z|})$. □

Define

$$\check{y}_k = \check{z}_k + \frac{\check{\ell}_k}{2\alpha}, \quad \check{\mathcal{F}} = \check{\mathcal{E}} + \sum_{k=1,2}|\check{\ell}_k|^2 + \frac{1}{2\mu}\sum_{k=1,2}(\check{a}_k^-)^2, \quad \check{b} = \sum_{k=1,2}(\check{a}_k^+)^2. \quad (4.14)$$

Lemma 4.8. *For $\mu > 0$ defined in Lemma 4.7, it holds.*

(i) Comparison.

$$\mu\check{\mathcal{N}}^2 \le \mu\|\vec{\check{\varepsilon}}\|_{H^1\times L^2} + \sum_{k=1,2}|\check{\ell}_k|^2 + \sum_{k=1,2}|\check{z}_k|^2 \le \check{\mathcal{F}} + \frac{\check{b}}{2\mu} + \sum_{k=1,2}|\check{z}_k|^2 \lesssim \check{\mathcal{N}}^2. \quad (4.15)$$

(ii) Positions.

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\check{y}_k\right| \lesssim \check{\mathcal{N}}(\mathcal{N} + \tilde{\mathcal{N}} + e^{-\frac{1}{2}|z|}) \lesssim \delta\check{\mathcal{N}}. \quad (4.16)$$

(iii) Damped components.

$$\frac{\mathrm{d}}{\mathrm{d}t}\check{\mathcal{F}} + 2\mu\check{\mathcal{F}} \lesssim \check{\mathcal{N}}^2(\mathcal{N} + \tilde{\mathcal{N}} + e^{-\frac{1}{2}|z|}) \lesssim \delta\check{\mathcal{N}}^2. \quad (4.17)$$

Proof. (i) follows from (4.13) (see also the proof of (2.48) in Lemma 2.7, (i)).
(ii) Estimate (4.16) follows from (4.9) and (4.10).
(iii) is a consequence of Lemmas 4.6 and 4.7 (see also Lemma 2.7), (iv)). □

Step 4. For the sake of contradiction, assume

$$\delta^{\frac{1}{2}}\check{\mathcal{N}}^2(0) < \check{b}(0). \tag{4.18}$$

We introduce the following bootstrap estimate

$$\delta^{\frac{3}{4}}\check{\mathcal{N}}^2 < \check{b} \tag{4.19}$$

and we define

$$T_* = \sup\{t > 0 \text{ such that (4.19) holds on } [0,t]\} > 0.$$

By (4.11) and then (4.19), there holds on $[0, T_*]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\check{b} \geq 2\nu^+\check{b} - C\delta\check{\mathcal{N}}\sqrt{\check{b}} \geq \nu^+\check{b}. \tag{4.20}$$

In particular, $\check{b}$ is positive, increasing on $[0, T_*)$ and

$$\forall t \in [0, T_*), \quad b(t) \geq e^{\nu^+ t}\check{b}(0). \tag{4.21}$$

By (4.17) and next (4.19),

$$\frac{\mathrm{d}}{\mathrm{d}t}[e^{2\mu t}\check{\mathcal{F}}] \lesssim e^{2\mu t}\delta\check{\mathcal{N}}^2 \lesssim e^{2\mu t}\delta^{\frac{1}{4}}\check{b}.$$

Integrating on $[0, t]$, for any $t \in [0, T_*)$ and using that $\check{b}$ is increasing, we obtain

$$\check{\mathcal{F}}(t) \lesssim e^{-2\mu t}\check{\mathcal{F}}(0) + \delta^{\frac{1}{4}}\check{b}(t),$$

and thus using (4.18),

$$\delta^{\frac{3}{4}}\check{\mathcal{F}}(t) \lesssim \delta^{\frac{3}{4}}\check{\mathcal{F}}(0) + \delta\check{b}(t) \lesssim \delta^{\frac{1}{4}}(\check{b}(0) + \check{b}(t)) \lesssim \delta^{\frac{1}{4}}\check{b}(t).$$

Using similar argument, we check that

$$\delta^{\frac{3}{4}}\left|\ell_k(t)\right|^2 \lesssim \delta^{\frac{1}{4}}\check{b}(t) \quad \text{for } k = 1, 2. \tag{4.22}$$

From (4.16), next (4.19) and then (4.20),

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}|\check{y}_k|^2\right| \lesssim \delta\check{\mathcal{N}}^2 \lesssim \delta^{\frac{1}{4}}\check{b} \lesssim \delta^{\frac{1}{4}}\frac{\mathrm{d}}{\mathrm{d}t}\check{b}.$$

Integrating on $[0, t]$, for any $t \in [0, T_*)$,

$$|\check{y}_k(t)|^2 \lesssim |\check{y}_k(0)|^2 + \delta^{\frac{1}{4}}\check{b}(t).$$

Thus, using (4.18) and (4.22),

$$\delta^{\frac{3}{4}}|\check{z}_k(t)|^2 \lesssim \delta^{\frac{3}{4}}|\check{y}_k(t)|^2 + \delta^{\frac{3}{4}}|\check{\ell}_k(t)|^2 \lesssim \delta^{\frac{1}{4}}\check{b}(t).$$

Using (4.15), we obtain

$$\delta^{3/4}\check{\mathcal{N}}^2 \lesssim \delta^{1/4}\check{b}$$

on $[0, T_*)$, and so we have strictly improved estimate (4.19) for δ small enough. By a continuity argument it follows that $T_* = +\infty$. This is contradictory since the exponential growth (4.21) would lead to unbounded $\check{b}$.
In conclusion, we have just proved that $\check{b}(0) \leq \delta^{\frac{1}{2}}\check{\mathcal{N}}^2(0)$. Observing

$$|a_k^+(0) - \tilde{a}_k^+(0)| \lesssim |\check{a}_k^+(0)| + |\langle\vec{\check{\varepsilon}}(0), (\vec{Z}_k^+(0) - \vec{\tilde{Z}}_k^+(0)\rangle| \lesssim \delta^{\frac{1}{4}}\check{\mathcal{N}}(0),$$

the proof of Proposition 4.4 is complete. □

4.4. **Modulation at initial time.** We introduce some notation related to initial data as written in the statement of Theorem 1.5

$$\Omega = (L, \vec{\phi}), \quad \|\Omega\| = |L| + \|\vec{\phi}\|_{H^1\times L^2}, \quad h = (h_1, h_2),$$

and as written in Propositions 4.2 and 4.4

$$\Gamma = (z_1, z_2, \ell_1, \ell_2, \vec{\varepsilon}_\perp), \quad \|\Gamma\| = |(z_1, z_2)| + |(\ell_1, \ell_2)| + \|\vec{\varepsilon}_\perp\|_{H^1\times L^2}, \quad a^+ = (a_1^+, a_2^+).$$

For $\delta > 0$, we denote by $\mathcal{V}_\delta^\perp$ the set of $\Omega = (L, \vec{\phi})$ satisfying

$$\begin{cases} L \in \mathbb{R}^N \text{ with } |L| > 10|\log \delta|, \\ \vec{\phi} \in \mathcal{B}_{H^1\times L^2}(\delta) \text{ with } \langle \vec{\phi}, \vec{Z}^+(\cdot \pm \frac{L}{2})\rangle = 0, \end{cases}$$

and by $\mathcal{W}_\delta^\perp$ the set of $\Gamma = (z_1, z_2, \ell_1, \ell_2, \vec{\varepsilon}_\perp)$ satisfying

$$\begin{cases} (z_1, z_2) \in \mathbb{R}^{2N} \text{ with } |z_1 - z_2| > 5|\log \delta|, \\ (\ell_1, \ell_2) \in \mathcal{B}_{\mathbb{R}^{2N}}(\delta), \\ \vec{\varepsilon}_\perp \in \mathcal{B}_{H^1\times L^2}(\delta) \text{ with (2.17) and } \langle \vec{\varepsilon}_\perp, \vec{Z}_k^+\rangle = 0 \text{ for } k = 1, 2. \end{cases}$$

In the statement of Theorem 1.5, we do not require the orthogonality conditions $\langle \vec{\phi}, \vec{Z}^+(\cdot \pm \frac{L}{2})\rangle = 0$ as in the definition of the set $\mathcal{V}_\delta^\perp$. Those conditions are obtained by adjusting (h_1, h_2) in a second step, see the proof of Theorem 1.5. We start with constructing the "manifold" for data in $\mathcal{V}_\delta^\perp$, which somehow corresponds to the tangent space and where the discrepancy is superlinear in δ.

Lemma 4.9. *Assume $\sigma_1 = 1$ and $\sigma_2 = -1$. There exists $C > 0$ such that for all $\delta > 0$ small enough, and for any $(\Omega, h) \in \mathcal{V}_\delta^\perp \times \mathcal{B}_{\mathbb{R}^2}(\delta)$, there exist unique $\Gamma[\Omega, h] \in \mathcal{W}_{C\delta}^\perp$ and $a^+[\Omega, h] \in \mathcal{B}_{\mathbb{R}^2}(C\delta)$ such that*

$$(\vec{Q} + h_1\vec{Y}^+)(\cdot - \tfrac{L}{2}) - (\vec{Q} + h_2\vec{Y}^+)(\cdot + \tfrac{L}{2}) + \vec{\phi} = \vec{Q}_1 + \vec{Q}_2 + \vec{W}(a^+) + \vec{\varepsilon}_\perp, \tag{4.23}$$

$$|\beta a^+ - h| \lesssim \delta^2. \tag{4.24}$$

Moreover, for any (Ω, h), $(\tilde{\Omega}, \tilde{h}) \in \mathcal{V}_\delta^\perp \times \mathcal{B}_{\mathbb{R}^2}(\delta)$, with $|L - \tilde{L}| < \delta$, it holds

$$\|\Gamma[\Omega, h] - \Gamma[\tilde{\Omega}, \tilde{h}]\| \lesssim \|\Omega - \tilde{\Omega}\| + |h - \tilde{h}|, \tag{4.25}$$

$$|(\beta a^+[\Omega, h] - h) - (\beta a^+[\tilde{\Omega}, \tilde{h}] - \tilde{h})| \lesssim \delta(\|\Omega - \tilde{\Omega}\| + |h - \tilde{h}|). \tag{4.26}$$

Proof. First, (i) of Lemma 2.2 implies the existence of $z_1, z_2, \ell_1, \ell_2, \vec{\varepsilon}$ such that

$$(\vec{Q} + h_1\vec{Y}^+)(\cdot - \tfrac{L}{2}) - (\vec{Q} + h_2\vec{Y}^+)(\cdot + \tfrac{L}{2}) + \vec{\phi} = \vec{Q}_1 + \vec{Q}_2 + \vec{\varepsilon} \tag{4.27}$$

where

$$\|\vec{\varepsilon}\|_{H^1\times L^2} + \sum_{k=1,2} |\ell_k| + e^{-2|z_1 - z_2|} \lesssim \delta$$

and $\vec{\varepsilon}$ satisfies the orthogonality relations (2.17). Using (2.27) and projecting (4.27) on $\partial_{x_j} Q_k$ for $k = 1, 2$, we find

$$|z_1 - \tfrac{L}{2}| + |z_2 + \tfrac{L}{2}| \lesssim \delta.$$

Hence $|z| \geq L - 2\delta^{\frac{1}{2}} \geq 5|\log \delta|$ for $\delta > 0$ small enough.
Moreover, (2.29) and (2.31) provide the Lipschitz estimates

$$\sum_{k=1,2} \{|z_k - \tilde{z}_k| + |\ell_k - \tilde{\ell}_k|\} + \|\vec{\varepsilon} - \tilde{\vec{\varepsilon}}\|_{H^1\times L^2} \lesssim \|\Omega - \tilde{\Omega}\| + |h - \tilde{h}|. \tag{4.28}$$

Second, define

$$a_k^+ = \langle \vec{\varepsilon}, \vec{Z}_k^+\rangle \quad \text{and} \quad \vec{\varepsilon}_\perp = \vec{\varepsilon} - \vec{W}(a_1^+, a_2^+) = \vec{\varepsilon} - \vec{W}(a^+),$$

so that $\vec{\varepsilon}_\perp$ satisfies $\langle \vec{\varepsilon}_\perp, \vec{Z}_k^+\rangle = 0$ for $k = 1, 2$. Using the definition of $\vec{\varepsilon}_\perp$ and (4.28), we obtain

$$\begin{aligned}&\sum_{k=1,2} \{|z_k - \tilde{z}_k| + |\ell_k - \tilde{\ell}_k|\} + \|\vec{\varepsilon}_\perp - \vec{\tilde{\varepsilon}}_\perp\|_{H^1\times L^2}\\ &\lesssim \sum_{k=1,2} \{|z_k - \tilde{z}_k| + |\ell_k - \tilde{\ell}_k|\} + \|\vec{\varepsilon} - \vec{\tilde{\varepsilon}}\|_{H^1\times L^2} \lesssim \|\Omega - \tilde{\Omega}\| + |h - \tilde{h}|,\end{aligned}$$

which is (4.25). Let us now project (4.27) on $\vec{Z}_1^+$. Using the expansion

$$Q(\cdot - \tfrac{L}{2}) - Q_1 = (z_1 - \tfrac{L}{2}) \cdot \nabla Q(\cdot - \tfrac{L}{2}) + O_{L^2}(\delta^2),$$

the fact that $\langle \partial_{x_j} Q, Y\rangle = 0$ for all $j = 1, \dots, N$, we get

$$|\langle \vec{Q}(\cdot - \tfrac{L}{2}) - \vec{Q}_1, \vec{Z}_1^+\rangle| \lesssim \delta^2.$$

Similarly,

$$|\langle \vec{Q}(\cdot + \tfrac{L}{2}) - \vec{Q}_2, \vec{Z}_1^+\rangle| = \langle (z_2 + \tfrac{L}{2}) \cdot \nabla Q(\cdot + \tfrac{L}{2}), \vec{Z}_1^+\rangle + O(\delta^2) \lesssim \delta e^{-\frac{1}{2}|z|} + O(\delta^2) \lesssim \delta^2.$$

Using the assumption $\langle \vec{\phi}, \vec{Z}^+(\cdot \pm \tfrac{L}{2})\rangle = 0$, we infer

$$\left| a_1^+ - h_1 \langle \vec{Y}^+, \vec{Z}^+\rangle \right| \lesssim \delta^2.$$

One can argue in the same way upon projecting (4.27) on $\vec{Z}_2^+$, and so derive (4.24). It remains to prove (4.26). From (4.25), we have

$$\sum_{k=1,2} |a_k^+ - \tilde{a}_k^+| + \|\vec{\varepsilon}_\perp - \vec{\tilde{\varepsilon}}_\perp\|_{H^1\times L^2} \lesssim \|\Omega - \tilde{\Omega}\| + |h - \tilde{h}|.$$

Using (2.27)-(2.28), observe as above that the following relation holds:

$$\begin{aligned}&\big[-\tfrac{\check{L}}{2} \cdot (\nabla \vec{Q} + h_1 \nabla \vec{Y}^+) + \check{h}_1 \vec{Y}^+\big](\cdot - \tfrac{\tilde{L}}{2}) + \big[\tfrac{\check{L}}{2} \cdot (\nabla \vec{Q} + h_2 \nabla \vec{Y}^+) - \check{h}_2 \vec{Y}^+\big](\cdot + \tfrac{\tilde{L}}{2})\\ &= \begin{pmatrix} \check{z}_1\cdot\nabla \\ -(\check{z}_1\cdot\nabla)(\ell_1\cdot\nabla)+(\check{\ell}_1\cdot\nabla)\end{pmatrix} Q(x - z_1) - \begin{pmatrix} \check{z}_2\cdot\nabla \\ -(\check{z}_2\cdot\nabla)(\ell_2\cdot\nabla)+(\check{\ell}_2\cdot\nabla)\end{pmatrix} Q(x - z_2)\\ &\quad + \vec{W}(\check{a}^+) - \vec{\tilde{W}}(\tilde{a}^+) + \vec{\tilde{\varepsilon}}_\perp - \vec{\tilde{\phi}} + O_{L^2}(|\check{L}|^2 + |\check{h}|^2 + |(\check{z}_1, \check{z}_2)|^2).\end{aligned}$$

Note that we use the notation from (4.5) and

$$\check{W}(a) := \sum_{k=1,2} \left\{ B_k(a) \check{Y}_k + \sum_{j=1}^N V_{k,j}(a) \partial_{x_j} \check{Q}_k \right\}, \quad \vec{\check{W}}(a) := \begin{pmatrix} \check{W}(a) \\ \nu \check{W}(a)\end{pmatrix}.$$

Projecting this relation on $\vec{Z}_k^+$, using the orthogonality relations on $\vec{\varepsilon}_\perp$, $\vec{\tilde{\varepsilon}}_\perp$, $\vec{\phi}$, $\vec{\tilde{\phi}}$ and the same argument as in the proof of (4.24), we find

$$|\beta(a_k^+ - \tilde{a}_k^+) - (h_k - \tilde{h}_k)| \lesssim \delta \left(\|\Omega - \tilde{\Omega}\| + |h - \tilde{h}| \right) \quad \text{for } k = 1, 2,$$

which completes the proof of the lemma. □

4.5. **End of the proof of Theorem 1.5.** We first classify 2-solitary wave whose initial data is chosen so that $\Omega = (L, \vec{\phi}) \in \mathcal{V}_\delta^\perp$.

Proposition 4.10. *For all $\delta > 0$ small enough, there exists a map $H : \mathcal{V}_\delta^\perp \to \bar{\mathcal{B}}_{\mathbb{R}^2}(\delta)$ such that, given $\Omega = (L, \vec{\phi}) \in \mathcal{V}_\delta^\perp$ and $h = (h_1, h_2) \in \mathcal{B}_{\mathbb{R}^2}(\delta)$, the solution $\vec{u}$ with initial data*

$$\vec{u}(0) = (\vec{Q} + h_1 \vec{Y}^+)(\cdot - \tfrac{L}{2}) - (\vec{Q} + h_2 \vec{Y}^+)(\cdot + \tfrac{L}{2}) + \vec{\phi}$$

is a 2-solitary wave if and only if $h = H(\Omega)$. Moreover, for any Ω, $\tilde{\Omega} \in \mathcal{V}_\delta^\perp$ such that $\|\Omega - \tilde{\Omega}\| < \delta$,

$$|H(\Omega) - H(\tilde{\Omega})| \lesssim \delta^{\frac{1}{4}} \|\Omega - \tilde{\Omega}\|. \tag{4.29}$$

Proof. First observe that from Propositions 4.2 and 4.4, for any $\Gamma(0) \in \mathcal{W}_\delta^\perp$, there exists a unique $A^+(\Gamma(0)) \in \mathcal{B}_{\mathbb{R}^2}(C\delta^{\frac{5}{4}})$ such that the solution $\vec{v}$ of (1.1) with initial data

$$\vec{v}(0) = \vec{Q}_1(0) + \vec{Q}_2(0) + \vec{W}(a_1^+(0), a_2^+(0)) + \vec{\varepsilon}_\perp(0)$$

with $(a_1^+(0), a_2^+(0)) \in \mathcal{B}_{\mathbb{R}^2}(\delta)$ is a 2-solitary wave if and only if $(a_1^+(0), a_2^+(0)) = A^+(\Gamma(0))$. Moreover, the following estimate holds

$$|A^+(\Gamma(0)) - A^+(\tilde{\Gamma}(0))| \lesssim \delta^{\frac{1}{4}} \|\Gamma(0) - \tilde{\Gamma}(0)\|.$$

Using the notation of Lemma 4.9, this means that

$$\vec{u} \text{ is a 2-solitary wave if and only if } a^+[\Omega, h] = A^+(\Gamma[\Omega, h]).$$

Let us show that this condition can be written $h = H(\Omega)$ for some function H. For $\Omega \in \mathcal{V}_\delta^\perp$ fixed, and $h \in \bar{\mathcal{B}}_{\mathbb{R}^2}(\delta)$, define

$$F_\Omega(h) = h - \beta \left\{ a^+[\Omega, h] - A^+(\Gamma[\Omega, h]) \right\}.$$

First, we observe that $F_\Omega : \bar{\mathcal{B}}_{\mathbb{R}^2}(\delta) \to \bar{\mathcal{B}}_{\mathbb{R}^2}(\delta)$ is a contraction for $\delta > 0$ small enough. Indeed, by Lemma 4.9 and Proposition 4.2,

$$|F_\Omega(h)| \le |h - \beta a^+[\Omega, h]| + |A^+(\Gamma[\Omega, h])| \lesssim \delta^{\frac{5}{4}},$$

and

$$\begin{aligned} |F_\Omega(h) - F_\Omega(\tilde{h})| &\le |(h - \tilde{h}) - \beta(a^+[\Omega, h] - a^+[\Omega, \tilde{h}])| \\ &\quad + \beta |A^+(\Gamma[\Omega, h]) - A^+(\Gamma[\Omega, \tilde{h}])| \\ &\lesssim \delta |h - \tilde{h}| + \delta^{\frac{1}{4}} \|\Gamma[\Omega, h] - \Gamma[\Omega, \tilde{h}]\| \lesssim \delta^{\frac{1}{4}} |h - \tilde{h}|. \end{aligned}$$

Therefore, by the fixed point theorem, there exists a unique $H = H(\Omega)$ in $\bar{\mathcal{B}}_{\mathbb{R}^2}(\delta)$ such that $F_\Omega(H) = H$, which is equivalent to $a^+[\Omega, H] = A^+(\Gamma[\Omega, H])$.
Second, for any Ω, $\tilde{\Omega} \in \mathcal{V}_\delta^\perp$,

$$\begin{aligned} |H(\Omega) - H(\tilde{\Omega})| &= |F_\Omega(H(\Omega)) - F_{\tilde{\Omega}}(H(\tilde{\Omega}))| \\ &\le |F_\Omega(H(\Omega)) - F_\Omega(H(\tilde{\Omega}))| + |F_\Omega(H(\tilde{\Omega})) - F_{\tilde{\Omega}}(H(\tilde{\Omega}))| \\ &\le \frac{1}{2} |H(\Omega) - H(\tilde{\Omega})| + |F_\Omega(H(\tilde{\Omega})) - F_{\tilde{\Omega}}(H(\tilde{\Omega}))| \end{aligned}$$

and so

$$|H(\Omega) - H(\tilde{\Omega})| \le 2 |F_\Omega(H(\tilde{\Omega})) - F_{\tilde{\Omega}}(H(\tilde{\Omega}))|.$$

By definition of $F_\Omega(h)$, Lemma 4.9 and Proposition 4.2, one has

$$\begin{aligned} |F_\Omega(h) - F_{\tilde{\Omega}}(h)| &\le \beta \left(|a^+[\Omega, h] - a^+[\tilde{\Omega}, h]| + |A^+(\Gamma[\Omega, h]) - A^+(\Gamma[\tilde{\Omega}, h])| \right) \\ &\lesssim \delta \|\Omega - \tilde{\Omega}\| + \delta^{\frac{1}{4}} \|\Gamma[\Omega, h] - \Gamma[\tilde{\Omega}, h]\| \lesssim \delta^{\frac{1}{4}} \|\Omega - \tilde{\Omega}\|, \end{aligned}$$

which yields $|H(\Omega) - H(\tilde{\Omega})| \lesssim \delta^{\frac{1}{4}} \|\Omega - \tilde{\Omega}\|$. □

Proof of Theorem 1.5. The map H was constructed in Proposition 4.10 (locally) on a subspace. Our goal is now to extend it to the full open set given in the statement of the Theorem. Let $\delta > 0$ to be fixed later. Given $(L, \vec{\phi})$ such that $|L| \ge 10 |\log \delta|$ and $\|\vec{\phi}\|_{H^1 \times L^2} < \delta$, we decompose

$$\vec{\phi} = -h_{1,\|} \vec{Y}^+(\cdot - \tfrac{L}{2}) + h_{2,\|} \vec{Y}^+(\cdot + \tfrac{L}{2}) + \vec{\phi}_\perp, \quad \langle \vec{\phi}_\perp, \vec{Z}^+(\cdot \pm \tfrac{L}{2}) \rangle = 0.$$

These conditions are a linear system on $h_{k,\|} = h_{k,\|}(L, \vec{\phi})$ of the form

$$\begin{cases} h_{1,\|} = -\beta \langle \vec{\phi}, \vec{Z}^+(\cdot - \frac{L}{2}) \rangle + O(e^{-L/2} h_{2,\|}) \\ h_{2,\|} = \beta \langle \vec{\phi}, \vec{Z}^+(\cdot + \frac{L}{2}) \rangle + O(e^{-L/2} h_{1,\|}) \end{cases}$$

which can be inverted for $\delta > 0$ small enough, and furthermore, for any such δ, one has

$$|h_{1,\|}| + |h_{2,\|}| + \|\vec{\phi}_\perp\|_{H^1\times L^2} \le C\|\vec{\phi}\|_{H^1\times L^2}, \tag{4.30}$$

and the Lipschitz estimates

$$\sum_{k=1,2} |h_{k,\|} - \tilde{h}_{k,\|}| + \|\vec{\phi}_\perp - \tilde{\vec{\phi}}_\perp\|_{H^1\times L^2} \le C\left(\left|L - \tilde{L}\right| + \|\vec{\phi} - \tilde{\vec{\phi}}\|_{H^1\times L^2}\right), \tag{4.31}$$

for some constant C independent of small $\delta > 0$. In particular, $\vec{\phi}_\perp \in \mathcal{V}^\perp_{C\delta}$ and up to lowering δ, we can assume that Proposition 4.10 applies on that set. Observe that our initial data can be written

$$\vec{u}(0) = \left(\vec{Q} + (h_1 - h_{1,\|})\vec{Y}^+\right)\left(\cdot - \tfrac{L}{2}\right) - \left(\vec{Q} + (h_2 - h_{2,\|})\vec{Y}^+\right)\left(\cdot + \tfrac{L}{2}\right) + \vec{\phi}_\perp.$$

Proposition 4.10 asserts that $\vec{u}$ is a 2-solitary wave if and only if $H(L, \vec{\phi}_\perp) = (h_1 - h_{1,\|}, h_2 - h_{2,\|})$ or equivalently that $(h_{1,\|}, h_{2,\|}) + H(L, \vec{\phi}_\perp) = (h_1, h_2)$. We are therefore led to define the extension of H as

$$H(L, \vec{\phi}) := (h_{1,\|}, h_{2,\|}) + H(L, \vec{\phi}_\perp).$$

(where $H(L, \vec{\phi}_\perp)$ is given in Proposition 4.10). Due to (4.29), (4.30) and (4.31), $H(L, \vec{\phi})$ is a Lipschitz map. In conclusion, H meets the requirements of Theorem 1.5. □

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