Improved uniqueness of multi-breathers of the modified Korteweg-de Vries equation

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December 26, 2022

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Dedicated to Professor Carlos Kenig on the occasion of his 70th birthday

Abstract

We consider multi-breathers of (mKdV). In [1], a smooth multi-breather was constructed, and proved to be unique in two cases: first, if the class of super-polynomial convergence to the profile (in the spirit of [2]), and second, under the assumption that all speeds of the breathers involved are positive (without rate of convergence).

The goal of this short note is to improve the second result: we show that uniqueness still holds if at most one velocity is negative or zero.

Key words: mKdV, breathers, solitons, multi-breathers, multi-solitons, uniqueness.

MSC Classification: Primary 35Q51, 35Q53, 35B40; Secondary 37K10, 37K40.

1 Setting of the problem

1.1 The modified Korteweg-de Vries equation

We consider the modified Korteweg-de Vries equation on $\mathbb{R}$:

$$\begin{cases} u_t + (u_{xx} + u^3)_x = 0 & (t,x) \in \mathbb{R}^2 \\ u(0) = u_0 & u(t,x) \in \mathbb{R} \end{cases} \tag{mKdV}$$

The (mKdV) equation appears as a model of some physical problems as plasma physics, electrodynamics [3], fluid mechanics, ferromagnetic vortices, and more; we refer to [1] for further information about the physical applications. Let us recall that (mKdV) is globally well-posed for any initial data in H^2 (see [4] for much stronger results), and for such data, three quantities are conserved in time:

$$\text{the } L^2 \text{ mass} \qquad M[u](t) := \frac{1}{2}\int u^2(t,x)dx, \tag{1}$$

$$\text{the energy} \qquad E[u](t) := \int \left(\frac{1}{2}u_x^2(t,x) - \frac{1}{4}u^4(t,x)\right)dx, \tag{2}$$

$$\text{the second energy} \qquad F[u](t) := \int \left(\frac{1}{2}u_{xx}^2(t,x) - \frac{5}{2}u^2(t,x)u_x^2(t,x) + \frac{1}{4}u^6(t,x)\right)dx. \tag{3}$$

Finally, (mKdV) is an integrable system, and there are (at least for smooth solutions) infinitely many conservation laws, but we point out that we will only use H^2 regularity and the above conservation laws.

1.2 Solitons and breathers of (mKdV)

The special "basic" solutions of (mKdV) that we consider here are solitons and breathers.

Definition 1. Let $c > 0$, $\kappa \in \{-1, 1\}$ and $x_0 \in \mathbb{R}$. A soliton $R_{c,\kappa}(x_0)$ of shape parameter (or velocity) c, of sign κ and of translation parameter (or initial position) x_0 is a solution of (mKdV), given by the following formula:

$$\forall (t,x) \in \mathbb{R}^2, \quad R_{c,\kappa}(t,x;x_0) := \kappa Q_c(x - x_0 - ct), \tag{4}$$

where Q_c is defined by the following formula:

$$\forall x \in \mathbb{R}, \quad Q_c(x) := \left(\frac{2c}{\cosh^2(c^{1/2}x)}\right)^{\frac{1}{2}}. \tag{5}$$

Remark 2. There exists a constant $C > 0$ that depends only on c such that

$$\forall (t,x) \in \mathbb{R}^2, \quad |R_{c,\kappa}(t,x;x_0)| \leqslant C\exp(-\sqrt{c}|x - x_0 - ct|). \tag{6}$$

Further properties of solitons (in particular, their H^2 variational structure) can be found in [1].

Definition 3. Let $\alpha, \beta > 0$ and $x_1, x_2 \in \mathbb{R}$. A breather $B_{\alpha,\beta}(x_1, x_2)$ of shape parameters α, β and of translation parameters x_1, x_2 a solution of (mKdV), given by the following formula:

$$\forall (t,x) \in \mathbb{R}^2, \quad B_{\alpha,\beta}(t,x;x_1,x_2) := 2\sqrt{2}\partial_x\left[\arctan\left(\frac{\beta}{\alpha}\frac{\sin(\alpha y_1)}{\cosh(\beta y_2)}\right)\right], \tag{7}$$

where

$$y_1 := x + \delta t + x_1 \quad \text{and} \quad y_2 := x + \gamma t + x_2, \tag{8}$$

$$\text{with} \quad \delta := \alpha^2 - 3\beta^2 \qquad \text{and} \qquad \gamma := 3\alpha^2 - \beta^2. \tag{9}$$

The velocity of $B_{\alpha,\beta}(x_1, x_2)$ is $-\gamma = \beta^2 - 3\alpha^2$ and its initial position is $-x_2$.

Remark 4. There exists a constant $C > 0$ that depends only on α and β such that

$$\forall (t,x) \in \mathbb{R}^2, \quad |B_{\alpha,\beta}(t,x;x_1,x_2)| \leqslant C\exp(-\beta|x + x_2 + \gamma t|). \tag{10}$$

Further properties of breathers can be found in [5] or [1].

1.3 Main result

We consider multi-breathers made of K breathers and L solitons in the sense of [1], which we recall now. Let $K, L \in \mathbb{N}$ and set $J = K + L$. For $1 \leqslant k \leqslant K$, let $\alpha_k, \beta_k > 0$ and $x^0_{1,k}, x^0_{2,k} \in \mathbb{R}$, and define the breathers

$$B_k := B_{\alpha_k,\beta_k}(x^0_{1,k}, x^0_{2,k}), \quad \text{with velocity} \quad v^b_k := \beta_k^2 - 3\alpha_k^2. \tag{11}$$

For $1 \leqslant l \leqslant L$, let $c_l > 0$, $\kappa_l \in \{-1, 1\}$ and $x^0_{0,l} \in \mathbb{R}$ and consider the solitons

$$R_l := R_{c_l,\kappa_l}(x^0_{0,l}), \quad \text{with velocity} \quad v^s_l := c_l. \tag{12}$$

An essential assumption in the present analysis is that *all the velocities of the considered objects, solitons or breathers, must be distinct*:

$$\forall k \neq k' \quad v^b_k \neq v^b_{k'}, \qquad \forall l \neq l' \quad v^s_l \neq v^s_{l'}, \qquad \forall k, l \quad v^b_k \neq v^s_l. \tag{13}$$

We may therefore order the speed and define an increasing function:

$$\underline{v} : \{1, \dots, J\} \to \{v^b_k, 1 \leqslant k \leqslant K\} \cup \{v^s_l, 1 \leqslant l \leqslant L\}. \tag{14}$$

The J-uple $(v_1, \dots, v_J)$ is thus the ordered set of all possible velocities of our objects. This allows us to write in a convenient way the sum of breathers and solitons: for $1 \leqslant j \leqslant J$, we define P_j as the object (either the soliton R_l or the breather B_k) that corresponds to the velocity v_j, so that $P_1, \dots, P_J$ are ordered by increasing velocity. We consider the sum of these breathers and solitons

$$P := \sum_{j=1}^{J} P_j = \sum_{k=1}^{K} B_k + \sum_{l=1}^{L} R_l, \tag{15}$$

and the associated multi-breather, that are solutions to (mKdV) which behave as P for large time as defined below

Definition 5. Given solitons and breathers (12), (11), whose sum is given by (15), a *multi-breather* associated to the sum P of solitons and breathers is a solution $p \in \mathcal{C}([T^*, +\infty), H^2(\mathbb{R}))$, for a constant $T^* > 0$, of (mKdV) such that

$$\|p(t) - P(t)\|_{H^2} \to 0 \quad \text{as} \quad t \to +\infty. \tag{16}$$

Let us recall from [1, Theorem 1.2] that as soon as the condition on the speeds (13) on P is satisfied, there exists a multi-breather p related to P, such that p is smooth and the convergence $p - P \to 0$ occurs at an exponential rate in any H^s, $s \in \mathbb{R}$.
Also, from [1, Theorem 1.4], under the extra assumption that all speeds are positive, that is $v_1 > 0$, this multi-breather is unique in the class of (16).

In this note, we revisit the proof given in [1], in order to improve this last uniqueness result: it actually holds if at most one of the velocities is non positive, that is under the assumption $v_2 > 0$. Here is the precise statement.

Theorem 6. *Given breathers* (11) *and solitons* (12)*, whose velocities satisfy* (13)*, let P be the sum of the considered solitons and breathers given in* (15)*.*
Assume that $v_2 > 0$, so that all velocities, except possibly one, are positive. Then the multi-breather of [1] $p \in \mathcal{C}([T^*, +\infty), H^2)$ associated to P is the unique *solution of* (mKdV) *such that* (16) *holds.*

Multi-solitons have been constructed for many dispersive models (see for example [6, 7, 8, 9] for (NLS), Klein-Gordon, or water-waves), but the question of uniqueness (or classification) is generally open. The examples that we are aware of, were such uniqueness is known, are the generalized Korteweg-de Vries equation [10], the generalized Benjamin-Bona-Mahony equation [11] and the Zakharov-Kuznetsov equation [12]. The underlying difficulty is the interaction of the nonlinear object and linear dispersion: for the two models, solitons move to the right, and dispersion to the left, which allows a very nice decoupling, that one can express via a monotonicity property. This feature is however absent in other dispersive models, which explains why uniqueness of multi-solitons remains an open problem in general.
In this note, we consider the more complex multi-breather of (mKdV). The point is that breathers may travel to the left, in the Airy dispersion zone: but if at most one moves there, our result shows that uniqueness still holds.

2 Proof

In order to prove Theorem (6), we consider P as in (15), we assume that $v_2 > 0$, and we consider a multi-breather $u \in \mathcal{C}([T, +\infty), H^2)$ such that

$$\|u(t) - P(t)\|_{H^2} \to 0 \quad \text{as} \quad t \to +\infty.$$

Our goal is to prove that $u \equiv p$, the multi-breather associated to P constructed in [1, Theorem 1.2]. For this, the main step is to prove that $u(t)$ converges actually exponentially fast to its profile $P(t)$.

Proposition 7. *There exists $\varpi > 0$, $T_0 \geqslant T$ and $C > 0$ such that*

$$\forall t \geqslant T_0, \quad \|u(t) - P(t)\|_{H^2} \leqslant Ce^{-\varpi t}. \tag{17}$$

This corresponds to Proposition 4.10 in [1] (where of course the assumption on P is different: there one supposes that $v_1 > 0$.).

2.1 Proof of Proposition 7

The proof follows mainly the lines of Section 4.2 of [1], with several changes that we will detail here. An important ingredient is (almost) monotonicity properties of localized quantities.
We do our best to treat breather and solitons together. To this end, we define the shape parameters as follows: for $j = 1, \dots, J$,

- if $P_j = B_k$ is a breather, then
$$(a_j, b_j) = (\alpha_k, \beta_k), \tag{18}$$
- if $P_j = R_l$ is a soliton, then
$$(a_j, b_j) = (0, \sqrt{c_l}). \tag{19}$$

The shape of the cut-off function that we will use is given by Ψ:

$$\Psi(x) := \frac{2}{\pi} \arctan\big(\exp(-\sqrt{\sigma}x/2)\big), \tag{20}$$

where $\sigma > 0$ is small enough (and precise conditions will be given in the proof).
We consider a cut-off function Φ_j given for $j = 1, \dots, J-1$ by

$$\Phi_j(t,x) := \Psi(x - m_j t), \tag{21}$$

and $\Phi_J \equiv 1$. For $j \in \{1, \dots, J-1\}$, Ψ_j tend to 1 at $-\infty$, to 0 at $+\infty$, with an exponentially localized transition between the centers of P_j and P_{j+1}. This requires that for all such j, $v_j < m_j < v_{j+1}$. However, in order to the monotonicity argument to work, we need to choose cut-off functions that move to the right, i.e. have positive velocities $m_j > 0$. For $j = 2, \dots, J-1$, we set

$$m_j := \frac{v_j + v_{j+1}}{2}. \tag{22}$$

(Then we indeed have $m_j > 0$ because $v_j \geqslant v_2 > 0$). On the other hand, m_1 needs better tuning: we define m_1 in the following way.

1. We first set $0 < \nu_1 < 1$ such that

$$(b_1^2 - a_1^2) + \nu_1(a_1^2 + b_1^2) > 0, \tag{23}$$

2. then we choose m_1 such that

$$\max(0, v_1) < m_1 < v_2, \quad \text{and} \tag{24}$$

$$m_1(b_1^2 - a_1^2) > \frac{1}{2}(\nu_1 - 1)(a_1^2 + b_1^2)^2. \tag{25}$$

Condition (23) corresponds to a choice of ν_1 sufficiently near 1. Condition (25) can be satisfied: indeed, if $b_1^2 - a_1^2 \leqslant 0$ then $v_1 < 0$ and it suffices to choose $m_1 > 0$ sufficiently small; if $b_1^2 - a_1^2 > 0$, it is always satisfied as the righthand side is negative.
These conditions will be used in order to derive suitable monotonicity properties, see Step 3.

We set $\tau_0 > 0$ the minimal distance between $\{v_1, \dots, v_J\}$ and $\{m_1, \dots, m_{J-1}\}$.
As mentioned, the scheme of the proof is here roughly the same as in [1]; however, some specifics change.
First we modulate the breathers P_j into $\widetilde{P}_j$ (by translation only), so that the default $w = u - \widetilde{P}$ (defined in Lemma 8) enjoys orthogonality properties. Then we prove that $\|w(t)\|_{H^2} \leqslant Ce^{-\varpi t}$ by induction, where $\varpi > 0$ is a constant depending on the data of the problem.
For $j = 1, \dots, J$, proposition $\mathcal{P}_j$ reads

$$\forall t \geqslant T', \quad \int \big(w^2 + w_x^2 + w_{xx}^2\big)\Phi_j + \sum_{i=1}^{j} \left| \int \widetilde{P}_i w \right| \leqslant C_j e^{-2\varpi t}, \tag{26}$$

where $T' \geqslant T$ is to be defined in the proof. $\mathcal{P}_0$ is the assertion "True".

Given $j \in \{1,\dots,J\}$, we assume $\mathcal{P}_{j-1}$, and our goal is to prove $\mathcal{P}_j$. We finally infer $\|u(t)-P(t)\|_{H^2} \leqslant Ce^{-\varpi t}$, in the concluding step of the proof.

One difference with [1] is that here we make our proof by induction on the modulated difference w and not on $u-P$. This is not crucial, but we find it nicer to perform a modulation for all the objects at once. One key difference compared to [1], though, is that we need and prove monotonicity for a functional which is slightly weaker than the natural Lyapunov functional required for the proof. When $v_1 < 0$, the proof requires a careful interpolation between positive terms in order to balance negative terms.

Step 1: Modulation

This step is devoted to the proof of the following modulation lemma.

Lemma 8. *There exists $C > 0$, $T_2 \geqslant T$, such that there exist unique $\mathcal{C}^1$ functions $y_{1,k}, y_{2,k}, y_{0,l} : [T_2,+\infty) \to \mathbb{R}$ such that if we set:*

$$w(t,x) := u - \widetilde{P}, \tag{27}$$

where

$$\widetilde{P}(t,x) := \sum_{k=1}^{K} \widetilde{B}_k(t,x) + \sum_{l=1}^{L} \widetilde{R}_l(t,x), \tag{28}$$

and

$$\widetilde{R}_l(t,x) := \kappa_l Q_{c_l}(x - x_{0,l}^0 + y_{0,l}(t) - c_l t), \tag{29}$$

$$\widetilde{B}_k(t,x) := B_{\alpha_k,\beta_k}(t,x;x_{1,k} + y_{1,k}(t), x_{2,k} + y_{2,k}(t)), \tag{30}$$

then, $w(t)$ satisfies, for any $t \in [T_2,+\infty)$,

$$\forall l = 1,\dots,L, \quad \int (\widetilde{R}_l)_x(t) w(t) = 0, \tag{31}$$

$$\forall k = 1,\dots,K \quad \int (\widetilde{B}_k)_1(t) w(t) = \int (\widetilde{B}_k)_2(t) w(t) = 0, \tag{32}$$

where we denote:

$$(\widetilde{B}_k)_1(t,x) := \partial_{x_1} \widetilde{B}_k, \quad (\widetilde{B}_k)_2(t,x) := \partial_{x_2} \widetilde{B}_k. \tag{33}$$

Moreover, for any $t \in [T_2,+\infty)$,

$$\|w(t)\|_{H^2} + |y_{1,k}(t)| + |y_{2,k}(t)| + |y_{0,l}(t)| \leqslant C\|v(t)\|_{H^2}, \tag{34}$$

and, if ϖ is small enough,

$$|y'_{1,k}(t)| + |y'_{2,k}(t)| + |y'_{0,l}(t)| \leqslant C\left(\int w(t)^2 \Phi_j \right)^{1/2} + Ce^{-\varpi t}. \tag{35}$$

Proof. The proof of this lemma can be performed in the same manner as in [1, Lemma 2.8]. □

As above for (15), we denote $\tilde{P}_j = \tilde{B}_k$ if $P_j = B_k$ is a breather, and $\tilde{P}_j = \tilde{R}_l$ if $P_j = R_l$ is a soliton, so that

$$\tilde{P} = \sum_{j=1}^{J} \tilde{P}_j.$$

The difference with [1] is that

- the modulation that we perform here does not modify any shape parameter (that is why there is only one modulation direction for each soliton here),
- we perform the modulation once and not on each step of the induction.

Step 2: Approximation of the Lyapunov functional

This step is devoted to define a localized Lyapunov functional. Let $j \in \{1, \ldots, J\}$. First, we define the localized quantities related to conservation laws as follows:

$$M_j(t) := \int u^2(t)\Phi_j(t), \tag{36}$$

$$E_j(t) := \int \left[\frac{1}{2}u_x^2 - \frac{1}{4}u^4\right]\Phi_j(t), \tag{37}$$

$$F_j(t) := \int \left[\frac{1}{2}u_{xx}^2 - \frac{5}{2}u^2u_x^2 + \frac{1}{4}u^6\right]\Phi_j(t). \tag{38}$$

Then the localized Lyapunov functional is

$$\mathcal{H}_j(t) := F_j(t) + 2\big(b_j^2 - a_j^2\big)E_j(t) + \big(a_j^2 + b_j^2\big)^2 M_j(t), \tag{39}$$

where a_j, b_j stand for generalized shape parameters defined in (18)-(19).
This Lyapunov functional was already introduced in [1]. We prove the following lemma that somehow quantifies how far is a modulated sum of solitons and breathers from being a critical point for $\mathcal{H}_j$.
First, there hold the following Taylor expansions.

Lemma 9. *There exists $C > 0$, $T_1 \geqslant T$ such that the following holds for any $t \geqslant T_1$:*

$$\left| M_j(t) - \sum_{i=1}^{j} M\big[\widetilde{P}_i\big] - \sum_{i=1}^{j} \int \widetilde{P}_i w - \frac{1}{2}\int w^2\Phi_j \right| \leqslant Ce^{-2\varpi t}, \tag{40}$$

$$\left| E_j(t) - \sum_{i=1}^{j} E\big[\widetilde{P}_i\big] - \sum_{i=1}^{j} \int \Big[(\widetilde{P}_i)_x w_x - \widetilde{P}_i^3 w\Big] - \int \Big[\frac{1}{2}w_x^2 - \frac{3}{2}\widetilde{P}^2 w^2\Big]\Phi_j \right| \leqslant Ce^{-2\varpi t} + o\left(\int w^2\Phi_j\right), \tag{41}$$

$$\begin{aligned} \Bigg| F_j(t) - \sum_{i=1}^{j} F\big[\widetilde{P}_i\big] - \sum_{i=1}^{j} \int \Big[(\widetilde{P}_i)_{xx} w_{xx} - 5\widetilde{P}_i(\widetilde{P}_i)_x^2 w - 5\widetilde{P}_i^2(\widetilde{P}_i)_x w_x + \frac{3}{2}\widetilde{P}_i^5 w\Big] \\ - \int \Big[\frac{1}{2}w_{xx}^2 - \frac{5}{2}w^2\widetilde{P}_x^2 - 10\widetilde{P}w\widetilde{P}_x w_x - \frac{5}{2}\widetilde{P}^2 w_x^2 + \frac{15}{4}\widetilde{P}^4 w^2\Big]\Phi_j(t) \Bigg| \end{aligned} \tag{42}$$

$$\leqslant Ce^{-2\varpi t} + o\left(\int (w^2 + w_x^2)\Phi_j\right). \tag{43}$$

Proof. See for example [1, Proposition 2.12]. We emphasize that we do not use here the induction assumption. □

The Lyapunov functional is constructed so as to make the linear terms in w cancel, as seen below.

Lemma 10. *There exists $T_2 \geqslant T_1$ such that the following holds for $t \geqslant T_2$:*

$$\mathcal{H}_j(t) = \sum_{i=1}^{j} F[\widetilde{P}_i] + 2\big(b_j^2 - a_j^2\big) \sum_{i=1}^{j} E[\widetilde{P}_i] + \big(a_j^2 + b_j^2\big)^2 \sum_{i=1}^{j} M[\widetilde{P}_i] + H_j(t) + O(e^{-2\varpi t}) + o\bigg(\int (w^2 + w_x^2)\Phi_j\bigg), \tag{44}$$

where

$$H_j(t) := \int \Big[\frac{1}{2}w_{xx}^2 - \frac{5}{2}w_x^2 \widetilde{P}_j^2 + \frac{5}{2}w^2(\widetilde{P}_j)_x^2 + 5w^2\widetilde{P}_j(\widetilde{P}_j)_{xx} + \frac{15}{4}w^2\widetilde{P}_j^4\Big]\Phi_j(t) + \big(b_j^2 - a_j^2\big)\int \Big[w_x^2 - 3w^2\widetilde{P}_j^2\Big]\Phi_j(t) + \frac{1}{2}\big(a_j^2 + b_j^2\big)^2 \int w^2 \Phi_j(t). \tag{45}$$

Proof. The proof of the lemma above can be performed as in [1, Proposition 2.12]: it uses Lemma 9, the elliptic equation satisfied by P_j or $\widetilde{P}_j$ (we do not need to make a distinction whether it is a soliton or a breather here), and the induction assumption for the contributions in the region $x \leqslant m_{j-1}t$. □

Step 3: Monotonicity

Let us first recall some monotonicity properties related to the localized conservations laws.

Lemma 11. *Let $\omega > 0$ as small as desired. There exists $T_3 = T_3(\omega) \geqslant T_2$ and $C > 0$ such that for $t \geqslant T_3$,*

$$\sum_{i=1}^{j} M[P_i] - M_j(t) \geqslant -Ce^{-2\varpi t}, \tag{46}$$

$$\sum_{i=1}^{j} \big(E[P_i] + \omega M[P_i]\big) - \big(E_j(t) + \omega M_j(t)\big) \geqslant -Ce^{-2\varpi t}, \tag{47}$$

$$\sum_{i=1}^{j} \big(F[P_i] + \omega M[P_i]\big) - \big(F_j(t) + \omega M_j(t)\big) \geqslant -Ce^{-2\varpi t}. \tag{48}$$

Proof. The lemma above may be proved in the same manner as in [1, Lemma 4.11]. □

We emphasize that some extra L^2 mass is needed in order to gain monotonicity for E_j or F_j: this fact was already noted in [10], and is related to a lack of control of non linear terms far away from the breathers/solitons.

We now turn to the main monotonicity result that we will use.
Let $0 < \nu < 1$ be close enough to 1, to be fixed later. We define, for $j = 1, \dots, J$, a functional $\mathcal{F}_j$ that is close to the Lyapunov functional $\mathcal{H}_j$:

$$\mathcal{F}_j(t) := F_j(t) + 2(b_j^2 - a_j^2)E_j(t) + \nu(a_j^2 + b_j^2)^2 M_j(t). \tag{49}$$

The following lemma states the almost-growth of $\mathcal{F}_j$:

Lemma 12. *There exists $0 < \nu < 1$ close enough to 1 such that there exists $T_4 \geqslant T_2$ and $C > 0$ such that for any $t \geqslant T_4$,*

$$\mathcal{F}_j(t) - \sum_{i=1}^{j} F[P_i] - 2(b_j^2 - a_j^2)\sum_{i=1}^{j} E[P_i] - \nu(a_j^2 + b_j^2)^2 \sum_{i=1}^{j} M[P_i] \leqslant Ce^{-2\varpi t}. \tag{50}$$

Remark 13. We emphasize the factor $\nu < 1$ in front of the M_j term, which make it a *weakened* version of the Lyapunov functional $\mathcal{H}_j$. Therefore, as M_j enjoys strong monotonicity, the monotonicity of $\mathcal{F}_j$ is a stronger result than merely that of $\mathcal{H}_j$. This improvement is needed in order to deal with $\int \widetilde{P}_j w$, see Step 5.

Proof. If $b_j^2 - a_j^2 \geqslant 0$, then Lemma 12 is an immediate consequence of Lemma 11. For the rest of the proof, we consider the case $b_j^2 - a_j^2 < 0$, which can only occur when $j = 1$, and which we assume for the rest of this proof. Let

$$\nu = \nu_1 + \frac{2}{3}(1 - \nu_1) < 1, \tag{51}$$

where ν_1 is defined in (23).
In the proof of Lemma 11 (see [1, Lemma 4.11]), there hold the more precise bounds: given $\omega > 0$, there exist $T_3' = T_3'(\omega) \geqslant T_2$ such that for all $t \geqslant T_3'$

$$\frac{d}{dt}F_1(t) \geqslant -Ce^{-2\varpi t} + \frac{3}{2}\int u_{xxx}^2|\Phi_{1x}| + \frac{m_1}{2}\int u_{xx}^2|\Phi_{1x}| - \omega\int (u_{xx}^2 + u_x^2 + u^2)|\Phi_{1x}|, \tag{52}$$

$$-\frac{d}{dt}E_1(t) \geqslant -Ce^{-2\varpi t} - \frac{3}{2}\int u_{xx}^2|\Phi_{1x}| - \frac{m_1}{2}\int u_x^2|\Phi_{1x}| - \omega\int (u_x^2 + u^2)|\Phi_{1x}|, \tag{53}$$

$$\frac{d}{dt}M_1(t) \geqslant -Ce^{-2\varpi t} + \frac{3}{2}\int u_x^2|\Phi_{1x}| + \frac{m_1}{2}\int u^2|\Phi_{1x}| - \omega\int u^2|\Phi_{1x}|. \tag{54}$$

Summing up the right linear combination, we infer:

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_1(t) \geqslant{}& -Ce^{-2\varpi t} + \frac{3}{2}\int u_{xxx}^2|\Phi_{1x}| \\ &+ \left(3(b_1^2 - a_1^2) + \frac{m_1}{2} - \omega\right)\int u_{xx}^2|\Phi_{1x}| \\ &+ \left(\frac{3}{2}\nu(a_1^2 + b_1^2)^2 + m_1(b_1^2 - a_1^2) - \omega\right)\int u_x^2|\Phi_{1x}| \\ &+ \left(\frac{m_1}{2}\nu(a_1^2 + b_1^2)^2 - \omega\right)\int u^2|\Phi_{1x}|.\end{aligned} \tag{55}$$

Heuristically, ω can be neglected, so that the coefficients of the terms in $\int u_{xxx}^2|\Phi_{1x}|$ and $\int u^2|\Phi_{1x}|$ are all positive. For the $\int u_x^2|\Phi_{1x}|$ term, we can play with ν and m_1 to ensure its positivity (which is one purpose of the conditions (23) and (25)). However the coefficient in front of $\int u_{xx}^2|\Phi_{1x}|$ could be trully negative: $m_1 < v_2$ could be only marginally positive, and $b_1^2 - a_1^2$ can take any real value (the only constraint is that it is larger than the speed v_1, on which we have no control, though). This term might be problematic, and will concentrate our efforts: roughly speaking, the idea is to bound it via interpolation with the other terms.

From the definition of m_1 given by (25), we have that

$$\frac{3}{2}\nu(a_1^2+b_1^2)^2 + m_1(b_1^2-a_1^2) > \frac{3}{2}\nu'(a_1^2+b_1^2)^2, \tag{56}$$

where

$$\nu' = \nu_1 + \frac{1-\nu_1}{3}. \tag{57}$$

We choose ω small enough with respect to the previous choice (by choosing T_4 large enough) so that

$$\frac{3}{2}\nu(a_1^2+b_1^2)^2 + m_1(b_1^2-a_1^2) - \omega \geqslant \frac{3}{2}\nu'(a_1^2+b_1^2)^2. \tag{58}$$

Knowing that $a_1^2+b_1^2>0$, we may choose ω even smaller (with respect to m_1 and ν) so that

$$\frac{m_1}{2}\nu(a_1^2+b_1^2)^2 - \omega \geqslant 0, \tag{59}$$

and

$$3(b_1^2-a_1^2) + \frac{m_1}{2} - \omega \geqslant 3(b_1^2-a_1^2). \tag{60}$$

In the case when with the chosen values of m_1, ν and ω, $3(b_1^2-a_1^2)+\frac{m_1}{2}-\omega$ is positive, the desired conclusion is straightforward by integration. From now on, we place ourselves in the case when

$$3(b_1^2-a_1^2) + \frac{m_1}{2} - \omega < 0. \tag{61}$$

Now, we want to bound above $\int u_{xx}^2|\Phi_{1x}|$. By integration by parts,

$$\begin{aligned}\int u_{xx}^2|\Phi_{1x}| &= -\int u_x u_{xxx}|\Phi_{1x}| - \int u_x u_{xx}|\Phi_{1xx}| \\ &\leqslant \sqrt{\int u_x^2|\Phi_{1x}| \int u_{xxx}^2|\Phi_{1x}|} + \frac{\sqrt{\sigma}}{2}\sqrt{\int u_x^2|\Phi_{1x}| \int u_{xx}^2|\Phi_{1x}|},\end{aligned} \tag{62}$$

because $|\Phi_{1xx}| \leqslant \frac{\sqrt{\sigma}}{2}|\Phi_{1x}|$. We denote:

$$X := \sqrt{\int u_{xx}^2|\Phi_{1x}|}, \tag{63}$$

and

$$A := \sqrt{\int u_x^2|\Phi_{1x}| \int u_{xxx}^2|\Phi_{1x}|}. \tag{64}$$

So, we have that

$$X^2 \leqslant A + \varepsilon X, \tag{65}$$

where

$$\varepsilon := \frac{\sqrt{\sigma}}{2}\sqrt{\int u_x^2 |\Phi_{1x}|} \leqslant \frac{\sqrt{\sigma}}{2}\|u\|_{\dot{H}^1}, \tag{66}$$

which can be as small as we want if we take σ small enough (for a given solution u). We deduce that

$$X \leqslant \frac{\varepsilon + \sqrt{\varepsilon^2 + 4A}}{2} \leqslant \varepsilon + \sqrt{A}. \tag{67}$$

Thus,

$$\begin{aligned} \int u_{xx}^2 |\Phi_{1x}| \leqslant \Bigg(\frac{\sigma}{4}\sqrt{\int u_x^2|\Phi_{1x}|} + \sqrt{\sigma}\Big(\int u_x^2 |\Phi_{1x}| \int u_{xxx}^2 |\Phi_{1x}| \Big)^{\frac{1}{4}} \\ + \sqrt{\int u_{xxx}^2 |\Phi_{1x}|} \Bigg) \sqrt{\int u_x^2 |\Phi_{1x}|}. \end{aligned} \tag{68}$$

So,

$$\begin{aligned} \Big(3(b_1^2 - a_1^2) + \frac{m_1}{2} - \omega\Big) \int u_{xx}^2 |\Phi_{1x}| \geqslant 3(b_1^2 - a_1^2)\sqrt{\int u_x^2 |\Phi_{1x}| \int u_{xxx}^2 |\Phi_{1x}|} \\ + 3(b_1^2 - a_1^2)\sqrt{\sigma}\Big(\int u_x^2 |\Phi_{1x}| \Big)^{\frac{3}{4}} \Big(\int u_{xxx}^2 |\Phi_{1x}| \Big)^{\frac{1}{4}} \\ + 3(b_1^2 - a_1^2)\frac{\sigma}{4} \int u_x^2 |\Phi_{1x}|. \end{aligned} \tag{69}$$

On the other hand, we have, for a choice of $\nu_2, \nu_3 > 0$ such that $\nu_1 + \nu_2 + \nu_3 = \nu'$ that

$$
\begin{aligned}
\frac{3}{2}\int u_{xxx}^2|\Phi_{1x}| &+ \Big(\frac{3}{2}\nu'(a_1^2+b_1^2)^2\Big)\int u_x^2|\Phi_{1x}| \\
&\geqslant \frac{3}{2}\nu_1\int u_{xxx}^2|\Phi_{1x}| + \Big(\frac{3}{2}\nu_1(a_1^2+b_1^2)^2\Big)\int u_x^2|\Phi_{1x}| \\
&+ \frac{3}{2}(1-\nu_1)\int u_{xxx}^2|\Phi_{1x}| + \Big(\frac{3}{2}\nu_2(a_1^2+b_1^2)^2\Big)\int u_x^2|\Phi_{1x}| \\
&+ \Big(\frac{3}{2}\nu_3(a_1^2+b_1^2)^2\Big)\int u_x^2|\Phi_{1x}| \\
&\geqslant 2\sqrt{\frac{3}{2}\nu_1\Big(\frac{3}{2}\nu_1(a_1^2+b_1^2)^2\Big)}\sqrt{\int u_x^2|\Phi_{1x}|\int u_{xxx}^2|\Phi_{1x}|} \\
&+ 4\Big(\frac{3}{2}(1-\nu_1)\int u_{xxx}^2|\Phi_{1x}|\Big)^{\frac{1}{4}}\Big(\Big(\frac{1}{2}\nu_2(a_1^2+b_1^2)^2\Big)\int u_x^2|\Phi_{1x}|\Big)^{\frac{3}{4}} \\
&+ \Big(\frac{3}{2}\nu_3(a_1^2+b_1^2)^2\Big)\int u_x^2|\Phi_{1x}| \\
&\geqslant 3\nu_1(a_1^2+b_1^2)\sqrt{\int u_x^2|\Phi_{1x}|\int u_{xxx}^2|\Phi_{1x}|} \\
&+ 2\cdot 3^{\frac{1}{4}}(1-\nu_1)^{\frac{1}{4}}\nu_2^{\frac{3}{4}}(a_1^2+b_1^2)^{\frac{3}{2}}\Big(\int u_x^2|\Phi_{1x}|\Big)^{\frac{3}{4}}\Big(\int u_{xxx}^2|\Phi_{1x}|\Big)^{\frac{1}{4}} \\
&+ \frac{3}{2}\nu_3(a_1^2+b_1^2)^2\int u_x^2|\Phi_{1x}|.
\end{aligned}
\tag{70}
$$

This is why, we deduce that

$$
\begin{aligned}
\frac{d}{dt}\mathcal{F}_1(t) &\geqslant -Ce^{-2\varpi t} \\
&+ \Big(3(b_1^2-a_1^2) + 3\nu_1(a_1^2+b_1^2)\Big)\sqrt{\int u_x^2|\Phi_{1x}|\int u_{xxx}^2|\Phi_{1x}|} \\
&+ \Big(3(b_1^2-a_1^2)\sqrt{\sigma} + 2\cdot 3^{\frac{1}{4}}(1-\nu_1)^{\frac{1}{4}}\nu_2^{\frac{3}{4}}(a_1^2+b_1^2)^{\frac{3}{2}}\Big)\Big(\int u_x^2|\Phi_{1x}|\Big)^{\frac{3}{4}}\Big(\int u_{xxx}^2|\Phi_{1x}|\Big)^{\frac{1}{4}} \\
&+ \Big(3(b_1^2-a_1^2)\frac{\sigma}{4} + \frac{3}{2}\nu_3(a_1^2+b_1^2)^2\Big)\int u_x^2|\Phi_{1x}|.
\end{aligned}
\tag{71}
$$

To finish, we remark that the coefficient in front of the integrals in (71) are all non negative: indeed

$$3(b_1^2-a_1^2) + 3\nu_1(a_1^2+b_1^2) \geqslant 0, \tag{72}$$

by definition of ν_1 given in (23);

$$3(b_1^2-a_1^2)\sqrt{\sigma} + 2\cdot 3^{\frac{1}{4}}(1-\nu_1)^{\frac{1}{4}}\nu_2^{\frac{3}{4}}(a_1^2+b_1^2)^{\frac{3}{2}} \geqslant 0, \tag{73}$$

and

$$3(b_1^2-a_1^2)\frac{\sigma}{4} + \frac{3}{2}\nu_3(a_1^2+b_1^2)^2 \geqslant 0, \tag{74}$$

by choosing $\sigma > 0$ small enough.
Thus,

$$\frac{d}{dt}\mathcal{F}_1(t) \geqslant -Ce^{-2\varpi t}. \tag{75}$$

We obtain the desired conclusion by integration. □

Step 4: Bound from above for $H_j(t)$

Lemma 14. *For any $t \geqslant T_4$, we have that*

$$H_j(t) \leqslant Ce^{-2\varpi t} + o\left(\int (w^2 + w_x^2)\Phi_j\right) \tag{76}$$

Proof. From Lemma 11, we know that for any $t \geqslant T_1$,

$$M_j(t) - \sum_{i=1}^{j} M[P_i] \leqslant Ce^{-2\varpi t}. \tag{77}$$

By summing this fact with the fact from the Lemma 12, we obtain that for any t large enough:

$$\mathcal{H}_j(t) - \sum_{i=1}^{j} F[P_i] - 2\big(b_j^2 - a_j^2\big) \sum_{i=1}^{j} E[P_i] - \big(a_j^2 + b_j^2\big)^2 \sum_{i=1}^{j} M[P_i] \leqslant Ce^{-2\varpi t}. \tag{78}$$

From (78) and Lemma 10, we obtain the desired conclusion for any $t \geqslant T_4$. □

Let us recall that H_j enjoys a crucial coercivity property:

Proposition 15 (Coercivity of H_j)**.** *There exists $\mu > 0$, and $T_5 \geqslant T_4$ such that, for $t \geqslant T_5$,*

$$H_j(t) \geqslant \mu \int (w_{xx}^2 + w_x^2 + w^2)\Phi_j(t) - \frac{1}{\mu}\left(\int \widetilde{P}_j w\sqrt{\Phi_j}\right)^2. \tag{79}$$

Proof. One can argue as in the proof of Proposition 4.10, Step 7 in [1] (see also Proposition 2.13). (We choose to keep the L^2 scalar product of $\widetilde{P}_j$ with $w\sqrt{\Phi_j}$ because coercivity is derived from the original coercivity (related to the linearization around soliton or breather of the relevant conservation law see [1, 5]) via localization argument on $w\sqrt{\Phi_j}$. Obviously, we could have stated coercivity up to the scalar product of $\int \widetilde{P}_j w$). □

Using the above two results, and the induction hypothesis we prove as in [1, Step 6] that for $t \geqslant T_5$,

$$\int (w^2 + w_x^2 + w_{xx}^2)\Phi_j \leqslant Ce^{-2\varpi t} + C\left(\int \widetilde{P}_j w\sqrt{\Phi_j}\right)^2. \tag{80}$$

Remark 16. Here the choice of modulating only by translation gives a writing simplification with respect to [1], where the scalar product only occured when P_j was a breather: we do not need to make this distinction now.

Step 5: Bound from above for $\left|\int \widetilde{P}_j w\sqrt{\Phi_j}\right|$

We now show that the seemingly problematic scalar product $\displaystyle\int \widetilde{P}_j w$ is actually quadratic (which the second part of the induction hypothesis $\mathcal{P}_j$).

Lemma 17. *For any* $t \geqslant T_5$*,*

$$\left|\int \widetilde{P}_j w\sqrt{\Phi_j}\right| \leqslant Ce^{-2\varpi t} + C\int \left(w^2 + w_x^2\right)\Phi_j \tag{81}$$

Proof. The proof follows the lines of that of Step 7 of Proposition 4.10 in [1, Section 4.2], and we only sketch it.
First we observe that it is enough to prove the same bound on

$$\left|\sum_{i=1}^{j}\int \widetilde{P}_i w\right|$$

because the induction hypothesis $\mathcal{P}_{j-1}$ (26) takes care of the terms $i \leqslant j-1$, and the localization $\sqrt{\Phi_j}$ causes an error of size $O(e^{-2\varpi t})$. Then, the idea is to go back to Lemma 11, and work on M_j and on E_j, F_j separately: both make the scalar products $\int \widetilde{P}_i w$ appear, but with opposite signs.
On the one hand, due to (40) and (46), we may deduce that from Lemma 11 and Lemma 10 that

$$\sum_{i=1}^{j}\int \widetilde{P}_i w \leqslant Ce^{-2\varpi t}. \tag{82}$$

On the other hand, due to (41)-(43), and using the elliptic equation satisfied by $\widetilde{P}_j$ and the induction assumption, we may infer that for $t \geqslant T_5$,

$$-(1-\nu)(a_j^2+b_j^2)^2\sum_{i=1}^{j}\int \widetilde{P}_i w \tag{83}$$

$$= O(e^{-2\varpi t}) + O\left(\int (w^2+w_x^2)\Phi_j\right) \tag{84}$$

$$+ \mathcal{F}_j(t) - \sum_{i=1}^{j}\left(F[P_i] + 2(b_j^2 - a_j^2)E[P_i] + \nu(a_j^2+b_j^2)^2 M[P_i]\right). \tag{85}$$

Observe the factor $1-\nu > 0$ that we placed: this make the functional $\mathcal{F}_j$ appear, and not merely the Lyapunov $\mathcal{H}_j$ (for which the linear terms cancel). Therefore, we are now in position of using the monotonicity (50), and we obtain that

$$-(1-\nu)(a_j^2+b_j^2)^2\sum_{i=1}^{j}\int \widetilde{P}_i w \leqslant Ce^{-2\varpi t} + C\int (w^2+w_x^2)\Phi_j. \tag{86}$$

Estimates (82) and (86) yield together that

$$\left|\sum_{i=1}^{j}\int \widetilde{P}_i w\right| \leqslant Ce^{-2\varpi t} + C\int (w^2+w_x^2)\Phi_j, \tag{87}$$

which gives the desired conclusion. □

Step 6: Conclusion

From (80) and (81), we deduce that for all $t \geqslant T_5$,

$$\int \big(w^2 + w_x^2 + w_{xx}^2\big)\Phi_j \leqslant Ce^{-2\varpi t}. \tag{88}$$

Plugging this in (81), an immediate localization argument shows that

$$\left|\int \widetilde{P}_j w\right| \leqslant Ce^{-2\varpi t}.$$

Using also $\mathcal{P}_{j-1}$, this proves $\mathcal{P}_j$ (26), and the induction is complete.

Thus, for $j = J$, we get

$$\forall t \geqslant T_5, \quad \|w(t)\|_{H^2} \leqslant Ce^{-\varpi t}. \tag{89}$$

Now, recall that from $\|u(t) - P(t)\|_{H^2} \to 0$ and (34), there hold

$$|y_{1,k}(t)| + |y_{2,k}(t)| + |y_{0,l}(t)| \to 0 \quad \text{as } t \to +\infty. \tag{90}$$

Then, by integration from (35) on $[t, +\infty)$ that for any $t \geqslant T_5$,

$$|y_{1,k}(t)| + |y_{2,k}(t)| + |y_{0,l}(t)| \leqslant Ce^{-\varpi t}. \tag{91}$$

So, by the mean-value theorem,

$$\begin{aligned} \|u(t) - P(t)\|_{H^2} &\leqslant \|w(t)\|_{H^2} + \|\widetilde{P}(t) - P(t)\|_{H^2} \\ &\leqslant \|w(t)\|_{H^2} + \sum_{k=1}^{K} \big(|y_{1,k}(t)| + |y_{2,k}(t)|\big) + \sum_{l=1}^{L} |y_{0,l}(t)| \\ &\leqslant Ce^{-\varpi t}. \end{aligned} \tag{92}$$

This concludes the proof of Proposition 7.

2.2 Proof of Theorem 6

Let us recall the uniqueness result of [1] in the class of super-polynomial convergence to the profile.

Proposition 18. *Given breathers (11) and solitons (12), whose velocities satisfy (13), let P be the sum of the considered solitons and breathers given in (15).*
There exists $N > 0$ large enough such that the multi-breather from [1] $p \in \mathcal{C}([T^*, +\infty), H^2)$ associated to P is the unique *solution $u \in \mathcal{C}([T_0, +\infty), H^2(\mathbb{R}))$ of (mKdV) such that*

$$\|u(t) - P(t)\|_{H^2} = O\left(\frac{1}{t^N}\right), \qquad \text{as} \quad t \to +\infty. \tag{93}$$

Theorem 6 is then an immediate consequence of Propositions 18 and 7.

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