SELF-SIMILAR SOLUTIONS AND CRITICAL SPACES FOR THE MODIFIED KORTEWEG-DE VRIES EQUATION

RAPHAËL CÔTE

ABSTRACT. We present some results obtained in collaboration with Simão Correia (University of Lisbon) and Luis Vega (University of Bilbao), regarding the understanding of self-similar solutions for the modified Korteweg-de Vries equation (mKdV).
We obtain the description of self-similar solutions in Fourier space, and we also prove a local well posedness result in a critical space where self-similar solutions live. As a consequence, we can study the flow of (mKdV) around self-similar solutions: in particular, we give an asymptotic description of small solutions as $t \to +\infty$ and construct solutions with a prescribed blow up behavior as $t \to 0$.

1. INTRODUCTION

1.1. **Motivation.** In this review paper, we give an account of our work in collaboration with Simão Correia and Luis Vega [3, 4]. We are interested in the dynamics near self-similar solutions of the modified Korteweg-de Vries equation:

$$\text{(mKdV)} \qquad \partial_t u + \partial^3_{xxx} u + \epsilon \partial_x (u^3) = 0, \quad u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}.$$

The signum $\epsilon \in \{\pm 1\}$ indicates whether the equation is focusing or defocusing. In our framework, ϵ will play no major role.
The (mKdV) equation enjoys a natural scaling: if u is a solution then

$$u_\lambda(t,x) := \lambda^{1/3} u(\lambda t, \lambda^{1/3} x)$$

is also a solution to (mKdV). As a consequence, the self-similar solutions, which preserve their shape under scaling, are of special interest: they are solutions of the form

$$S(t,x) = t^{-1/3} V(t^{-1/3} x) \quad \text{for } t > 0, x \in \mathbb{R},$$

where $V : \mathbb{R} \to \mathbb{R}$ is the self-similar profile, so that $S_\lambda = S$. After an integration we see that the profile V solves a Painlevé II-type equation

$$V'' = \frac{1}{3} y V - \epsilon V^3 + \alpha. \tag{1}$$

for some $\alpha \in \mathbb{R}$.
Self-similar solutions play an important role for the (mKdV) flow: they exhibit an explicit blow up behavior, and they appear in the description of solutions for large time. Even for small and smooth initial data, solutions display a modified scattering where self-similar solutions naturally occur: we refer to the works by Hayashi and Naumkin [14, 13], which were revisited by Germain, Pusateri and Rousset [8] and Harrop-Griffiths [11].

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Another example where self-similar solutions of the (mKdV) equation are relevant is in the long time asymptotics of the so-called Intermediate Long Wave (ILW) equation. This equation occurs in the propagation of waves in a one-dimensional stratified fluid in two limiting cases. In the shallow water limit, the propagation reduces to the KdV equation, while in the deep water limit, it reduces to the so-called Benjamin-Ono equation. In a recent work, Bernal-Vilchis and Naumkin [2] study the large-time behavior of small solutions of the (modified) ILW, and they prove that in the so-called self-similar region the solutions tend at infinity to a self-similar solution of (mKdV).
Self-similar solutions and the (mKdV) flow are also relevant as a model for the behavior of vortex filament in fluid dynamics. Goldstein and Petrich [9] proposed the following geometric flow for the description of the evolution of the boundary of a vortex patch in the plane under the Euler equations:

$$\partial_t z = -\partial_{sss} z + \partial_s \bar{z} (\partial_{ss} z)^2, \quad |\partial_s z|^2 = 1,$$

where $z = z(t,s)$ is complex valued and parametrize by its arctlength s a plane curve which evolves in time t. A direct computation shows that its curvature solves the focusing (mKdV) (with $\epsilon = 1$), and self-similar solutions with a profile V solution to (1) correspond to logarithmic spirals making a corner (of angle $\pi - 2\alpha$): this kind of spirals are observed in a number of fluid dynamics phenomena. We refer to [19] and the references therein for more details. Let us also mention that we were also motivated by works by Banica and Vega for related questions, modeled by nonlinear Schrödinger type equations, see [1] for example.

1.2. **Description in Fourier space.** In the defocusing case $\epsilon = -1$, equation (1) actually corresponds to the Painlevé II equation, which has its own interest and was intensively studied. Very precise asymptotics where obtained for its solutions. For example (see [12]), in the case $\epsilon = -1$, $\alpha = 0$, for any $\kappa \in \mathbb{R}$, there exist a unique self similar solution V_κ defined for large enough $y \gg 1$ such that

$$V_\kappa(y) = \kappa \operatorname{Ai}(y) + O\left(y^{-1/4} e^{-\frac{4}{3\sqrt{3}} y^{3/2}} \right) \quad \text{as} \quad y \to +\infty, \tag{2}$$

where Ai is the Airy function

$$\operatorname{Ai}(y) := \frac{1}{\pi} \int_0^{+\infty} \cos\left(\xi^3 + y\xi\right) d\xi.$$

Also, any solution to (1) which tends to 0 as $y \to +\infty$ is one of the V_κ. If furthermore $\kappa \in (-1,1)$, V_κ is defined on $\mathbb{R}$ and

$$V_\kappa(y) = \frac{2\sqrt{\rho}}{|3y|^{1/4}} \cos\left(\frac{2}{3\sqrt{3}} |y|^{3/2} - \frac{3}{2} \rho \ln|y| + \theta \right) + O\left(|y|^{-5/4} \ln|y| \right) \quad \text{as} \quad y \to -\infty \tag{3}$$

$$\text{where} \quad \rho = \frac{1}{2\pi} \ln\left(\frac{1}{1-\kappa^2} \right) \quad \text{and} \quad \theta = -3\rho \left(\ln 2 + \frac{1}{4} \ln 3 \right) + \ln \Gamma(i\rho) + \frac{\pi}{2} \operatorname{sgn} \kappa - \frac{\pi}{4}.$$

(Γ denotes the Gamma function). Recall for comparison the asymptotics of the Airy function:

$$\operatorname{Ai}(y) = \frac{1}{\sqrt{\pi}(3y)^{1/4}} e^{-\frac{2}{3\sqrt{3}} y^{3/2}} + O\left(y^{-5/4} e^{-\frac{2}{3\sqrt{3}} y^{3/2}} \right) \qquad \text{as} \quad y \to +\infty,$$

$$\operatorname{Ai}(y) = \frac{1}{\sqrt{\pi}|3y|^{1/4}} \cos\left(\frac{2}{3\sqrt{3}} |y|^{3/2} - \frac{\pi}{4} \right) + O\left(|y|^{-5/4} \ln|y| \right) \qquad \text{as} \quad y \to -\infty.$$

One can actually have a full asymptotic expansion (at $\pm\infty$) for the functions above.
If $|\kappa| = 1$, V_κ is still global but is no longer oscillatory as $y \to -\infty$ (it is equivalent to $\sqrt{|y|/2}$ and has a full asymptotic expansion); when $|\kappa| > 1$, V is no longer defined on $\mathbb{R}$ (it has an infinite number of poles).
We refer to the works by Hastings and McLeod [12] and Deift and Zhou [5] and the reference therein for the above results, and more (see also [6] and the book [7]).
In the work of Perelman and Vega [19], related results were obtained in the focusing case $\epsilon = 1$, using only ODE techniques. ((1) is scaled differently compared to the presentation in those works, this accounts for the difference in the constants).

However, little is known on the Fourier side, and this ought to be relevant when studying the (mKdV) flow, because dispersive properties are better captured in Fourier space. Our first result give the leading terms of the Fourier transform of a self-similar profile V.

Theorem 1. *Given $c, \alpha \in \mathbb{R}$ small enough, there exists unique $a \in \mathbb{R}$, $A, B \in \mathbb{C}$ and a self-similar solution $S(t,x) = t^{-1/3}V(t^{-1/3}x)$ to* (mKdV)*, where V satisfies*

$$\text{(4)} \qquad \text{for } \xi \geqslant 2, \quad e^{-it\xi^3}\hat{V}(\xi) = Ae^{ia\ln|\xi|} + B\frac{e^{3ia\ln|\xi| - i\frac{8}{9}\xi^3}}{\xi^3} + z(\xi),$$

$$\text{(5)} \qquad \text{for } |\xi| \leqslant 1, \quad e^{-it\xi^3}\hat{V}(\xi) = c + \frac{3i\alpha}{2\pi}\operatorname{sgn}(\xi) + z(\xi),$$

where $z \in W^{1,\infty}(\mathbb{R})$, $z(0) = 0$ and for any $k < \frac{4}{7}$, $|z(\xi)| + |\xi z'(\xi)| = O(|\xi|^{-k})$ as $|\xi| \to +\infty$.
The constant a and B are related to A by

$$\text{(6)} \qquad a = -\frac{3}{4\pi}|A|^2, \quad B = -\frac{3i\epsilon}{16\pi\sqrt{2}}e^{ia\ln 3}|A|^2 A.$$

Finally, the map $(c, \alpha) \mapsto A$ is one-to-one onto an adequate neighbourhood of $0 \in \mathbb{C}$*, bi-Lipschitz, and maps* $(0,0)$ *to* 0*.*

First observe that the knowledge of $\hat{V}$ for positive frequencies $\xi > 0$ gives a complete description: for $\xi < 0$, $\hat{V}(\xi) = \overline{\hat{V}(-\xi)}$ and $z(\xi) = \overline{z(-\xi)}$.
In particular, V is continuous if and only if $\alpha = 0$, and otherwise has a jump discontinuity of size $\dfrac{3i}{\pi}\alpha$ at $\xi = 0$. Due to the control on the remainder z, the self-similar solution S generated by V satisfies

$$\text{(7)} \qquad S(t) \rightharpoonup c\delta_0 + \alpha\,\text{v.p.}\left(\frac{1}{x}\right) \quad \text{as } t \to 0^+, \quad \text{where} \quad c = \int V(y)dy,$$

(and the mean c of V is well defined).
In the description of $\hat{V}$, the term in B plays a role only in the expansion of the derivative: even though it decays, its high oscillations mean that it is also a leading order term for the derivative $\hat{V}'$, with decay $1/\xi$ like the term in A.
Let us also notice that the parameters A, B and a may vary, but the phase $-8\xi^3/9$ in the term in B is completely constrained. A is related to c, α by an integral expression which is explicit but not very tractable (see (20) and (21)): it would be nice to have a more computable link.

As a consequence of the explicit Fourier expansion, we are able to link the profile constructed in Theorem 3, with the V_κ defined in (2).

Proposition 2. *Fix $\epsilon = -1$ and $\alpha = 0$. Then the solution V constructed in Theorem 3 coincides with V_κ defined in* (2)*, where A and κ are related via the relation*

$$|A|^2 = 2\ln\left(\frac{1}{1-\kappa^2}\right), \quad \text{and } \operatorname{Re} A \text{ and } \kappa \text{ have same sign.} \tag{8}$$

1.3. **Well posedness of** (mKdV) **in a critical space.** Now that self similar solutions are well understood, we can start our study of the flow of (mKdV) around them.
We first need a few notations. We denote by $\mathcal{G}(t)$ the linear KdV group:

$$\widehat{\mathcal{G}(t)v}(\xi) = e^{it\xi^3}\hat{v}(\xi),$$

for any $v \in \mathcal{S}'(\mathbb{R})$. Given a (space-time) function u, we denote $\tilde{u}$ the function defined by

$$\tilde{u}(t,\xi) := \widehat{\mathcal{G}(-t)u(t)}(\xi) = e^{-it\xi^3}\hat{u}(t,\xi). \tag{9}$$

For $v \in \mathcal{S}'(\mathbb{R})$ such that $\hat{v} \in L^\infty \cap \dot{H}^1$, and for $t > 0$, we define the norm (depending on t):

$$\|v\|_{\mathcal{E}(t)} := \|\widehat{\mathcal{G}(-t)v}\|_{L^\infty(\mathbb{R})} + t^{-1/6}\|\partial_\xi \widehat{\mathcal{G}(-t)v}\|_{L^2((0,+\infty))}. \tag{10}$$

As before, the knowledge of frequencies $\xi > 0$ is enough to completely determine a solution $u(t)$, and in the above definition, the purpose of considering $L^2((0,+\infty))$ is to allow a jump at 0, as is exhibited in the self similar profiles V of Theorem 1 when $\alpha \neq 0$.
Let us emphasize that the $\mathcal{E}(t)$ norm is scaling invariant, in the following sense:

$$\|u_\lambda(t)\|_{\mathcal{E}(t)} = \|u(\lambda t)\|_{\mathcal{E}(\lambda t)}.$$

In particular, self-similar solution have constant $\mathcal{E}(t)$ norm for $t \in (0,+\infty)$.
We define the functional space

$$\mathcal{E}(1) := \{u \in \mathcal{S}'(\mathbb{R}) : \|u\|_{\mathcal{E}(1)} < +\infty\},$$

and for $I \subset (0,+\infty)$,

$$\mathcal{E}(I) = \{u : I \to \mathcal{S}'(\mathbb{R}) : \tilde{u} \in \mathscr{C}(I, \mathscr{C}_b((0,+\infty))), \partial_p \tilde{u} \in L^\infty(I, L^2((0,+\infty)))\},$$

endowed with the norm $\sup_{t\in I} \|\cdot\|_{\mathcal{E}(t)}$ ($\mathscr{C}_b$ means continuous and bounded).
We can now state our results. The first is a local well-posedness result in the space $\mathcal{E}(I)$, for initial data $u_1 \in \mathcal{E}(1)$ at time $t = 1$ (in this setting, initial data at time 0 make little sense).

Theorem 3. *Let $u_1 \in \mathcal{E}(1)$. Then there exist $T > 1$ and a solution $u \in \mathcal{E}([1/T, T])$ to* (mKdV) *such that $u(1) = u_1$.*
Furthermore, one has forward uniqueness. More precisely, let $0 < t_1 < t_2$ and u and v be two solutions to (mKdV) *such that $u, v \in \mathcal{E}([t_1,t_2])$. If $u(t_1) = v(t_1)$, then for all $t \in [t_1,t_2]$, $u(t) = v(t)$.*

For small data in $\mathcal{E}(1)$, the solution is actually defined for large times, and one can describe the asymptotic behavior. This is the content of our second result.

Theorem 4. *There exists $\delta > 0$ small enough such that the following holds.*
If $\|u_1\|_{\mathcal{E}(1)} \leqslant \delta$, the corresponding solution satisfies $u \in \mathcal{E}([1,+\infty))$. Furthermore, let S be the self-similar solution such that

$$\hat{S}(1,0^+) = \hat{u}_1(0^+) \in \mathbb{C}.$$

Then $\|u(t)-S(t)\|_{L^\infty} \lesssim \|u_1\|_{\mathscr{E}(1)} t^{-5/6^-}$ *and there exists a profile* $U_\infty \in \mathscr{C}_b(\mathbb{R}\setminus\{0\},\mathbb{C})$*, with* $|U_\infty(0^+)| = \lim_{p\to+\infty} |\hat{S}(1,p)|$ *is well-defined, and*

$$\left|\tilde{u}(t,p) - U_\infty(p)\exp\left(\frac{i}{4\pi}|U_\infty(p)|^2 \log t\right)\right| \lesssim \frac{\delta}{\langle p^3 t\rangle^{\frac{1}{12}}} \|u_1\|_{\mathscr{E}(1)}.$$

As a consequence, we infer the asymptotic for large times in the physical space.

Corollary 5. *We use the notations of Theorem 4, and let*

$$y = \begin{cases} \sqrt{-x/3t}, & \text{if } x<0,\\ 0, & \text{if } x>0. \end{cases}$$

One has, for all $t \geqslant 1$ *and* $x \in \mathbb{R}$*,*

$$\left|u(t,x) - \frac{1}{t^{1/3}}\operatorname{Ai}\left(\frac{x}{t^{1/3}}\right) U_\infty(y)\exp\left(\frac{i}{6}|U_\infty(y)|^2\log t\right)\right| \lesssim \frac{\delta}{t^{1/3}\left\langle x/t^{1/3}\right\rangle^{3/10}}. \tag{11}$$

In proving Theorem 3 and 4, we use a framework derived from the work of Hayashi and Naumkin [13], improved so that only critically invariant quantities are involved. In particular, we use very similar multiplier identities and vector field estimates. An important new difficulty though is that to perform such energy-type inequalities, the precise algebraic structure of the problem has to be respected (for example, when performing integration by parts): the method truly requires nonlinear solutions. It thus seems that one cannot use a perturbative argument like fixed point. On the other hand, the rigorous derivation of such inequalities at our level of regularity is quite nontrivial.
This problem does not appear in [13] as the authors work in a (weighted) subspace of H^1, for which a nice local (and global) well-posedness result hold ((mKdV) is actually well-posed in H^s for $s \geqslant 1/4$, see [16]). However, no nontrivial self-similar solution belongs to these spaces (even without jump). Let us also mention the work by Grünrock and Vega [10], where local well-posedness is proved in

$$\widehat{H}^s_r = \{u \in \mathcal{S}'(\mathbb{R}) : \|\langle p\rangle^s \hat{u}\|_{L^{r'}} < +\infty\} \quad \text{for } 1<r\leqslant 2,\ s\geqslant \frac{1}{2}-\frac{1}{2r}.$$

This framework is not suitable for our purpose: self-similar belong to $\widehat{H}^0_1$ but not better. When finding a remedy for this, due to the jump at frequency 0 for self-similar solutions (5), we must take extra care on the choice of the functional setting. In particular, smooth functions *are not dense* in $\mathscr{E}$ spaces (and they can not approximate self-similar solutions).
In a nutshell, we face antagonist problems coming low and high frequencies, and we were fortunate enough to find an amenable approximate problem which take care of both simultaneously, and for which it is possible to derive uniform estimates in the spirit of [13]: we will give further details in Section 3.
At this point, we pass to the limit in n (Section 5), and a delicate but standard compactness argument allow to prove the existence part of Theorem 3 and Theorem 4. The description for large time (the second part of Theorem 4 and Corollary 5) is then a byproduct of the above analysis.
For the forward uniqueness, we look at the L^2 norm of the difference of two solutions. Of course, this does not make sense a priori, we don't have enough integrability., The point is to use the pointwise decay for functions in $\mathscr{E}(t)$, for $x \gg 1$ (a taste of the decay of the Airy

function on the right). For $x \ll -1$, the decay is too slow (like the Airy function, again), and we use a cut-off function to get rid of it. When computing the variation in (forward) time of the L^2 norm, this cut-off make a wild boundary term appear; but fortunately it comes with the right sign. This is related to a monotonicity property first observed and used by Kato [15], and a key feature in the study of the dynamics of solitons by Martel and Merle [18].
Backward uniqueness in Theorem 3 is open, although we would be surprised if it weren't the case. One can recover it under some extra decay information, namely that $u_1 \in L^2(\mathbb{R})$ (of course this is no longer a critical space), see [3, Proposition 4].

1.4. **A few further consequences.** The stability of self-similar solutions at blow-up time $t=0$, or more generally the behavior of solutions with initial data in $\mathcal{E}(1)$ near $t=0$ is a challenging question. In this direction, let us state two results.
As a byproduct of our analysis, we can construct solutions to (mKdV) with a given self-similar blow up profile as $t \to 0^+$, as shown below.

Proposition 6 (Blow-up solutions with a given profile). *For δ sufficiently small, given $g_0 \in \mathcal{E}(1)$ with $\|g_0\|_{\mathcal{E}(1)} < \delta$, there exists a solution $u \in \mathcal{E}((0,+\infty))$ of* (mKdV) *such that*

$$\forall t > 0, \quad \left\| t^{1/3} u(t, t^{1/3} x) - \widehat{g_0}(x) \right\|_{H^{-1}(\mathbb{R})} \lesssim \delta t^{1/3}. \tag{12}$$

In fact, even the description of the effects of small and smooth perturbations of self-similar solutions *for small time* is not trivial. For example, consider the toy problem of the linearized equation

$$\partial_t v + \partial_{xxx} v + \partial_x (K^2 v) = 0.$$

near the fundamental solution $K(t,x) = t^{-1/3} \operatorname{Ai}(t^{-1/3} x)$ of the linear Korteweg-de Vries equation (which is, in some sense, the self-similar solution to the linear problem). The most natural move is to use the estimates of Kenig, Ponce and Vega [17], which allows to recover the loss of a derivative:

$$\|v\|_{L^\infty_t L^2_x} \lesssim \|v(0)\|_{L^2_x} + \|K^2 v\|_{L^1_x L^2_t}.$$

Now one can essentially only use Hölder estimate:

$$\|K^2 v\|_{L^1_x L^2_t} \leqslant \|K\|^2_{L^4_x L^\infty_t} \|v\|_{L^2_{x,t}},$$

but due to the slow decay for $x \ll -1$, $K(t) \notin L^4_x$ for any t, and the argument can not be closed.
As a first step, we however prove a stability result of self-similar solutions at blow-up time, for low frequency perturbations. Given $\alpha > 0$ and a sequence $(a_k)_{k \in \mathbb{N}} \subset \mathbb{R}^+$ satisfying

$$a_0, a_1 = 1, \quad \text{and for all } k \geqslant 0, \quad a_k \leqslant \alpha a_{2k+1},$$

let us define the norm for the remainder:

$$\|v\|_{\mathcal{R}_\alpha} = \left(\sup_{k \geqslant 0} a_k \|\partial_x^k v\|^2_{L^2} \right)^{1/2}. \tag{13}$$

Then we have:

Proposition 7 (Stability of the self-similar blow-up under $\mathcal{R}_\alpha$-perturbations). *There exists $\delta > 0$ sufficiently small such that, if $w_1 \in \mathcal{R}_\alpha$ and S is a self-similar solution with*

$$\|w_1\|^2_{\mathcal{R}_\alpha} + \alpha \|S(1)\|^2_{\mathcal{E}(1)} < \delta,$$

then the solution u of (mKdV) *with initial data $u_1 = S(1) + w_1$ is defined on* $(0,1]$ *and*

$$\sup_{t\in(0,1)} \|u(t) - S(t)\|^2_{\mathscr{R}_\alpha} < 2\delta.$$

Obviously, we shrank considerably the critical space by taking smooth perturbations of self-similar solutions, but the above result still shows some kind of stability of self-similar blow up; observe in particular that the blow up time is *not* affected by the perturbation.

In what follows, we will give some insight of the main steps and of the main ideas of the proofs of Theorems 1, 3 and 4. Proposition 2, Corollary 5, Proposition 6 and Proposition 7 follow easily from the analysis developed for the theorems: we will not give further details on these matters and rather refer directly to [3, 4].

2. Outline of the proof of Theorem 1

We took a truly PDE perspective to prove Theorem 1. Although one could probably proceed using directly the asymptotics of self-similar solutions and performing adequate stationary phase estimates, we believe that PDE techniques would be useful when considering perturbation of self-similar solutions under the (mKdV) flow. Another reason is that the full asymptotic description is only known for the Painlevé II equation (with $\epsilon = -1$), and we are motivated by vortex filament dynamics, which is related to the focusing $\epsilon = 1$ (mKdV) equation.

In Fourier space, equation (1) takes the form

$$-\frac{i}{3}\hat{V}' = \xi^2\hat{V} - \epsilon\widehat{|V|^2V} + \frac{1}{2\pi}\alpha\delta_{\xi=0}.$$

(The choice of the nonlinearity $|V|^2V$ is by pure convenience: its allows for complex valued V, with no extra cost on the computations). Denote $v(\xi) = e^{-i\xi^3}\hat{V}(\xi)$. Then

$$v' = e^{-i\xi^3}(\hat{V}' - 3i\xi^2\hat{V}) = -3i\epsilon e^{-i\xi^3}\widehat{|V|^2V} + \frac{3i}{2\pi}\alpha\delta_{\xi=0}.$$

If we integrate, and write $\widehat{|V|^2V}$ in integral form, we see that we are looking for a fixed point

$$v = \Psi(v), \quad \text{with} \quad \Psi(v)(\xi) = c + \frac{3i}{2\pi}\alpha - \frac{3i\epsilon}{4\pi^2}\int_0^\xi I(v)(\eta)d\eta \quad \text{for } \xi > 0, \tag{14}$$

(and $\Psi(v)(\xi) = \overline{\Psi(v)(-\xi)}$ for $\xi < 0$) and $I(v) = \underline{I}(v,v,v)$, where $\underline{I}$ is the trilinear operator:

$$\underline{I}(f,g,h)(\xi) := e^{-i\xi^3}\iint_{\eta_1+\eta_2+\eta_3=\xi} e^{i(\eta_1^3+\eta_2^3+\eta_3^3)}f(\eta_1)g(\eta_2)\bar{h}(-\eta_3)d\eta_1 d\eta_2. \tag{15}$$

We will consider our fixed point to be of the form

$$v = W + z,$$

where W is our ansatz and z is small in some adequate functional space.

2.1. **Choosing a correct ansatz.** To find a good ansatz W, we proceed by iteration, in a way quite analoguous to the Picard scheme. First we considered $\Psi(\mathbb{1})$, because constants for v correspond to the Airy function for V, which is a solution to the linear part

$$\mathrm{Ai}'' = \frac{1}{3} y \,\mathrm{Ai}$$

of the Painlevé equation (1). In fact, the leading term of $\Psi(\mathbb{1})$ presents slow oscillations, of the form $e^{ia\ln|\xi|}$ for large ξ (this can be seen by computing the leading term of $I(\mathbb{1})$.
Then we compute the leading term of $\Psi(e^{ia\ln|\xi|})$: at least formally and for a correct choice of a, it is $e^{ia\ln|\xi|}$ itself!
The computations essentially relie on stationary phase type arguments. One of the main difficulties in completing this program is to obtain a correct estimation of the remainder terms. In the integrals involved in I, we see that the phases are *quadratic* (or cubic), which naturally leads to stationary phase estimates. This means a rather slow decay, and also the need to develop efficient bounds on the errors on the stationary phase. This should be done preferably in L^∞ based spaces: indeed, we have pointwise estimates on the main order terms, and the problem is critical in some sense (the ansatz has no decay at infinity for example), so that we can not afford to lose information.
This is in sharp contrast with the analogous problem for the nonlinear Schrödinger equation: the phases appearing in the integrals in that case are linear, thus are never stationary, and the analysis is much simpler.
We mainly use the following elementary lemma to bound the errors:

Lemma 8. *For any $\xi \neq 0$, and function $g \in W^{1,\infty}(\mathbb{R})$,*

$$\left| \int e^{i\xi\eta^2} g(\eta) d\eta - \sqrt{\frac{\pi}{|\xi|}} e^{i\frac{\pi}{4}\operatorname{sgn}(\xi)} g(0) \right| \lesssim \frac{\|g\|_\infty}{\sqrt{|\xi|}} + \frac{\|g'\|_\infty}{|\xi|} \tag{16}$$

Furthermore, if there exists $R > 0$ such that $\operatorname{supp} g \subset [-\xi R, \xi R]$, then for some $C = C(R)$,

$$\left| \int e^{i\xi\eta^2} g(\eta) d\eta - \sqrt{\frac{\pi}{|\xi|}} e^{i\frac{\pi}{4}\operatorname{sgn}(\xi)} g(0) \right| \leqslant C \frac{\ln|\xi|}{|\xi|} \|g'\|_\infty . \tag{17}$$

Lemma 8 is very thrifty regarding the number of derivatives it needs, but it still requires one. Now, when we consider the derivatives $\partial_\xi I(e^{ia\ln|\xi|})$, we see another term at leading order:

$$e^{3ia\ln|\xi|} \frac{e^{-i8\xi^3/9}}{\xi^3}.$$

This second term can not be avoided: it contributes $O(|\xi|^{-1})$ to the derivative, even if it has high decay (it is highly oscillating). Also, it requires that we do a distinct analysis for low and high frequencies. Fortunately, the introduction of this second term in the ansatz does not lead to a different asymptotic development for $\Psi(S)$ and we are able to complete the proof with the two terms ansatz given by

$$\forall \xi > 0, \quad W(\xi) := \chi(\xi) e^{ia\ln|\xi|} \left(A + B e^{2ia\ln|\xi|} \frac{e^{-i8\xi^3/9}}{\xi^3} \right), \tag{18}$$

with constants $A, B \in \mathbb{C}$ and $a \in \mathbb{R}$ to be chosen.

After all these computations are performed (with careful control on the error), one needs to adjust the constants A, B, a to get a consistent anstatz indeed. For this, we match the asymptotic of $\Psi(W)$ at 0 and ∞: this gives the condition (6). Hence the only remaining freedom in the ansatz W is now A, and we denote it W_A. These (lenghty) computations can be summarized in the following statement on the derivative of $\Psi(W_A)$.

Proposition 9. *Let $\gamma \in (6/7, 1)$. One has*

$$\forall \xi \in \mathbb{R}, \quad \left| -\frac{3i\epsilon}{4\pi^2} I(W_A)(\xi) - W_A'(\xi) \right| \lesssim \min(1, |\xi|^{-2+\gamma/2}). \tag{19}$$

Taking care of the constants is more delicate, because the fixed point is of the form $W_A + z$: although the small remainder z will not affect the oscillating terms, it does affect the constants c and α, and so the function $\Psi = \Psi_{c,\alpha}$: see (14), we now need to make the dependence explicit.
In other words, given $c, \alpha \in \mathbb{R}$, our goal is to find $A \in \mathbb{C}$ and a function z such that

$$\Psi_{c,\alpha}(W_A + z) = W_A + z.$$

Matching the constants in the asymptotic for $W_A + z$ and for $\Psi_{c,\alpha}(W_A + z)$ yields

$$c + \frac{3i}{2\pi}\alpha - \frac{\epsilon}{4\pi^2}(3i\mathscr{I}(A,z) - A) = 0 \quad \text{where}$$

$$\mathscr{I}(A,z) := \int_0^1 I(W_A+z)(\eta)d\eta + \int_1^\infty \left(I(W_A+z)(\eta) - \pi|A|^2 A \frac{e^{i\alpha \ln|\eta|}}{|\eta|} \right) d\eta. \tag{20}$$

Taking real and imaginary part in the above relation, we are led to solve the system

$$c = -\epsilon \operatorname{Re} A - \frac{3\epsilon}{4\pi^2} \operatorname{Im} \mathcal{I}(A,z) \quad \text{and} \quad \alpha = -\frac{2\pi\epsilon}{3} \operatorname{Im} A + \frac{\epsilon}{2\pi} \operatorname{Re} \mathscr{I}(A,z), \tag{21}$$

(and $\Psi_{c,\alpha}(W_A + z) = W_A + z$).

2.2. **Finding the fixed point.** First, we assume $A \in \mathbb{C}$ is given, and we construct a fixed point for the function

$$\tilde{\Psi}_A : z \mapsto \Psi_{c(A,z),\alpha(A,z)}(W_A + z) - W_A \tag{22}$$

where $\Psi_{c,\alpha}$ is defined in (14) and $c(A,z)$ and $\alpha(A,z)$ are *defined* by (21).
We work in weighted $W^{1,\infty}$-type spaces: define the Z^k norm by

(23)
$$\|z\|_{Z^k} := \|z(\xi)(1+|\xi|^k)\|_{L^\infty(\mathbb{R})} + \|z'(1+|\xi|^{k+1})\|_{L^\infty(0,+\infty)} + \|z'(1+|\xi|^{k+1})\|_{L^\infty(-\infty,0)}.$$

The key point are multilinear estimates.

Lemma 10. *Fix $k \in \left(\frac{1}{2}, \frac{4}{7}\right)$. Let $z, w, u \in Z^k$ and $A \in \mathbb{C}$, $|A| < 1$. Then*

(1) *(Linear estimate)*

$$|\underline{I}(z, S_A, S_A)(\xi)| \lesssim |A|^2 \|z\|_{Z^k} \min\{1, |\xi|^{-k-1}\}.$$

(2) *(Quadratic estimate)*

$$|\underline{I}(S_A, z, w)(\xi)| \lesssim |A| \|z\|_{Z^k} \|w\|_{Z^k} \min\{1, |\xi|^{-k-1}\}.$$

(3) *(Cubic estimate)*

$$|\underline{I}(z,w,u)(\xi)| \lesssim \|z\|_{Z^k}\|\|w\|_{Z^k}\|u\|_{Z^k}\min\{1,|\xi|^{-k-1}\}.$$

The proofs of these estimates are actually a byproduct of the computations performed in determining the leading term in the ansatz S_A. Also, $\underline{I}$ is *not* symmetric, but has weaker form of symmetry, and the above estimates are enough for our purposes: they allow to show that the map $(A,z) \mapsto (c(A,z), \alpha(A,z))$ is locally Lipschitz (essentially, it is cubic). As a consequence, a standard argument yields a fixed point for $\tilde{\Psi}_A$, as stated below.

Proposition 11. *For $A \in \mathbb{C}$, $|A| < \epsilon_1$ sufficiently small, the map $\tilde{\Psi}_A$ admits a (unique) fixed point which we denote $z_A \in Z^k$, and such that*

$$\|z_A\|_{Z^k} + \left\| z_A - \left(c(A,z_A) + \frac{3i}{2\pi}\alpha(A,z_A) \right) \eta^{-1} \right\|_{L^\infty(\eta\in(0,1))} < 3|A|.$$

(The second term on the left hand side is required to close the fixed point argument, we do need a better control on the low frequencies than what is provided by the Z^k norm).

Second, we prove that the map $A \mapsto (c(A,z_A), \alpha(A,z_A))$ is bijective locally around 0 (heuristically, it is because $\mathscr{I}(A,z)$ is cubic in A, z). Given c and α, its inverse provides the amplitude A to define the ansatz, and thus desired self-similar profile.
The key point for this is to get a Lipschitz continuity of the the function $\mathcal{I}(A,z)$, and more precisely, the statement below:

Lemma 12. *Fix $k \in \left(\frac{1}{2}, \frac{4}{7}\right)$. For any $\epsilon, \delta > 0$ sufficiently small, the following holds true. Let $A_1, A_2 \in \mathbb{C}$ with $|A_1|, |A_2| < \varepsilon$, and $z, w \in Z^k$ such that $\|z\|_{Z^k}, \|w\|_{Z^k} \leqslant 3\varepsilon$ then*

$$|\mathscr{I}(A_1,z) - \mathscr{I}(A_2,w)| \leqslant C\varepsilon^2(|A_1 - A_2| + \|z-w\|_{Z^k}), \tag{24}$$

$$\|z_{A_1} - z_{A_2}\|_{Z^{k-\delta}} \leqslant C(\delta)\varepsilon^2|A_1 - A_2|, \tag{25}$$

where $\mathcal{I}$ is as in (20) *and $C(\delta)$ only depends on δ.*

When computing the difference to prove the above result, one sees that there is a logarithmic loss $\ln|\xi|$: it us due to the oscillating phase in $e^{ia\ln|\xi|}$, in view of the expression of a in terms of A (luckily, the phase in the highly oscillating term $e^{-8i/9|\xi|^3}$ does not depend on A, this is crucial!). This loss can be compensated by decreasing slightly the parameter k to absorb the log loss – this is the purpose of introducing $\delta > 0$. Then we recover Lipschitz continuity for the maps we are interested in.

This δ loss is not problematic, because we are now only interested in the finite dimensional map

$$A \mapsto (c(A,z_A), \alpha(A,z_A)).$$

With Lemma 12 in hand, one easily sees that this maps is a Lipschitz perturbation (with Lipschitz constant $O(|A|^2)$) of an invertible $\mathbb{R}$-linear map $\mathbb{C} \to \mathbb{R}^2$, and so is invertible near $A = 0$. This concludes the proof of Theorem 1.

3. OULINE OF THE PROOF OF THEOREM 3 AND 4

If one takes the Fourier transform (in space) of (mKdV), we see that this equation writes

$$\partial_t \tilde{u}(t,\xi) = -\frac{1}{4\pi^2}\mathcal{N}[u](t,\xi), \tag{26}$$

where the nonlinearity is

(27)

$$\mathcal{N}[u](t,\xi) = i\xi^3 \iint_{\eta_1+\eta_2+\eta_3=1} e^{-itp\xi^3(1-\eta_1^3-\eta_2^3-\eta_3^3)}\tilde{u}(t,\xi\eta_1)\tilde{u}(t,\xi\eta_2)\tilde{u}(t,\xi\eta_3)d\eta_1 d\eta_2.$$

3.1. The leading term in the nonlinearity.

An important computation is a stationary phase lemma for $\mathcal{N}[u]$.

Lemma 13 (Asymptotics of the nonlinearity on the Fourier side)**.** *Let $u \in \mathcal{E}(I)$. One has the following asymptotic development for $\mathcal{N}[u]$: for all $t \in I$ and $\xi > 0$,*

$$\mathcal{N}[u](t,\xi) = \frac{\pi p^3}{\langle p^3 t\rangle}\left(i|\tilde{u}(t,\xi)|^2\tilde{u}(t,p) - \frac{1}{\sqrt{3}}e^{-\frac{8itp^3}{9}}\tilde{u}^3\left(t,\frac{\xi}{3}\right)\right) + R[u](t,\xi) \tag{28}$$

where the remainder R satisfies the bound

$$|R[u](t,\xi)| \lesssim \frac{p^3\|u(t)\|^3_{\mathcal{E}(t)}}{(\xi^3 t)^{5/6}\langle\xi^3 t\rangle^{1/4}}. \tag{29}$$

Similar statements may be found in [13, Lemma 2.4] and [8]. The specificity of our result is that we have limited spatial decay (we emphasize that we work in a critical space), and so computations and the estimations of the errors have to be performed very carefully, with as few integration by parts as possible.

In the same spirit, we have pointwise decay estimates, which somehow transcribe the decay of the Airy function.

Lemma 14 (Decay estimates)**.** *Let $I \subset (0,+\infty)$ be a time interval, and $u \in \mathcal{C}(I,\mathcal{S}')$ such that $\|u\|_{\mathcal{E}(I)} < +\infty$. Then there hold $u \in \mathcal{C}(I, L^\infty_{\mathrm{loc}}(\mathbb{R}))$ and more precisely, for $t \in I$ and $x \in \mathbb{R}$, one has*

$$|u(t,x)| \lesssim \frac{1}{t^{1/3}\left\langle |x|/t^{1/3}\right\rangle^{1/4}}\|u(t)\|_{\mathcal{E}(t)}, \tag{30}$$

$$|\partial_x u(t,x)| \lesssim \frac{1}{t^{2/3}}\left\langle |x|/t^{1/3}\right\rangle^{1/4}\|u(t)\|_{\mathcal{E}(t)}, \tag{31}$$

$$\text{and if } x \geqslant t^{1/3}, \quad |u(t,x)| \lesssim \frac{1}{t^{1/12}x^{3/4}}\|u(t)\|_{\mathcal{E}(t)}. \tag{32}$$

3.2. The approximate problem.

This approximate problem is actually a variant of the Friedrichs scheme where we filter out high frequency via a cut-off function χ_n in Fourier space. We solve this approximate problem using a fixed point argument: the cut-off takes care of the lack of decay for large frequency, but again, smooth functions are not dense in the space X_n where the fixed point is found (X_n is a version of $\mathcal{E}$ where high frequencies are tamed). Finally, in order to obtain uniform estimates, it turns out that, due to the absence of decay for large frequencies of self-similar solutions, boundary terms cannot be neglected unless the cut-off function χ_n is chosen in a very particular way.

Let us now be more specific. One can convince oneself that there exists a sequence of even decreasing functions $(\chi_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R})$ such that

- for all $n \in \mathbb{N}$, $0 < \chi_n \leqslant 1$, $\chi_n^{1/2} \in \mathcal{S}(\mathbb{R})$,
- $\chi_n(\xi) = 1$ for $|\xi| \leqslant n$.
- $\sup_{\xi\in\mathbb{R}} |\xi(\chi_n^{1/2})'(\xi)| \to 0$ as $n \to +\infty$.

(The third condition is most delicate). We consider the approximate problem

$$(\Pi_n\text{-mKdV}) \qquad \begin{cases} \partial_t u + \partial_{xxx} u + \Pi_n \partial_x(u^3) = 0, \\ u(1) = \Pi_n u_1, \end{cases} \qquad \widehat{\Pi_n u} := \chi_n \hat{u}.$$

where $u_1 \in \mathscr{E}(1)$ is given. Define the norm

$$\|u\|_{X_n(t)} := \|\widehat{\mathcal{G}(-t)u}\chi_n^{-1}\|_{L^\infty} + \left\|\partial_\xi(\widehat{\mathcal{G}(-t)u})\chi_n^{-1/2}\right\|_{L^2((0,+\infty))} \tag{33}$$

and the space

$$X_n(I) := \left\{u \in \mathscr{C}(I, \mathcal{S}'(\mathbb{R})) : \tilde{u}\chi_n^{-1} \in \mathscr{C}(I, \mathscr{C}_b((0,+\infty))),\ \partial_p \tilde{u}\chi_n^{-1/2} \in \mathscr{C}(I, L^2((0,+\infty)))\right\}.$$

We emphasize that functions in $X_n(I)$ are allowed to have a Fourier transform with a jump at 0, in particular, they are not Schwarz class. Observe that if $u \in \mathscr{E}(1)$, then

$$\|\Pi_n u_1\|_{X_n(1)} \leqslant \|u_1\|_{\mathscr{E}(1)}.$$

The first step is the following existence result for the approximate problem.

Proposition 15. *Given any $u_1 \in \mathscr{E}(1)$, there exists $T_{-,n} < 1$, $T_{+,n} > 1$ and a unique $u_n \in X_n((T_{-,n}, T_{+,n}))$ maximal solution of* $(\Pi_n$-mKdV). *Moreover, if $T_{+,n} < \infty$, then*

$$\lim_{t\to T_{+,n}} \|u(t)\|_{X_n(t)} = +\infty.$$

(A similar statement holds at $T_{-,n}$). In particular, $u_n \in \mathscr{E}((T_{-,n}, T_{+,n}))$.

This is done by a standard fixed point argument. Then we need to show some uniform bounds on u_n. This is done by considering the multiplier operator

$$\widehat{\mathcal{I}u}(t,\xi) = i\partial_\xi \hat{u}(t,\xi) - \frac{3it}{p}\partial_t \hat{u}(t,p) = ie^{itp^3}\left(\partial_p \tilde{u} - \frac{3t}{\xi}\partial_t \tilde{u}\right).$$

It corresponds to the formal dilation operator

$$x + 3t\int_{-\infty}^{x} \partial_t dx',$$

which was already used by Hayashi and Naumkin (but in a setting where its formulation in physical space was well defined). The operator $\mathcal{I}$ is very convenient because

- it enjoys nice commutation relation with the linear KdV operator $\partial_t + \partial_{xxx}$,
- it interacts well with the nonlinearity of (mKdV), (because it is "first order").
- $\mathcal{I}u$ relates nicely to $\partial_\xi \tilde{u}$.

We furthermore observed that $\mathcal{I}$ behaves well with the approximate problem (Π_n-mKdV) too, *provided that the cut-off χ_n is chosen as above*, with errors term which can be controlled. As a consequence, we obtain uniform bounds on $\mathcal{I}u_n$ related quantities. This is summarized below.

Lemma 16 ($\dot{H}^1$ bound for (Π_n-mKdV)). *There exist $\kappa > 0$ such that the following holds. Given $u_1 \in \mathscr{E}(1)$, the corresponding solution u_n of* (Π_n-mKdV) *is defined on $[T_{-,n}, T_{+,n}]$ and satisfies*

$$\widehat{\mathcal{I}u_n} \in \mathscr{C}^1([T_{-,n}, T_{+,n}], L^2((0,+\infty), \chi_n^{-1} dp)).$$

and

$$\forall t \in [1, T_{+,n}], \quad \|\widehat{\mathcal{I}u_n}(t)\chi_n^{-1/2}\|_{L^2((0,+\infty))} \leqslant \|\widehat{\mathcal{I}u_n}(1)\chi_n^{-1/2}\|_{L^2((0,+\infty))} t^{\kappa\|u_n\|^2_{\mathscr{E}([1,t])}} + o_n(1)\|u_n\|^3_{\mathscr{E}([1,t])} t^{1/6} \tag{34}$$

$$\forall t \in [T_{-,n}, 1], \quad \|\widehat{\mathcal{I}u_n}(t)\chi_n^{-1/2}\|_{L^2((0,+\infty))} \leqslant \|\widehat{\mathcal{I}u_n}(1)\chi_n^{-1/2}\|_{L^2((0,+\infty))} t^{-\kappa\|u_n\|^2_{\mathscr{E}([t,1])}} + o_n(1)\|u_n\|^3_{\mathscr{E}([t,1])} t^{1/6}. \tag{35}$$

This lemma is the crux of the proof of Theorem 3 and 4. Here, the third condition on χ_n is crucially needed, and we also relie on a nice cancellation.
Using the relation between $\mathcal{I}u$ and $\partial_t \tilde{u}$, we see that the above control can be bootstraped to a control of $\mathscr{E}(t)$, and we get uniform bound on a uniform interval:

Proposition 17 (Uniform local existence for large data). *There exist a function $C : \mathbb{R}_+ \to \mathbb{R}_+$ such that the following holds. Given $u_1 \in \mathscr{E}(1)$, there exists $T_-(u_1) < 1$ and $T_+(u_1) > 1$ such that, for large n, the corresponding solution u_n of* (Π_n-mKdV) *is defined on $[T_-(u_1), T_+(u_1)]$ and*

$$\|u_n\|_{\mathscr{E}([T_-(u_1), T_+(u_1)])} \leqslant C(\|u_1\|_{\mathscr{E}(1)}).$$

Surprisingly, the proof of this proposition goes by contradiction, because our estimates give bounds but no uniform continuity: this is one of the side effect of working in critical space. If we furthermore assume that $\|u_1\|_{\mathscr{E}(1)}$ is small, a similar statement holds with $T_+(u_1) = +\infty$.

3.3. **Construction of the solution and forward uniqueness.** Proposition 17 is what is needed to develop a somewhat standard compactness argument, which yields the existence part in Theorems 3 and 4.

It remains the forward uniqueness property which we consider now. It proceeds via a completely different and independent argument. Without loss of generality, we can assume that $t_1 = 1$. As mentioned in the introduction, given two solutions u and v which coincide at $t = 1$, we consider the difference $w = u - v$. w satisfies

$$(\partial_t + \partial_{xxx})w = ((w+v)^3 - v^3)_x, \quad w(1) = 0.$$

We consider a cut-off function $\phi \in C^\infty(\mathbb{R})$, non-decreasing and such that $\phi(x) = 0$ for $x < 0$ and $\phi(x) = 1$ for $x > -1$, and for $n \geqslant 1$, let $\phi_n(x) = \phi(1 + x/n)$, so that

$$\phi_n(x) \to 1, \quad \|\partial_{xx}\phi_n\|_{L^1} \to 0.$$

Via a regularization by mollification, we can establish an identity for the (localized L^2) norm of w:

$$\frac{1}{2}\int w^2 \phi_n dx = \int_1^t \int \left(-\frac{3}{2}(\partial_x w)^2 \partial_x \phi_n - (w \partial_x w)\partial_{xx}\phi_n + \partial_x((w+v)^3 - v^3) w \phi_n \right) dx ds \tag{36}$$

This takes advantage of the pointwise decays obtained in Lemma 14, which show (among other things) that

$$w \in L^2(\phi_n dx), \quad \partial_x w \in L^2(\partial_x \phi_n dx).$$

Now, as ϕ is chosen to be non-decreasing, the first term of right-hand side of (36) is negative, and so we get that for $t \geqslant 1$,

$$\frac{1}{2}\int w(t)^2 \phi_n dx \leqslant \int_1^t \left(\|w\partial_x w\|_{L^\infty} \|\partial_{xx}\phi_n\|_{L^1} + \int \partial_x((w+v)^3 - v^3) w \phi_n dx \right) ds$$
$$\lesssim \|w\|^2_{\mathscr{E}([t_1,t_2])} \|\partial_{xx}\phi_n\|_{L^1} \log t + \frac{\|u\|^2_{\mathscr{E}([t_1,t_2])} + \|v\|^2_{\mathscr{E}([t_1,t_2])}}{t} \int w(t)^2 \phi_n dx$$

We are in a position to apply a Gronwall argument. Letting $n \to +\infty$, we get that for $t \geqslant 1$, $\|w(t)\|_{L^2} = 0$, and the proof is complete.

Actually, from there, we can give some partial continuity for $\partial_\xi \tilde{u}$. A little further work with (34) and (35) gives continuity at $t = 1$. For $t_1 \neq 1$, we consider the solution v constructed as before with given data at t_1 (instead of 1), and $v(t_1) = u(t_1)$. As we just saw, v is continuous at t_1, and, using forward uniqueness, u and v coincide for $t \geqslant t_1$: hence the map $[1/T, T] \to L^2$, $t \mapsto \partial_\xi \tilde{u}$ is continuous to the right.
Backward uniqueness would similarly yield continuity to the left, and so the full continuity $\partial_\xi \tilde{u} \in \mathscr{C}([1/T, T], L^2)$.

References

[1] V. Banica and L. Vega. Evolution of polygonal lines by the binormal flow. *arXiv:1807.06948*, 2018.

[2] Fernando Bernal-Vilchis and Pavel I. Naumkin. Self-similar asymptotics for solutions to the intermidiate long wave equation. *J.of Evol. Eqns., to appear*, 2019.

[3] Simão Correia, Raphaël Côte, and Luis Vega. Asymptotics in fourier space of self-similar solutions to the modified korteweg-de vries equation. *Preprint, arXiv:1807.02302*, 2018.

[4] Simão Correia, Raphaël Côte, and Luis Vega. Self-similar dynamics for the modified korteweg-de vries equation. *Preprint, arXiv:1904.04524*, 2019.

[5] P. A. Deift and X. Zhou. Asymptotics for the Painlevé II equation. *Comm. Pure Appl. Math.*, 48(3):277–337, 1995.

[6] A. S. Fokas and M. J. Ablowitz. On the initial value problem of the second Painlevé transcendent. *Comm. Math. Phys.*, 91(3):381–403, 1983.

[7] Athanassios S. Fokas, Alexander R. Its, Andrei A. Kapaev, and Victor Yu. Novokshenov. *Painlevé transcendents*, volume 128 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2006. The Riemann-Hilbert approach.

[8] Pierre Germain, Fabio Pusateri, and Frédéric Rousset. Asymptotic stability of solitons for mKdV. *Adv. Math.*, 299:272–330, 2016.

[9] R.E. Goldstein and D.M. Petrich. Solitons, Eulers equations, and vortex patch dynamics. *Phys. Rev. Lett.*, 69(4):555–558, 1992.

[10] Axel Grünrock and Luis Vega. Local well-posedness for the modified kdv equation in almost critical $\widehat{H^s_r}$-spaces. *Trans. Amer. Math. Soc.*, 361:5681–5694, 2009.

[11] Benjamin Harrop-Griffiths. Long time behavior of solutions to the mKdV. *Comm. Partial Differential Equations*, 41(2):282–317, 2016.

[12] S. P. Hastings and J. B. McLeod. A boundary value problem associated with the second Painlevé transcendent and the Korteweg-de Vries equation. *Arch. Rational Mech. Anal.*, 73(1):31–51, 1980.

[13] Nakao Hayashi and Pavel Naumkin. On the modified Korteweg-de Vries equation. *Math. Phys. Anal. Geom.*, 4(3):197–227, 2001.

[14] Nakao Hayashi and Pavel I. Naumkin. Large time behavior of solutions for the modified Korteweg-de Vries equation. *Internat. Math. Res. Notices*, (8):395–418, 1999.
[15] Tosio Kato. On the Cauchy problem for the (generalized) Korteweg-de Vries equation. In *Studies in applied mathematics*, volume 8 of *Adv. Math. Suppl. Stud.*, pages 93–128. Academic Press, New York, 1983.
[16] Carlos E. Kenig, Gustavo Ponce, and Luis Vega. Well-posedness and scattering result for the generalized Korteweg-De Vries equation via contraction principle. *Comm. Pure Appl. Math.*, 46:527–620, 1993.
[17] Carlos E. Kenig, Gustavo Ponce, and Luis Vega. On the concentration of blow up solutions for the generalized KdV equation critical in L^2. In *Nonlinear wave equations (Providence, RI, 1998)*, volume 263 of *Contemp. Math.*, pages 131–156. Amer. Math. Soc., Providence, RI, 2000.
[18] Yvan Martel and Frank Merle. A Liouville theorem for the critical generalized Korteweg-de Vries equation. *J. Math. Pures Appl. (9)*, 79:339–425, 2000.
[19] G. Perelman and L. Vega. Self-similar planar curves related to modified Korteweg-de Vries equation. *J. Differential Equations*, 235(1):56–73, 2007.

RAPHAËL CÔTE
Université de Strasbourg
CNRS, IRMA UMR 7501
F-67000 Strasbourg, France
cote@math.unistra.fr