PERTURBATION AT BLOW-UP TIME OF SELF-SIMILAR SOLUTIONS FOR THE MODIFIED KORTEWEG-DE VRIES EQUATION

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Abstract. We prove a first stability result of self-similar blow-up for the modified Korteweg-de Vries equation on the line. More precisely, given a self-similar solution and a sufficiently small regular profile, there is a unique global solution which behaves at $t = 0$ as the sum of the self-similar solution and the smooth perturbation.

1. Introduction

1.1. **Description of the problem and motivation.** We consider the modified Korteweg-de Vries equation on the whole line

$$\partial_t u + \partial_{xxx} u = \pm\partial_x(u^3), \quad (t,x) \in \mathbb{R}^2, \quad u(t,x) \in \mathbb{R}. \tag{mKdV}$$

Throughout this work, the specific sign of the nonlinearity is irrelevant. To simplify the exposition, we treat the focusing case (with the $+$ sign), even though the results presented also hold for the defocusing one.

This equation admits a scaling invariance: if u is a solution, so is $u_\lambda(x,t) = \lambda u(\lambda^3 t, \lambda x)$, for any $\lambda > 0$. As a consequence, one may look for self-similar solutions of (mKdV), which are invariant under scaling. A simple computation shows that these solutions are of the form

$$S(t,x) = \frac{1}{t^{1/3}} V\left(\frac{x}{t^{1/3}}\right) \quad \text{where} \quad V'' - \frac{y}{3}V = V^3 + \alpha,\ \alpha \in \mathbb{R}. \tag{1.1}$$

The existence of profiles V can be studied using either ODE techniques ([9, 10, 17, 22]) or stationary phase arguments ([6]). Very precise asymptotics were obtained in both physical and frequency space. Generally speaking, self-similar profiles have the same behavior as the Airy function (which solves the linear equation), up to some logarithmic corrections. In physical space, the profiles have weak decay and strong oscillations as $x \to -\infty$. On Fourier side, a jump discontinuity at the zero frequency appears for $\alpha \neq 0$ and no decay is available for large frequencies (see Proposition 4 for a precise description).

As it turns out, self-similar solutions determine the behavior of small solutions for large times. This was first seen by Deift and Zhou in [8] using inverse scattering techniques, under strong smoothness and decay assumptions. In Hayashi and Naumin's works [18, 19], the phenomena was proven as a consequence of modified scattering. This was later revisited in [11] and [15]. On the other hand, self-similarity

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induces a natural blow-up behavior at $t = 0$. This singularity is directly connected to some geometric flows. Indeed, (mKdV) appears in the modeling of the evolution of the boundary of a vortex patch on the plane subject to Euler's equations ([12]) and in the study of vortex filaments in $\mathbb{R}^3$. In these models, self-similar blow-up is connected to the formation of logarithmic spirals (if $\alpha = 0$, one observes a sharp corner).

For geometric flows modeled by the cubic nonlinear Schrödinger equation, we advise the reader to look at the series of papers [1, 2, 3] by Banica and Vega, and references therein. Both the cubic (NLS) and the (mKdV) equations are L^1-critical. This feature translates a critical polynomial behavior of the nonlinearity at $t = 0$. In the (NLS) case, using the pseudo-conformal transformation, one can reduce the self-similar blow-up analysis at $t = 0$ to a problem at $t = +\infty$. Furthermore, self-similar solutions are transformed to constants, which is of course a nice simplification. However, for the (mKdV) equation, no such transformation exists. One must handle the critical behavior and truly understand what happens at the blow-up time.

The Cauchy problem for (mKdV) is notoriously difficult at low regularity, even subcritical. In the context of H^s spaces, a long-standing threshold was $s \geqslant 1/4$ ([20]), which is actually sharp in terms of the uniform continuity of the flow (see [5, 21]). Recently, Harrop-Griffith, Killip and Visan [16] were able to show sharp local well-posedness in $H^s(\mathbb{R})$ for $s > -1/2$, right above the critical exponent, relying heavily on the complete integrability of the equation. Moreover, for the critical case $s = -1/2$, the authors prove ill-posedness due to an instantaneous norm inflation mechanism. Outside of the usual H^s framework, Grünrock and Vega [13] proved local well-posedness in the Fourier-Lesbesgue spaces

$$\widehat{H}^s_r = \{u \in \mathcal{S}'(\mathbb{R}) : \|\langle\xi\rangle^s \hat{u}\|_{L^{r'}} < +\infty\} \quad \text{for } 1 < r \leqslant 2,\ s \geqslant \frac{1}{2} - \frac{1}{2r},$$

once again barely missing the critical space $\widehat{H}^0_1$. In any case, a well-posedness result at critical regularity remains out of reach.

In a previous work with Luis Vega [7], we built a critical space on which self-similar solutions naturally exist and proved local and global well-posedness for strictly positive times. This is an even more constrained problem that the mere Cauchy problem at critical regularity: on the one hand, as noted above, the loss of derivative in the nonlinear term is difficult to handle at low regularity; on the other hand, the rough properties of self-similar profiles imply very mild conditions on the functional space. These ingredients had to be carefully balanced in order to achieve a suitable framework on which we could analyze self-similar solutions for large times. This framework will once again play a major role in the analysis at blow-up time, as we will see later on.

The goal of this work is to give a first step in understanding the (mKdV) flow near self-similar solutions at time $t = 0$. There are two intertwined stability problems which one may consider. The first is to start with a perturbed self-similar solution at time $t = 1$ and to study the behavior as $t \to 0$. The second, on which we focus here, is to construct a solution u of (mKdV), defined on a small time interval around $t = 0$, and such that, given a perturbation z,

$$u(t) - S(t) \to z \quad \text{as } t \to 0 \quad \text{in some appropriate norm,}$$

We shall prove that it is possible to construct such a solution u, for a large (open) class of perturbations z, thus showing a first result on the stability of self-similar blow-up for (mKdV).

This is in the same spirit as Bourgain and Wang [4] for the L^2-critical (NLS), and Gutierrez and Vega [14] for the cubic 1D (NLS). However, (mKdV) self-similar solutions are localized neither in physical nor Fourier space, as opposed to solitons (as in [4]), or constant solutions (as in [14]). Even further, the L^1-criticality of the equation leads to modified scattering, involving logarithmic spirals (see [22]). These critical features in both space and time create substantial obstacles in the analysis of the linearized problem around self-similar solutions. To our knowledge, our result is the first to directly construct solutions under such a rough background.

1.2. **Definitions and statement of the main result.** Given a function $v : I \subset \mathbb{R} \to \mathcal{S}'(\mathbb{R})$, we define the profile

$$\tilde{v}(t,\xi) := e^{it\xi^3}\hat{v}(t,\xi)$$

(we denote by $\hat{\cdot}$ or $\mathcal{F}$ the Fourier transform in the space variable). Observe that, if v is a solution of the Airy equation, then $\tilde{v}$ is constant in time. On the other hand, a self-similar solution S will satisfy (with a slight abuse of notation) $\tilde{S}(t,\xi) = \tilde{S}(t^{1/3}\xi)$. By canceling the linear evolution, the oscillatory behavior in frequency is completely concentrated on the nonlinear term: the equation (mKdV) writes for the profile $\tilde{u}$

$$\partial_t \tilde{u} = N[\tilde{u}](t), \qquad \text{where}$$

$$N[\tilde{u}](t) := \frac{i\xi}{4\pi^2} \iint_{\xi_1+\xi_2+\xi_3=\xi} e^{it(\xi^3-\xi_1^3-\xi_2^3-\xi_3^3)}\tilde{u}(t,\xi_1)\tilde{u}(t,\xi_2)\tilde{u}(t,\xi_3)d\xi_1 d\xi_2.$$

One may use stationary phase arguments (pointwise in time) to extract the main contributions of the nonlinear term. To bound properly the remainder in such an expansion, we define

$$\|u\|_{\mathscr{E}(t)} := \|\tilde{u}(t)\|_{L^\infty} + t^{-1/6}\|\partial_\xi \tilde{u}(t)\|_{L^2(\mathbb{R}\backslash\{0\})}$$

and, for any interval $I \subset (0,+\infty)$,

$$\mathscr{E}(I) = \left\{u : I \to \mathcal{S}'(\mathbb{R}) : \tilde{u} \in \mathscr{C}(I, L^\infty(\mathbb{R})),\ \partial_\xi \tilde{u} \in L^\infty(I, L^2(\mathbb{R}\backslash\{0\}))\right\}$$

endowed with the norm

$$\|u\|_{\mathscr{E}(I)} = \sup_{t\in I} \|u(t)\|_{\mathscr{E}(t)}.$$

Remark 1. By $L^2(\mathbb{R}\backslash\{0\})$, we mean the set of distributions whose restriction to $\mathbb{R}\backslash\{0\}$ identifies with an L^2 function. From Sobolev's embedding, a function in $\mathscr{E}$ is 1/2-Hölder continuous in frequency, with the possible exception of $\xi = 0$, where a jump discontinuity may occur. One needs to allow this behavior in order to include self-similar solutions (1.1) with $\alpha \neq 0$. Fortunately, the following proofs are sufficiently robust to handle the extra difficulty related to this jump (observe that the zero frequency is preserved by the (mKdV) flow).

Remark 2. We point out that the time weight $t^{-1/6}$ is necessary to render the $\mathscr{E}$-norm scaling-invariant. Indeed, given $u \in \mathscr{E}((0,+\infty))$, the rescaling $u_\lambda(t,x) = u(\lambda^3 t, \lambda x)$, $\lambda > 0$, satisfies $\|u_\lambda\|_{\mathscr{E}(t)} = \|u\|_{\mathscr{E}(\lambda^3 t)}$.

As it was proven in [7], the space $\mathscr{E}$ is sufficient to perform the stationary phase analysis (see also [18] for a similar development using a slightly stronger norm). Here and below, $\langle x\rangle := \sqrt{1+|x|^2}$ stands for the Japanese bracket.

Lemma 3 (The profile equation, [7, Lemma 7])**.** *Let $u \in \mathscr{E}(I)$. For all $t \in I$ and $\xi > 0$,*

$$N[\tilde{u}](t,\xi) = \frac{\pi\xi^3}{\langle \xi^3 t\rangle}\left(i|\tilde{u}(t,\xi)|^2\tilde{u}(t,\xi) - \frac{1}{\sqrt{3}}e^{-8it\xi^3/9}\tilde{u}^3\left(t,\frac{\xi}{3}\right)\right) + R[u](t,\xi) \tag{1.2}$$

$$(1.3) \qquad \textit{with} \qquad |R[u](t,\xi)| \lesssim \frac{\xi^3\|u(t)\|^3_{\mathscr{E}(t)}}{(\xi^3 t)^{5/6}\langle \xi^3 t\rangle^{1/4}}.$$

Consequently, if u is a distributional solution of (mKdV) *on I,*

$$(1.4) \qquad \forall t\in I, \quad \|\partial_t\tilde{u}(t)\|_{L^\infty} \lesssim \frac{1}{t}\|u(t)\|^3_{\mathscr{E}(t)}.$$

One of the main observations in [7] is that the $\mathscr{E}$ norm is enough to bound both the nonlinear term and self-similar solutions:

Proposition 4 (Existence of self-similar solutions, [6, Theorem 1])**.** *Given $c,\alpha\in\mathbb{R}$ sufficiently small, there exists a unique self-similar solution $S\in\mathscr{E}((0,+\infty))$ with*

$$\|S(t)\|_{\mathscr{E}(t)} = \|S(1)\|_{\mathscr{E}(1)} \lesssim c^2+\alpha^2 \qquad \textit{for all } t>0, \textit{and}$$

$$S(t) \rightharpoonup c\delta_{x=0} + \alpha\,\text{p.v.}\left(\frac{1}{x}\right) \quad \textit{in } \mathscr{D}'(\mathbb{R}) \textit{ as } t\to 0.$$

Furthermore, there exist $A,B\in\mathbb{C}$ such that

$$\tilde{S}(t^{1/3}\xi) \sim \begin{cases} Ae^{ia\ln|t^{1/3}\xi|} + B\frac{e^{ia\ln|t\xi^3| - i\frac{8}{9}t\xi^3}}{t\xi^3}, & |t\xi^3|\gg 1,\\ c+\frac{3i\alpha}{2\pi}\operatorname{sgn}(t\xi^3), & |t\xi^3|\ll 1.\end{cases}$$

Remark 5. The $\mathscr{E}(t)$ norm of self-similar solutions is preserved due to this norm being scale-invariant.

Our goal in this paper is to construct solutions u to (mKdV) which blow up at time 0 as the sum of such a self-similar solution S and a prescribed (more regular) perturbation.

Let us outline the scheme to derive a precise statement and its proof. Using estimate (1.4), one can hope to bootstrap the L^∞ norm of $\tilde{u}$. In order to control the $\mathscr{E}$ norm, we need another key ingredient: the scaling operator Iu, formally defined as

$$Iu := xu + 3t\int_{-\infty}^{x}\partial_t u dx',$$

or equivalently in Fourier variable:

$$(1.5) \qquad \widehat{Iu}(t,\xi) := i\partial_\xi\hat{u} - \frac{3it}{\xi}\partial_t\hat{u}.$$

As it can be seen in Fourier variables, the L^2 norm of Iu is intimately related to that of $\partial_\xi\tilde{u}$. A direct computation yields

$$(\partial_t+\partial_x^3)Iu = 3u^2(Iu)_x$$

and thus

$$(1.6) \qquad \frac{d}{dt}\|Iu(t)\|^2_{L^2} \lesssim \left|\int u^2(Iu)_x Iu dx\right| \lesssim \left|\int uu_x(Iu)^2dx\right| \lesssim \frac{\|u\|^2_{\mathscr{E}(t)}}{t}\|Iu(t)\|^2_{L^2}.$$

(recall [7, Lemma 6]). As it is clear from (1.4) and (1.6), the problem is marginally singular at $t=0$. This should not come as a surprise, due to the L^1-critical nature of the (mKdV) equation. For positive times away from $t=0$, these estimates are sufficient to construct a solution over the space $\mathscr{E}$ (see [7]). To explain how to improve the behavior at $t=0$, let us look closely to (1.2) and forget the R term. If, for some reason, one had $|\tilde{u}(t,\xi)|\lesssim\langle\xi\rangle^{-\epsilon}$ for some $\epsilon>0$, then

$$|\partial_t\tilde{u}(t,\xi)| \lesssim \frac{\xi^{3-\epsilon}}{\langle\xi^3 t\rangle}\sup_t\left\{\|\tilde{u}(t)\|^2_{L^\infty}\|\langle\cdot\rangle^\epsilon\tilde{u}(t)\|_{L^\infty}\right\} \lesssim \frac{1}{t^{1-\epsilon/3}},$$

which can now be integrated in $(0,t)$ to produce an L^∞ bound on $\tilde{u}$. There are two problems with this approach: first, as one may expect, self-similar solutions do not enjoy any extra decay in ξ; second, an *a priori* bound for the extra decay would have to go through the profile equation, where one finds once again the $1/t$ behavior at $t = 0$. On the other hand, if one had $|\tilde{u}(t,\xi)| \lesssim t^\epsilon$, then

$$|\partial_t \tilde{u}(t,\xi)| \lesssim \frac{\xi^3 t^\epsilon}{\langle \xi^3 t \rangle} \sup_t \left\{ t^{-\epsilon} \|\tilde{u}(t)\|_{L^\infty}^3 \right\} \lesssim \frac{1}{t^{1-\epsilon}},$$

and the integration becomes possible on $(0,t)$. Unfortunately, this assumption is even more problematic, since it implies that $\tilde{u}(0,\xi) \equiv 0$. It becomes clear that an extra decay in either frequency or time would suffice to derive an L^∞_ξ bound. The key idea is to decompose u as

$$\tilde{u}(t,\xi) = \tilde{S}(t,\xi) + \tilde{z}(t,\xi) + \tilde{w}(t,\xi), \tag{1.7}$$

where z has extra smoothness and we aim at bootstrapping information on $\|w\|_{\mathscr{E}}$. The self-similar solution, despite its singular behavior, is an exact solution with precise asymptotics in both space and frequency. The regular term $\tilde{z}$ can be chosen sufficiently smooth in space and frequency: in fact, as no polynomial bound in time is necessary, we will assume $\tilde{z}$ constant in time (that is, it corresponds to the linear evolution of the perturbation). The remainder term $\tilde{w}$ will satisfy a bound $\|w(t)\|_{\mathscr{E}(t)} \lesssim t^\epsilon$ and it will measure the interaction between the self-similar solution and the localized linear solution. The equation for the remainder w is

$$\partial_t w + \partial_{xxx} w = \partial_x (u^3 - S^3), \quad w(0) = 0. \tag{1.8}$$

Observe that, since the evolution of the regular part z is linear, no *a priori* decay and smoothness estimates are necessary. The problem is completely reduced to the existence of w over $\mathscr{E}$ with a polynomial bound in time. From the above discussion, the L^∞ bound on $\tilde{w}$ should hold and we are left with the *a priori* bound on Iw, for which the equation is

$$(\partial_t + \partial_x^3) Iw = 3\left(u^2 (Iu)_x - S^2 (IS)_x\right).$$

It is at this point that another decisive feature is revealed: due to the self-similar nature of S, $(IS)_x \equiv 0$. Thus

$$(\partial_t + \partial_x^3) Iw = 3u^2 (Ie^{-t\partial_x^3} z)_x + 3u^2 (Iw)_x$$

A direct integration yields

$$\begin{aligned} \frac{d}{dt}\|Iw\|_{L^2}^2 &\lesssim \left| \int u^2 (Ie^{-t\partial_x^3} z)_x Iw dx \right| + \left| \int u^2 (Iw)_x Iw dx \right| \\ &\lesssim \frac{\|u\|_{\mathscr{E}(t)}^2}{t^{2/3}} \|(Ie^{-t\partial_x^3} z)_x\|_{L^2} \|Iw\|_{L^2} + \frac{\|u\|_{\mathscr{E}(t)}^2}{t} \|Iw\|_{L^2}^2. \end{aligned}$$

Since

$$\mathcal{F}(Ie^{-t\partial_x^3} z)_x = -i\xi \mathcal{F}(Ie^{-t\partial_x^3} z) = \xi \left(\partial_\xi - \frac{3t}{\xi} \partial_t \right) (e^{it\xi^3} \hat{z}) = \xi e^{it\xi^3} \partial_\xi \hat{z},$$

the factor $\|(Ie^{-t\partial_x^3} z)_x\|_{L^2} = \|\xi \partial_\xi \hat{z}\|_{L^2}$ causes no further singular behavior at $t = 0$. As $Iw \equiv 0$ at $t = 0$, this inequality can now be integrated to produce a polynomial bound on Iw. Here we see the importance of I: it provides essential *a priori* bounds while completely canceling out the self-similar background.

The decomposition (1.7) of u is quite natural. If $S \equiv 0$, then w is just the Duhamel integral term, for which one may indeed expect a polynomial bound by applying the H^s local well-posedness theory. The point of this work is that self-similar solutions do not disrupt the classical theory, even though they do not belong to the usual spaces involved in the Cauchy problem. A solution with a self-similar background can still be obtained as a perturbation of the linear flow.

Remark 6. Speaking loosely, self-similar solutions appear from the underlying structure of the *equation* and not from any specific balance between nonlinearity and dispersion (as it is for solitons). Their blow-up behavior is caused by the equation itself. Being unavoidable, it should also be stable. This is in strong contrast with soliton-related blow-up, where the singularity comes from the precise structure of the *solution*. There, small perturbations may obviously lead to strong unstable behavior.

We now state the main result of this paper. Define the space of admissible perturbations

$$\mathcal{Z} := \left\{ z \in \mathcal{S}'(\mathbb{R}) : \|\!|z|\!\|:= \|z\|_{L^1} + \|\langle\xi\rangle^2 \hat{z}\|_{L^1} + \|\langle\xi\rangle\partial_\xi \hat{z}\|_{L^1} + \|\xi\partial_\xi\hat{z}\|_{L^2} < +\infty \right\}.$$

Theorem 7 (Stability of self-similar solutions at blow-up time)**.** *There exists $\delta_0 > 0$ and $C > 0$, such that, given $z \in \mathcal{Z}$ and a self-similar solution $S \in \mathscr{E}((0,+\infty))$ with*

$$\delta := \|\!|z|\!\| + \|S\|_{\mathscr{E}((0,+\infty))} \leqslant \delta_0, \tag{1.9}$$

there exists a unique $w \in \mathscr{E}((0,+\infty)) \cap L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))$ satisfying

$$\forall t > 0, \quad \|w(t)\|_{\mathscr{E}(t)} \leqslant C\delta^3 t^{1/9} \quad \text{and} \quad \|w(t)\|_{L^2(\mathbb{R})} \leqslant C\delta^3 t^{1/18}, \tag{1.10}$$

such that $u(t) = S(t) + e^{-t\partial_x^3} z + w(t)$ is a distributional solution of (mKdV) *on $\mathbb{R}^+ \times \mathbb{R}$ and*

$$\begin{cases} u(t) - S(t) \to z & \text{in } L^2(\mathbb{R}) \\ \hat{u}(t) - \hat{S}(t) \to \hat{z} & \text{in } L^\infty(\mathbb{R}) \end{cases} \quad \text{as} \quad t \to 0^+.$$

Remark 8. From time reversibility, one may solve the problem for negative times and glue the solutions together. Thus one may actually go *beyond* the blow-up time. After some careful considerations, this is not that surprising: over $\mathscr{E}$, the self-similar solution does not present any sort of blow-up behavior at $t = 0$.

In order to prove this result, we first need to understand how the various components of u interact in the nonlinear term. This is done in Section 2. Afterwards in Section 3, we construct an approximation sequence by cutting off high frequencies (Proposition 13) and prove the necessary *a priori* bounds in $\mathscr{E}$ through a careful and well-suited *bootstrap* argument (Proposition 15), where the structure of the approximation is crucial. Finally, in Section 4, the limiting procedure yields the claimed solution on a small time interval, which can then be extended for all positive times using the global results of [7]. The uniqueness statement follows from a direct energy argument (Proposition 16).

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2. Linear and multilinear estimates

In the following, the variables ξ, ξ_1, ξ_2 and ξ_3 are linked via the relation

$$\xi = \xi_1 + \xi_2 + \xi_3.$$

We will perform a stationary phase analysis, with the phase

$$\Phi = \Phi(\xi, \xi_1, \xi_2) := \xi^3 - (\xi_1^3 + \xi_2^3 + \xi_3^3) = 3(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3).$$

Consider the trilinear version of N defined by

$$N[\tilde{f}, \tilde{g}, \tilde{h}](t, \xi) := \frac{i\xi}{4\pi^2} \iint_{\xi_1+\xi_2+\xi_3=\xi} e^{it\Phi} \tilde{f}(t,\xi_1)\tilde{g}(t,\xi_2)\tilde{h}(t,\xi_3) d\xi_1 d\xi_2.$$

Before we give some bound on N with terms in $\tilde{w}$ or $\tilde{z}$, we first study an anisotropic version of N. Indeed it turns out in the energy estimates involving the dilation operator I, that some terms can not be interpreted as a full derivative (mainly because the equation for w has a source term). Therefore, we will also need a bound on a term of the form $N[\tilde{S}, \tilde{S}, v]$, but where the weight ξ (associated to the derivative in physical space) only falls on the v term.

Lemma 9. *For any $0 < t < 1$, if $f, g, h \in \mathscr{E}(t)$,*

$$\left\| \iint e^{it\Phi} \tilde{f}(\xi_1)\tilde{g}(\xi_2)\xi_3\tilde{h}(\xi_3)d\xi_1 d\xi_2 \right\|_{L^\infty_\xi} \lesssim \frac{1}{t}\|f\|_{\mathscr{E}(t)}\|g\|_{\mathscr{E}(t)}\|h\|_{\mathscr{E}(t)}. \tag{2.1}$$

Compared with the expression N, the difference is the multiplier ξ_3 instead of ξ. We could present an almost identical proof as done in [7, Lemma 7 and Appendix A]: there, the estimates are more precise, with an expression of the leading term and a sharper bound on the remainder. For the convenience of the reader, we provide a full proof of (2.1) in Appendix A.
As an immediate consequence, there hold a trilinear version of Lemma 3.

Lemma 10 (L^∞ bounds in $\mathscr{E}$)**.** *For any $t > 0$ and $f, g, h \in \mathscr{E}(t)$,*

$$|N[\tilde{f}, \tilde{g}, \tilde{h}](t, \xi)| \lesssim \frac{1}{t}\|f\|_{\mathscr{E}(t)}\|g\|_{\mathscr{E}(t)}\|h\|_{\mathscr{E}(t)}. \tag{2.2}$$

Proof. As $\xi = \xi_1 + \xi_2 + \xi_3$, it suffices to invoke (2.1) and the symmetry in the variables.
Of course, estimate (2.2) can also be derived from polarizing (1.2)-(1.3). We refer to [7, Lemma 7]: actually its proof (in the appendix there) is done for the trilinear version $N[\tilde{f}, \tilde{g}, \tilde{h}]$, and gives in particular (2.2). □

The $1/t$ decay in (2.2) cannot be improved, in view of the leading terms in (1.2); one can also notice that the $1/t$ rate is optimal because the bound only involves only the critical norm $\mathscr{E}(t)$. For the same reason, the $1/t$ decay is optimal in (2.1) as well. However, if one of the functions involved in N is better behaved, namely belongs to $\mathcal{Z}$, we can gain some decay in time. This our next result.

Lemma 11 (L^∞ bounds on terms with z)**.** *For any $0 < t \leqslant 1$, $z \in \mathcal{Z}$ and $v \in \mathscr{E}(t)$, one has*

$$\|N[\tilde{z}, \tilde{v}, \tilde{v}](t)\|_{L^\infty_\xi} \lesssim \frac{1}{t^{8/9}}|||z|||\|v\|^2_{\mathscr{E}(t)}, \tag{2.3}$$

$$\|N[\tilde{z}, \tilde{z}, \tilde{v}](t)\|_{L^\infty_\xi} \lesssim \frac{1}{t^{2/3}}|||z|||^2\|v\|_{\mathscr{E}(t)}, \tag{2.4}$$

$$\|N[\tilde{z}, \tilde{z}, \tilde{z}](t)\|_{L^\infty_\xi} \lesssim |||z|||^3. \tag{2.5}$$

Proof. Estimate (2.5) is direct : we simply bound by

$$\begin{aligned} |N[\tilde{z}, \tilde{z}, \tilde{z}](t, \xi)| &\leqslant \left(\int_{\xi_1+\xi_2+\xi_3=\xi} (|\xi_1| + |\xi_2| + |\xi_3|)|\hat{z}(\xi_1)\hat{z}(\xi_2)\hat{z}(\xi_3)|d\xi_1 d\xi_2 \right) \\ &\lesssim \|\hat{z}\|^2_{L^1}\|\xi\hat{z}\|_{L^\infty} \lesssim |||z|||^3. \end{aligned}$$

We now prove (2.3), (2.4) simultaneously. For each fixed $t \in (0, 1]$ and $\xi \in \mathbb{R}$, we split $\mathbb{R}^2$ into several domains $\mathscr{A}, \mathscr{B}$, etc.. For each of them, we consider various cases depending on the relative size of the frequencies involved with respect to t (of course, the implicit constants do not depend on (t, ξ)).
To shorten notation, we denote

$$I_1 = N[\tilde{z}, \tilde{v}, \tilde{v}](t, \xi) \quad \text{and} \quad I_2 = N[\tilde{z}, \tilde{z}, \tilde{v}](t, \xi),$$

and, if $\mathscr{D} \subset \mathbb{R}^2$, we denote $I_1(\mathscr{D})$, $I_2(\mathscr{D})$ the corresponding integral where the domain of integration is $\mathscr{D}$ instead of $\mathbb{R}^2$.

Case $\mathscr{A}$. Let $\mathscr{A} := \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1| \geqslant \max(|\xi|, |\xi_2|)/100\}$.
The bound in this case is direct. Indeed,

$$|I_1(\mathscr{A})| \lesssim |\xi| \int_{|\xi_1| \geqslant |\xi|/10} |\hat{z}(\xi_1)| \left(\int_{|\xi_2| \leqslant 10|\xi_1|} d\xi_2 \right) d\xi_1 \|\tilde{v}\|_{L^\infty}^2 \lesssim \int_{\mathbb{R}} |\xi_1|^2 |\hat{z}(\xi_1)| d\xi_1 \|\tilde{v}\|_{L^\infty}^2$$
$$\lesssim \|\langle \xi \rangle^2 \hat{z}\|_{L^1} \|v\|_{\mathscr{E}(t)}^2.$$

Similarly,

$$|I_2(\mathscr{A})| \lesssim |\xi| \int_{\mathscr{A}} |\hat{z}(\xi_1)\hat{z}(\xi_2)| d\xi_1 d\xi_2 \|\tilde{v}\|_{L^\infty} \lesssim \|\langle \xi \rangle \hat{z}\|_{L^1} \|\hat{z}\|_{L^1} \|v\|_{\mathscr{E}(t)} \lesssim \|\!|\!| z |\!|\!\|^2 \|v\|_{\mathscr{E}(t)}.$$

Case $\mathscr{B}$. Let $\mathscr{B} := \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi| \geqslant \max(|\xi_2|/10, 10|\xi_1|)\}$. Here we consider several subcases depending on the size of $t\xi^3$.
Step ($\mathscr{B}$.0). If $|t\xi^3| < 10^9$, then

$$|I_1(\mathscr{B})| \lesssim |\xi| \int_{|\xi_2| \leqslant 10|\xi|} |\tilde{v}| d\xi_2 \|\hat{z}\|_{L^1} \|\tilde{v}\|_{L^\infty} \lesssim |\xi|^2 \|\hat{z}\|_{L^1} \|\tilde{v}\|_{L^\infty}^2 \lesssim \frac{1}{t^{2/3}} \|\!|\!| z |\!|\!\| \|v\|_{\mathscr{E}(t)}^2.$$

We bound similarly

$$|I_2(\mathscr{B})| \lesssim |\xi| \|\hat{z}\|_{L^1}^2 \|\tilde{v}\|_{L^\infty} \lesssim \frac{1}{t^{1/3}} \|\!|\!| z |\!|\!\|^2 \|v\|_{\mathscr{E}(t)}.$$

For the remaining computations in Case $\mathscr{B}$, we assume that

$$|t\xi^3| \geqslant 10^9,$$

and we further split the domain $\mathscr{B}$ by letting

$$\mathscr{B}_1 = \{(\xi_1, \xi_2) \in \mathscr{B} : ||\xi_3| - |\xi_2|| \leqslant a\} \quad \text{and} \quad \mathscr{B}_2 = \mathscr{B} \backslash \mathscr{B}_1,$$

for some $0 < a < |\xi|/10$ (depending on ξ) to be fixed later. We will perform an integration by parts using

$$e^{it\Phi} = \partial_{\xi_j}(e^{it\Phi}) \frac{1}{it\partial_{\xi_j}\Phi},$$

where $j = 1, 2$ and recall that

$$\partial_{\xi_j}\Phi = 3(\xi_3^2 - \xi_j^2) = 3(\xi_3 + \xi_j)(\xi_3 - \xi_j).$$

Notice that on $\mathscr{B}$, $|\partial^2_{\xi_j\xi_j}\Phi| \lesssim |\xi|$. Also, an extra care should be taken with the boundary terms, as $\tilde{v}$ may have a jump at frequency 0. To this end, the domains of integration are meant to be deprived from the lines $\xi_2 = 0$ or $\xi_3 = 0$, while the boundary terms are always meant to contain the corresponding portion of these lines. This is why, throughout this proof, we change from the standard notation and denote by $\partial\Delta$ the boundary of $\Delta \backslash (\{\xi_2 = 0\} \cup \{\xi_3 = 0\})$. This does not weigh on the estimates, as we will use the $\|\tilde{v}\|_{L^\infty}$ bound to control the boundary terms.

Step ($\mathscr{B}$.1). On $\mathscr{B}_1$, we have $|\xi_2 + \xi_3 - \xi| = |\xi_1| \leqslant |\xi|/10$ so that $|\xi_2 + \xi_3| \geqslant 9|\xi|/10$. On the other side, $||\xi_2| - |\xi_3|| \leqslant a \leqslant |\xi|/10$ is small relative to $|\xi_2 + \xi_3|$: this implies that $|\xi_2 - \xi_3| = ||\xi_2| - |\xi_3|| \leqslant |\xi|/10$, and we infer

$$|\xi_2 - \xi/2|, |\xi_3 - \xi/2| \leqslant |\xi|/10.$$

As a consequence, $|\xi_3| - |\xi_1| \geqslant |\xi|/2 - |\xi|/5 \geqslant |\xi|/4$ and so $|\partial_{\xi_1}\Phi| \gtrsim |\xi|^2$. Therefore, we perform an IBP with respect to ξ_1:

$$|I_1(\mathscr{B}_1)| \leqslant \left| \xi \int_{\mathscr{B}_1} e^{it\Phi} \partial_{\xi_1} \left(\frac{1}{it\partial_{\xi_1}\Phi} \hat{z}(\xi_1) \tilde{v}(t, \xi_3) \right) \tilde{v}(t, \xi_2) d\xi_1 d\xi_2 \right|$$

$$
\begin{aligned}
&+|\xi|\int_{\partial\mathscr{B}_1}\frac{1}{t|\partial_{\xi_1}\Phi|}|\hat z(\xi_1)\tilde v(t,\xi_2)\tilde v(t,\xi_3)|d\sigma(\xi_1,\xi_2)\\
\lesssim\ &|\xi|\int_{\mathscr{B}_1}\frac{|\xi|}{t|\xi|^4}|\hat z(\xi_1)|\|\tilde v\|_{L^\infty}^2+\frac{1}{t|\xi|^2}|\partial_\xi\hat z(\xi_1)|\|\tilde v\|_{L^\infty}^2d\xi_1d\xi_2\\
&+|\xi|\int_{\mathscr{B}_1}\frac{1}{t|\xi|^2}|\hat z(\xi_1)||\partial_\xi\tilde v(\xi_3)|\|\tilde v\|_{L^\infty}d\xi_1d\xi_2\\
&+|\xi|\int_{\partial\mathscr{B}_1}\frac{1}{t|\xi|^2}|\hat z(\xi_1)|\|\tilde v\|_{L^\infty}^2d\sigma(\xi_1,\xi_2).
\end{aligned}
$$

On $\mathscr{B}_1$, for fixed ξ_1, $|\xi_2-(\xi-\xi_1)/2|=|\xi_2-\xi_3|/2\leqslant a/2$, so that

$$
\begin{aligned}
|\xi|\int_{\mathscr{B}_1}\frac{|\xi|}{t|\xi|^4}|\hat z(\xi_1)|d\xi_1d\xi_2&\lesssim\frac{1}{t|\xi|^2}\int_{\mathbb{R}}|\hat z(\xi_1)|\left(\int_{|\xi_2-(\xi-\xi_1)/2|\leqslant a/2}d\xi_2\right)d\xi_1\\
&\lesssim\frac{a}{t|\xi|^2}\|\hat z\|_{L^1},\\
|\xi|\int_{\mathscr{B}_1}\frac{1}{t|\xi|^2}|\partial_\xi\hat z(\xi_1)|d\xi_1d\xi_2&\lesssim\frac{1}{t|\xi|}\int_{\mathbb{R}}|\partial_\xi\hat z(\xi_1)|\left(\int_{|\xi_2-(\xi-\xi_1)/2|\leqslant a/2}d\xi_2\right)d\xi_1\\
&\lesssim\frac{a}{t|\xi|}\|\partial_\xi\hat z\|_{L^1},
\end{aligned}
$$

and

$$
\begin{aligned}
&|\xi|\int_{\mathscr{B}_1}\frac{1}{t|\xi|^2}|\hat z(\xi_1)||\partial_\xi\tilde v(\xi_3)|d\xi_1d\xi_2\\
&\quad\lesssim\frac{1}{t|\xi|}\int_{\mathbb{R}}|\hat z(\xi_1)|\left(\int_{|\xi_2-(\xi-\xi_1)/2|\leqslant a/2}|\partial_\xi\tilde v(\xi-\xi_1-\xi_2)|d\xi_2\right)d\xi_1\\
&\quad\lesssim\frac{1}{t|\xi|}\|\hat z\|_{L^1}a^{1/2}\|\partial_\xi\tilde v\|_{L^2}\lesssim\frac{a^{1/2}}{t^{5/6}|\xi|}\|\hat z\|_{L^1}\|v\|_{\mathscr{E}(t)}.
\end{aligned}
$$

We see on the second bound that one requires $a\ll|\xi|$ in order to gain over the $1/t$ bound.
For the boundary term, we have

$$
|\xi|\int_{\partial\mathscr{B}_1}\frac{1}{t|\xi|^2}|\hat z(\xi_1)|d\sigma(\xi_1,\xi_2)\lesssim\frac{1}{t|\xi|}\|\hat z\|_{L^1}.
$$

Therefore,

$$
|I_1(\mathscr{B}_1)|\lesssim\left(\frac{a}{t|\xi|}+\frac{a^{1/2}}{t^{5/6}|\xi|}+\frac{1}{t|\xi|}\right)\|\hat z\|_{W^{1,1}}\|v\|_{\mathscr{E}(t)}^2. \tag{2.6}
$$

Step ($\mathscr{B}$.2). On $\mathscr{B}_2$, $||\xi_2|-|\xi_3||\geqslant a$. Also, as $|\xi_1|\leqslant|\xi|/10$, $|\xi_2|+|\xi_3|\geqslant 9|\xi|/10$ and so $|\partial_{\xi_2}\Phi|\gtrsim a|\xi|$. Here, we perform an IBP in ξ_2:

$$
\begin{aligned}
|I_1(\mathscr{B}_2)|\leqslant\ &\left|\xi\int_{\mathscr{B}_2}e^{it\Phi}\partial_{\xi_2}\left(\frac{1}{it\partial_{\xi_2}\Phi}\tilde v(t,\xi_2)\tilde v(t,\xi_3)\right)\hat z(\xi_1)d\xi_1d\xi_2\right|\\
&+|\xi|\int_{\partial\mathscr{B}_2}\frac{1}{t|\partial_{\xi_2}\Phi|}|\hat z(\xi_1)\tilde v(t,\xi_2)\tilde v(t,\xi_3)|d\sigma(\xi_1,\xi_2)\\
\lesssim\ &|\xi|\int_{\mathscr{B}_2}\frac{|\xi|}{t|a\xi|^2}|\hat z(\xi_1)|\|\tilde v\|_{L^\infty}^2+\frac{1}{t|a\xi|}|\hat z(\xi_1)||\partial_\xi\tilde v(\xi_2)|\|\tilde v\|_{L^\infty}d\xi_1d\xi_2\\
&+|\xi|\int_{\mathscr{B}_2}\frac{1}{t|a\xi|}|\hat z(\xi_1)|\|\tilde v\|_{L^\infty}|\partial_\xi\tilde v(\xi_3)|d\xi_1d\xi_2\\
&+|\xi|\int_{\partial\mathscr{B}_2}\frac{1}{t|a\xi|}|\hat z(\xi_1)|\|\tilde v\|_{L^\infty}^2d\sigma(\xi_1,\xi_2).
\end{aligned}
$$

Observe that all derivatives fall on $\tilde{v}$ (or Φ, but not $\hat{z}$), the point being that $\|\partial_\xi \tilde{v}\|_{L^2}$ is better behaved than $\|\tilde{v}\|_{L^\infty}$. To complete the bounds, we now only use that $|\xi_1|, |\xi_2| \lesssim |\xi|$ on $\mathscr{B}_2$ as follows

$$
\begin{aligned}
&|\xi| \int_{\mathscr{B}_2} \frac{1}{ta^2|\xi|} |\hat{z}(\xi_1)| d\xi_1 d\xi_2 \lesssim \frac{|\xi|}{ta^2} \|\hat{z}\|_{L^1},\\
&|\xi| \int_{\mathscr{B}_1} \frac{1}{t|a\xi|} |\hat{z}(\xi_1)| (|\partial_\xi \tilde{v}(\xi_2)| + |\partial_\xi \tilde{v}(\xi_3)| d\xi_1 d\xi_2\\
&\qquad \lesssim \frac{1}{ta} \int_{\mathbb{R}} |\hat{z}(\xi_1)| \left(\int_{|\xi_2| \lesssim |\xi|} (|\partial_\xi \tilde{v}(\xi_2)| + |\partial_\xi \tilde{v}(\xi - \xi_1 - \xi_2)| d\xi_2 \right) d\xi_1\\
&\qquad \lesssim \frac{1}{ta} \|\hat{z}\|_{L^1} |\xi|^{1/2} \|\partial_\xi \tilde{v}\|_{L^2} \lesssim \frac{|\xi|^{1/2}}{t^{5/6}|a|} \|\hat{z}\|_{L^1} \|v\|_{\mathscr{E}(t)}.
\end{aligned}
$$

For the boundary term, we simply have

$$\int_{\partial \mathscr{B}_2} \frac{1}{ta} |\hat{z}(\xi_1)| d\sigma(\xi_1, \xi_2) \lesssim \frac{1}{ta} \|\hat{z}\|_{L^1}.$$

Therefore,

$$I_1(\mathscr{B}_2)| \lesssim \left(\frac{|\xi|}{ta^2} + \frac{|\xi|^{1/2}}{t^{5/6}a} + \frac{1}{ta} \right) \|\hat{z}\|_{W^{1,1}} \|v\|^2_{\mathscr{E}(t)}. \tag{2.7}$$

Step ($\mathscr{B}$.3). We now optimize in a, choosing $a = |\xi|^{2/3}$. As $|\xi|^{1/3} \geqslant 10t^{-1/9} \geqslant 10$, $a \leqslant |\xi|/10$, which justifies the above computations. Using (2.6) and (2.7), and that $|\xi|^{-1} \lesssim t^{1/3}$, we get

$$|I_1(\mathscr{B})| \leqslant |I_1(\mathscr{B}_1)| + |I_1(\mathscr{B}_2)| \lesssim \frac{1}{t^{8/9}} \|\hat{z}\|_{W^{1,1}} \|v\|^2_{\mathscr{E}(t)}.$$

We now bound $I_2(\mathscr{B})$. The bounds are obtained in a similar fashion as for $I_1(\mathscr{B})$ (they are in fact simpler). However, to sharpen the bound, the frequency splitting is slightly different:

$$\mathscr{B}_4 = \{(\xi_1, \xi_2) \in \mathscr{B} : ||\xi_3| - |\xi_2|| \leqslant |\xi|/10\} \quad \text{and} \quad \mathscr{B}_5 = \mathscr{B} \backslash \mathscr{B}_4,$$

(this corresponds to the choice $a = |\xi|/10$).

Step ($\mathscr{B}$.4). For $I_2(\mathscr{B}_4)$, as $|\partial_{\xi_1} \Phi| \gtrsim |\xi|^2$ on $\mathscr{B}_4$, we perform an IBP in ξ_1:

$$
\begin{aligned}
|I_2(\mathscr{B}_4)| \leqslant &\left| \xi \int_{\mathscr{B}_4} e^{it\Phi} \partial_{\xi_1} \left(\frac{1}{it\partial_{\xi_1} \Phi} \hat{z}(\xi_1) \tilde{v}(t, \xi_3) \right) \hat{z}(\xi_2) d\xi_1 d\xi_2 \right|\\
&+ |\xi| \int_{\partial \mathscr{B}_4} \frac{1}{t|\partial_{\xi_1} \Phi|} |\hat{z}(\xi_1) \hat{z}(\xi_2) \tilde{v}(t, \xi_3)| d\sigma(\xi_1, \xi_2)\\
\lesssim &|\xi| \int_{\mathscr{B}_4} \frac{|\xi|}{t|\xi|^4} |\hat{z}(\xi_1)||\hat{z}(\xi_2)| \|\tilde{v}\|_{L^\infty} + \frac{1}{t|\xi|^2} |\partial_\xi \hat{z}(\xi_1) \hat{z}(\xi_2)| \|\tilde{v}\|_{L^\infty} d\xi_1 d\xi_2\\
&+ |\xi| \int_{\mathscr{B}_4} \frac{1}{t|\xi|^2} |\hat{z}(\xi_1) \hat{z}(\xi_2) \partial_\xi \tilde{v}(\xi_3)| d\xi_1 d\xi_2\\
&+ |\xi| \int_{\partial \mathscr{B}_4} \frac{1}{t|\xi|^2} |\hat{z}(\xi_1)| \|\hat{z}\|_{L^\infty} \|\tilde{v}\|_{L^\infty} d\sigma(\xi_1, \xi_2).
\end{aligned}
$$

The gain over the case ($\mathscr{B}$.1) comes from the two factors in z which insure an $L^1(d\xi_1 d\xi_2)$ bound:

$$|I_2(\mathscr{B}_4)| \lesssim \frac{1}{t|\xi|^2} \|\hat{z}\|^2_{L^1} \|\tilde{v}\|_{L^\infty} + \frac{1}{t|\xi|} \|\partial_\xi \hat{z}\|_{L^1} \|\hat{z}\|_{L^1} \|\tilde{v}\|_{L^\infty}$$

$$
\begin{aligned}
&\quad + \frac{1}{t|\xi|}\|\hat z\|_{L^1}\|\partial_\xi v\|_{L^2}\|\hat z\|_{L^2} + \frac{1}{t|\xi|}\|\hat z\|_{L^1}\|\hat z\|_{L^\infty}\|\tilde v\|_{L^\infty}\\
&\lesssim \frac{1}{t|\xi|}\|\hat z\|_{L^1}\|\hat z\|_{W^{1,1}}\|v\|_{\mathscr{E}(t)},
\end{aligned}
$$

where we used the Sobolev embedding $\|\hat z\|_{L^2} \lesssim \|\hat z\|_{W^{1,1}}$ in the last estimate. As we assumed $|\xi| \gtrsim t^{-1/3}$ here, we infer

$$
|I_2(\mathscr{B}_4)| \lesssim \frac{1}{t^{2/3}}\|\hat z\|_{L^1}\|\hat z\|_{W^{1,1}}\|v\|_{\mathscr{E}(t)}. \tag{2.8}
$$

Step ($\mathscr{B}$.5). For $I_2(\mathscr{B}_5)$, $|\partial_{\xi_2}\Phi| \gtrsim |\xi|^2$, so that we perform an IBP in ξ_2:

$$
\begin{aligned}
|I_2(\mathscr{B}_5)| &\leqslant \left|\xi \int_{\mathscr{B}_5} e^{it\Phi}\partial_{\xi_2}\left(\frac{1}{it\partial_{\xi_2}\Phi}\hat z(\xi_2)\tilde v(t,\xi_3)\right)\hat z(\xi_1)d\xi_1 d\xi_2\right|\\
&\quad + |\xi|\int_{\partial\mathscr{B}_5}\frac{1}{t|\partial_{\xi_1}\Phi|}|\hat z(\xi_1)\hat z(\xi_2)\tilde v(t,\xi_3)|d\sigma(\xi_1,\xi_2)\\
&\lesssim |\xi|\int_{\mathscr{B}_5}\frac{|\xi|}{t|\xi|^4}|\hat z(\xi_1)||\hat z(\xi_2)|\|\tilde v\|_{L^\infty} + \frac{1}{t|\xi|^2}|\hat z(\xi_1)||\partial_\xi \hat z(\xi_2)|\|\tilde v\|_{L^\infty}d\xi_1 d\xi_2\\
&\quad + |\xi|\int_{\mathscr{B}_5}\frac{1}{t|\xi|^2}|\hat z(\xi_1)||\hat z(\xi_2)||\partial_\xi \tilde v(\xi_3)|d\xi_1 d\xi_2\\
&\quad + |\xi|\int_{\partial\mathscr{B}_5}\frac{1}{t|\xi|^2}|\hat z(\xi_1)|\|\hat z\|_{L^\infty}\|\tilde v\|_{L^\infty}d\sigma(\xi_1,\xi_2)\\
&\lesssim \frac{1}{t|\xi|^2}\|\hat z\|_{L^1}^2\|\tilde v\|_{L^\infty} + \frac{1}{t|\xi|}\|\hat z\|_{L^1}\|\partial_\xi \hat z\|_{L^1}\|v\|_{L^\infty}\\
&\quad + \frac{1}{t|\xi|}\|\hat z\|_{L^1}\|\hat z\|_{L^2}\|\partial_\xi \tilde v\|_{L^2} + \frac{1}{t|\xi|}\|\hat z\|_{L^1}\|\hat z\|_{L^\infty}\|\tilde v\|_{L^\infty}\\
&\lesssim \frac{1}{t|\xi|}\|\hat z\|_{L^1}\|\hat z\|_{W^{1,1}}\|\tilde v\|_{\mathscr{E}(t)}.
\end{aligned}
$$

(recall that $0 < t \leqslant 1$). Together with (2.8), we infer

$$
|I_2(\mathscr{B})| \leqslant |I_2(\mathscr{B}_4)| + |I_2(\mathscr{B}_5)| \lesssim \frac{1}{t^{2/3}}\|\hat z\|_{L^1}\|\hat z\|_{W^{1,1}}\|\tilde v\|_{\mathscr{E}(t)}.
$$

Case $\mathscr{C}$. We finally consider

$$
\begin{aligned}
\mathscr{C} &= \mathbb{R}^2\backslash(\mathscr{A}\cup\mathscr{B})\\
&= \{(\xi_1,\xi_2)\in\mathbb{R}^2 : |\xi_1| < \max(|\xi|,|\xi_2|)/100,\ |\xi| < \max(|\xi_2|/10, 10|\xi_1|)\}.
\end{aligned}
$$

Observe that on $\mathscr{C}$, $|\xi| \leqslant \max(|\xi_2|/10, \max(|\xi|,|\xi_2|)/10) = \max(|\xi|/10, |\xi_2|/10)$ so that $|\xi| \leqslant |\xi_2|/10$, and therefore $|\xi_1| \leqslant |\xi_2|/100$. Hence $|\xi_3+\xi_2| \leqslant |\xi_2|/5$, $|\xi_3| \geqslant 4|\xi_2|/5$ and ξ_2 and ξ_3 are the highest frequencies (of the same magnitude). In particular, $|\partial^2_{\xi_i\xi_j}\Phi| \lesssim |\xi_2|$ on $\mathscr{C}$.

We argue in $\mathscr{C}$ in the same spirit as we did for case $\mathscr{B}$. We split

$$
\begin{aligned}
\mathscr{C}_0 &= \{(\xi_1,\xi_2)\in\mathscr{C} : |t\xi_2^3| \leqslant 10^9\}\\
\mathscr{C}_1 &= \{(\xi_1,\xi_2)\in\mathscr{C} : |t\xi_2^3| \geqslant 10^9 \text{ and } |\xi-\xi_1| \leqslant |\xi_2|^{2/3}\},\\
\mathscr{C}_2 &= \{(\xi_1,\xi_2)\in\mathscr{C} : |t\xi_2^3| \geqslant 10^9 \text{ and } |\xi-\xi_1| \geqslant |\xi_2|^{2/3}\}.
\end{aligned}
$$

(ξ_2 is now playing the role of ξ in Case $\mathscr{B}$).

Step ($\mathscr{C}$.0). On $\mathscr{C}_0$, we bound as in ($\mathscr{B}$.0):

$$
I_1(\mathscr{C}_0) \leqslant |\xi|\int_{\mathscr{C}_0}|\hat z(\xi_1)|d\xi_1 d\xi_2\|\tilde v\|_{L^\infty}^2 \lesssim \|\hat z\|_{L^1}\|\tilde v\|_{L^\infty}^2\int_{|\xi_2|\lesssim t^{-1/3}}|\xi_2|d\xi_2
$$

$$\lesssim \frac{1}{t^{2/3}}\|\hat{z}\|_{L^1}\|\tilde{v}\|_{L^\infty}^2, \tag{2.9}$$

$$I_2(\mathscr{C}_0) \leqslant |\xi| \int_{\mathscr{C}_0} |\hat{z}(\xi_1)| d\xi_1 d\xi_2 \|\hat{z}\|_{L^\infty}\|\tilde{v}\|_{L^\infty} \lesssim \|\hat{z}\|_{L^1}\|\hat{z}\|_{L^\infty}\|\tilde{v}\|_{L^\infty} \int_{|\xi_2|\lesssim t^{-1/3}} |\xi_2| d\xi_2$$

$$\lesssim \frac{1}{t^{2/3}}\|\hat{z}\|_{L^1}\|\hat{z}\|_{L^\infty}\|\tilde{v}\|_{L^\infty}. \tag{2.10}$$

Step ($\mathscr{C}$.1). On $\mathscr{C}_1$, we integrate by parts in ξ_1: observe that in this domain $|\xi_1| \leqslant |\xi_2|/100$ and $|\xi_3| \geqslant 4|\xi_2|/5$ so that

$$|\partial_{\xi_1}\Phi| = 3|\xi_1^2 - \xi_3^2| \gtrsim |\xi_2|^2.$$

Hence,

$$\begin{aligned}
|I_1(\mathscr{C}_1)| \leqslant &\left| \xi \int_{\mathscr{C}_1} e^{it\Phi} \partial_{\xi_1}\left(\frac{1}{it\partial_{\xi_1}\Phi}\hat{z}(\xi_1)\tilde{v}(t,\xi_3)\right)\tilde{v}(t,\xi_2) d\xi_1 d\xi_2 \right| \\
&+ |\xi| \int_{\partial\mathscr{C}_1} \frac{1}{t|\partial_{\xi_1}\Phi|} |\hat{z}(\xi_1)\tilde{v}(t,\xi_2)\tilde{v}(t,\xi_3)| d\sigma(\xi_1,\xi_2) \\
\lesssim &|\xi| \int_{\mathscr{C}_1} \frac{|\xi_2|}{t|\xi_2|^4}|\hat{z}(\xi_1)|\|\tilde{v}\|_{L^\infty}^2 + \frac{1}{t|\xi_2|^2}|\partial_\xi \hat{z}(\xi_1)|\|\tilde{v}\|_{L^\infty}^2 d\xi_1 d\xi_2 \\
&+ |\xi| \int_{\mathscr{C}_1} \frac{1}{t|\xi_2|^2}|\hat{z}(\xi_1)||\partial_\xi \tilde{v}(\xi_3)|\|\tilde{v}\|_{L^\infty} d\xi_1 d\xi_2 \\
&+ |\xi| \int_{\partial\mathscr{C}_1} \frac{1}{t|\xi_2|^2}|\hat{z}(\xi_1)|\|\tilde{v}\|_{L^\infty}^2 d\sigma(\xi_1,\xi_2).
\end{aligned}$$

On $\mathscr{C}_1$, $|\xi| \lesssim |\xi_2|$, and for fixed ξ_2, $|\xi - \xi_1| \leqslant |\xi_2|^{2/3}$, so that

$$\begin{aligned}
|\xi| \int_{\mathscr{C}_1} \frac{|\xi_2|}{t|\xi_2|^4}|\hat{z}(\xi_1)| d\xi_1 d\xi_2 &\lesssim \frac{1}{t}\int_{|\xi_2|\gtrsim t^{-1/3}} \frac{d\xi_2}{|\xi_2|^2}\|\hat{z}\|_{L^1} \lesssim \frac{1}{t^{2/3}}\|\hat{z}\|_{L^1}, \\
|\xi| \int_{\mathscr{C}_1} \frac{1}{t|\xi_2|^2}|\partial_\xi \hat{z}(\xi_1)| d\xi_1 d\xi_2 &\lesssim \frac{1}{t}\int_{|\xi_2|\gtrsim t^{-1/3}} \int (|\xi - \xi_1| + |\xi_1||\partial_\xi \hat{z}(\xi_1)| d\xi_1) \frac{d\xi_2}{|\xi_2|^2} \\
&\lesssim \frac{1}{t}\int_{|\xi_2|\gtrsim t^{-1/3}} \left(|\xi_2|^{2/3}\|\partial_\xi \hat{z}\|_{L^1} + \|\xi\partial_\xi \hat{z}\|_{L^1}\right)\frac{d\xi_2}{|\xi_2|^2} \\
&\lesssim \frac{1}{t^{8/9}}\|\langle\xi\rangle\partial_\xi \hat{z}\|_{L^1}, \\
|\xi| \int_{\mathscr{C}_1} \frac{1}{t|\xi_2|^2}|\hat{z}(\xi_1)||\partial_\xi \tilde{v}(\xi_3)| d\xi_1 d\xi_2 &\lesssim \frac{1}{t}\int_{\mathbb{R}} |\hat{z}(\xi_1)| \left(\int_{|\xi_2|\gtrsim t^{-1/3}} |\partial_\xi \tilde{v}(\xi_3)|\frac{d\xi_2}{|\xi_2|}\right) d\xi_1 \\
&\lesssim \frac{1}{t}\|\hat{z}\|_{L^1}\|\partial_\xi \tilde{v}\|_{L^2}\left(\int_{|\xi_2|\gtrsim t^{-1/3}} \frac{d\xi_2}{|\xi_2|^2}\right)^{1/2} \\
&\lesssim \frac{1}{t^{5/6}}\|\hat{z}\|_{L^1}\|\partial_\xi \tilde{v}\|_{L^2} \lesssim \frac{1}{t^{2/3}}\|\hat{z}\|_{L^1}\|v\|_{\mathscr{E}(t)}.
\end{aligned}$$

On $\partial\mathscr{C}_1$, since $|\xi_2| \gtrsim |\xi|$ and $|\xi_2| \gtrsim t^{-1/3}$, one has $\dfrac{|\xi|}{|\xi_2|^2} \lesssim t^{1/3}$. Thus

$$|\xi| \int_{\partial\mathscr{C}_1} \frac{1}{t|\xi_2|^2}|\hat{z}(\xi_1)| d\sigma(\xi_1,\xi_2) \leqslant \frac{1}{t^{2/3}}\|\hat{z}\|_{L^1}, \tag{2.11}$$

Therefore, we get

$$|I_1(\mathscr{C}_1)| \lesssim \frac{1}{t^{8/9}}(\|\langle\xi\rangle\partial_\xi \hat{z}\|_{L^1} + \|\hat{z}\|_{L^1})\|v\|_{\mathscr{E}(t)}^2. \tag{2.12}$$

Step ($\mathscr{C}$.2). On $\mathscr{C}_2$, we integrate by parts in ξ_2: observe that in this domain

$$|\partial_{\xi_2}\Phi| = 3|\xi_2^2 - \xi_3^2| = 3|\xi_2 - \xi_3||\xi - \xi_1| \gtrsim |\xi_2|^{5/3},$$

Hence we can estimate

$$\begin{aligned}
|I_1(\mathscr{C}_2)| &\leqslant \left|\xi \int_{\mathscr{C}_2} e^{it\Phi}\partial_{\xi_1}\left(\frac{1}{it\partial_{\xi_2}\Phi}\tilde{v}(t,\xi_2)\tilde{v}(t,\xi_3)\right)\hat{z}(\xi_1)d\xi_1 d\xi_2\right| \\
&\quad + |\xi| \int_{\partial\mathscr{C}_2} \frac{1}{t|\partial_{\xi_2}\Phi|}|\hat{z}(\xi_1)\tilde{v}(t,\xi_2)\tilde{v}(t,\xi_3)|d\sigma(\xi_1,\xi_2) \\
&\lesssim |\xi| \int_{\mathscr{C}_2} \frac{|\xi_2|}{t|\xi_2|^{10/3}}|\hat{z}(\xi_1)|\|\tilde{v}\|_{L^\infty}^2 d\xi_1 d\xi_2 \\
&\quad + |\xi| \int_{\mathscr{C}_2} \frac{1}{t|\xi_2|^{5/3}}|\hat{z}(\xi_1)|(|\partial_\xi \tilde{v}(\xi_2)| + |\partial_\xi \tilde{v}(\xi_3)|)\|\tilde{v}\|_{L^\infty} d\xi_1 d\xi_2 \\
&\quad + |\xi| \int_{\partial\mathscr{C}_2} \frac{1}{t|\xi_2|^{5/3}}|\hat{z}(\xi_1)|\|\tilde{v}\|_{L^\infty}^2 d\sigma(\xi_1,\xi_2).
\end{aligned}$$

On $\mathscr{C}_2$, $|\xi| \lesssim |\xi_2|$, so that

$$\begin{aligned}
&|\xi| \int_{\mathscr{C}_2} \frac{1}{t|\xi_2|^{7/3}}|\hat{z}(\xi_1)|d\xi_1 d\xi_2 \lesssim \frac{1}{t}\int_{|\xi_2| \gtrsim t^{-1/3}} \frac{d\xi_2}{|\xi_2|^{4/3}}\|\hat{z}\|_{L^1} \lesssim \frac{1}{t^{8/9}}\|\hat{z}\|_{L^1}, \\
&|\xi| \int_{\mathscr{C}_2} \frac{1}{t|\xi_2|^{5/3}}|\hat{z}(\xi_1)\partial_\xi \tilde{v}(\xi_2)|d\xi_1 d\xi_2 \lesssim \frac{1}{t}\int_{\mathbb{R}} |\hat{z}(\xi_1)| \left(\int_{|\xi_2| \gtrsim t^{-1/3}} |\partial_\xi \tilde{v}(\xi_2)| \frac{d\xi_2}{|\xi_2|^{2/3}}\right) \\
&\quad \lesssim \frac{1}{t}\|\hat{z}\|_{L^1}\|\partial_\xi \tilde{v}\|_{L^2}\left(\int_{|\xi_2| \gtrsim t^{-1/3}} \frac{d\xi_2}{|\xi_2|^{4/3}}\right)^{1/2} \lesssim \frac{1}{t}\|\hat{z}\|_{L^1}\|\partial_\xi \tilde{v}\|_{L^2} t^{1/18} \\
&\quad \lesssim \frac{1}{t^{7/9}}\|\hat{z}\|_{L^1}\|v\|_{\mathscr{E}(t)}.
\end{aligned}$$

On $\partial\mathscr{C}_2$, we have, as in (2.11), $\dfrac{|\xi|}{|\xi_2|^{5/3}} \lesssim t^{2/9}$ so that

$$|\xi| \int_{\partial\mathscr{C}_2} \frac{1}{t|\xi_2|^{5/3}}|\hat{z}(\xi_1)|d\sigma(\xi_1,\xi_2) \leqslant \frac{1}{t^{7/9}}\|\hat{z}\|_{L^1}.$$

Hence, we obtain

$$|I_1(\mathscr{C}_2)| \lesssim \frac{1}{t^{8/9}}\|\hat{z}\|_{L^1}\|v\|_{\mathscr{E}(t)}^2.$$

Together with (2.9) and (2.12), we infer that

$$|I_1(\mathscr{C})| \lesssim \frac{1}{t^{8/9}}\|\!|z|\!\|\|v\|_{\mathscr{E}(t)}^2.$$

Step ($\mathscr{C}$.3). We now bound I_2 on

$$\mathscr{C}_3 = \mathscr{C}\backslash\mathscr{C}_0 = \{(\xi_1,\xi_2) \in \mathscr{C} : |t\xi_2|^3 > 10^9\}.$$

For I_2, we don't need to further split the domain. As for $\mathscr{C}_1$, on $\mathscr{C}_3$ we integrate by parts with respect to ξ_1. In this region, $|\xi_3| \geqslant 4|\xi_2|/5$ and $|\xi_1| \leqslant |\xi_2|/5$ so that $|\partial_{\xi_1}\Phi| = 3|\xi_1^2 - \xi_3|^2 \gtrsim |\xi_2|^2$. Hence we bound:

$$\begin{aligned}
|I_2(\mathscr{C}_3)| &\leqslant \left|\xi \int_{\mathscr{C}_3} e^{it\Phi}\partial_{\xi_1}\left(\frac{1}{it\partial_{\xi_1}\Phi}\hat{z}(\xi_1)\tilde{v}(t,\xi_3)\right)\hat{z}(\xi_2)d\xi_1 d\xi_2\right| \\
&\quad + |\xi| \int_{\partial\mathscr{C}_3} \frac{1}{t|\partial_{\xi_1}\Phi|}|\hat{z}(\xi_1)\hat{z}(\xi_2)\tilde{v}(t,\xi_3)|d\sigma(\xi_1,\xi_2) \\
&\lesssim |\xi| \int_{\mathscr{C}_3} \frac{|\xi_2|}{t|\xi_2|^4}|\hat{z}(\xi_1)\hat{z}(\xi_2)|\|\tilde{v}\|_{L^\infty} + \frac{1}{t|\xi_2|^2}|\partial_\xi \hat{z}(\xi_1)\hat{z}(\xi_2)|\|\tilde{v}\|_{L^\infty} d\xi_1 d\xi_2
\end{aligned}$$

$$+ \int_{\mathscr{C}_3} \frac{1}{t|\xi_2|^2} |\hat{z}(\xi_1)\hat{z}(\xi_2)\partial_\xi \tilde{v}(\xi_3)| d\xi_1 d\xi_2$$
$$+ |\xi| \int_{\partial\mathscr{C}_3} \frac{1}{t|\xi_2|^2} |\hat{z}(\xi_1)| \|\hat{z}\|_{L^\infty} \|\tilde{v}\|_{L^\infty} d\sigma(\xi_1, \xi_2).$$

On $\mathscr{C}_3$, $|\xi| \lesssim |\xi_2|$ and $\frac{1}{|\xi_2|} \lesssim t^{1/3}$. Hence

$$|\xi| \int_{\mathscr{C}_3} \frac{|\xi_2|}{t|\xi_2|^4} |\hat{z}(\xi_1)\hat{z}(\xi_2)| d\xi_1 d\xi_2 \lesssim \frac{1}{t^{1/3}} \int |\hat{z}(\xi_1)\hat{z}(\xi_2)| d\xi_1 d\xi_2 \lesssim \frac{1}{t^{1/3}} \|\hat{z}\|_{L^1}^2,$$
$$|\xi| \int_{\mathscr{C}_3} \frac{1}{t|\xi_2|^2} |\partial_\xi \hat{z}(\xi_1) z(\xi_2)| d\xi_1 d\xi_2 \lesssim \frac{1}{t^{2/3}} \int_{\mathbb{R}} |\partial_\xi \hat{z}(\xi_1) z(\xi_2)| d\xi_1 d\xi_2 \lesssim \frac{1}{t^{2/3}} \|\partial_\xi \hat{z}\|_{L^1} \|\hat{z}\|_{L^1}$$

and

$$|\xi| \int_{\mathscr{C}_3} \frac{1}{t|\xi_2|^2} |\hat{z}(\xi_1) z(\xi_2) \partial_\xi \tilde{v}(t, \xi_3)| d\xi_1 d\xi_2 \lesssim \frac{1}{t^{2/3}} \int |\hat{z}(\xi_1)| \left(\int |z(\xi_2) \partial_\xi \tilde{v}(t, \xi_3)| \right) d\xi_2$$
$$\lesssim \frac{1}{t^{2/3}} \|\hat{z}\|_{L^1} \|\hat{z}\|_{L^2} \|\partial_\xi \tilde{v}\|_{L^2} \lesssim \frac{1}{t^{1/2}} \|\hat{z}\|_{L^1} \|\hat{z}\|_{L^2} \|v\|_{\mathscr{E}(t)}.$$

On $\partial\mathscr{C}_3$, we have similarly

$$|\xi| \int_{\partial\mathscr{C}_3} \frac{1}{t|\xi_2|^2} |\hat{z}(\xi_1)| d\sigma(\xi_1, \xi_2) \leqslant \frac{1}{t^{2/3}} \|\hat{z}\|_{L^1}.$$

Therefore

$$|I_2(\mathscr{C}_3)| \lesssim \frac{1}{t^{2/3}} \|\hat{z}\|_{L^1} \|\hat{z}\|_{W^{1,1}} \|v\|_{\mathscr{E}(t)}.$$

Together with (2.10), we conclude that

$$|I_2(\mathscr{C})| \lesssim \frac{1}{t^{2/3}} \|\hat{z}\|_{L^1} \|\hat{z}\|_{W^{1,1}} \|v\|_{\mathscr{E}(t)}.$$

Conclusion.
Summing up the bounds obtained in cases $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$, and observing that these cover $\mathbb{R}^2$, we conclude that

$$|I_1| \lesssim \frac{1}{t^{8/9}} \|\langle \xi \rangle \hat{z}\|_{W^{1,1}} \|v\|_{\mathscr{E}(t)}^2,$$
$$|I_2| \lesssim \frac{1}{t^{2/3}} \|\hat{z}\|_{L^1} \|\hat{z}\|_{W^{1,1}} \|v\|_{\mathscr{E}(t)}. \qquad \square$$

Lemma 12. *Given $t \in (0,1)$, $z \in \mathcal{Z}$ and $v \in \mathscr{E}(0,1)$,*

$$|v(t,x)| \lesssim \frac{1}{t^{1/3} \langle x/t^{1/3} \rangle^{1/4}} \|v(t)\|_{\mathscr{E}(t)}, \qquad |e^{-t\partial_x^3} z(x)| \lesssim \min\left(1, \frac{1}{t^{1/3} \langle x/t^{1/3} \rangle^{1/4}}\right) \|z\|,$$
$$|\partial_x v(t,x)| \lesssim \frac{\langle x/t^{1/3} \rangle^{1/4}}{t^{2/3}} \|v(t)\|_{\mathscr{E}(t)}, \qquad |\partial_x e^{-t\partial_x^3} z(x)| \lesssim \|z\|,$$

In particular, $u = v + e^{-t\partial_x^3} z$ satisfies

$$\|u u_x(t)\|_{L^\infty} \lesssim \frac{1}{t} \|v(t)\|_{\mathscr{E}(t)} (\|v(t)\|_{\mathscr{E}(t)} + \|z\|) + \|z\|^2,$$
$$\|u(t)\|_{L^\infty} \lesssim \frac{1}{t^{1/3}} \|v(t)\|_{\mathscr{E}(t)} + \|z\|,$$
$$\text{and} \qquad \|u(t)\|_{L^6} \lesssim \frac{1}{t^{5/18}} \|v(t)\|_{\mathscr{E}(t)} + \|z\|.$$

In the above bounds on u, observe the gain in powers of t on the z terms.

Proof. Since $\langle\xi\rangle\hat{z} \in L^1(\mathbb{R})$, one has directly $e^{-t\partial_x^3}z \in W^{1,\infty}(\mathbb{R})$. Also, $\hat{z} \in L^2(\mathbb{R})$, so that $e^{-t\partial_x^3}z \in L^2(\mathbb{R})$. In particular,

$$\|e^{-t\partial_x^3}z\|_{L^6} \lesssim |||z|||.$$

Furthermore, by [7, Lemma 6],

$$|v(t,x)| \lesssim \frac{1}{t^{1/3}\langle x/t^{1/3}\rangle^{1/4}}\|v(t)\|_{\mathscr{E}(t)}, \quad |v_x(t,x)| \lesssim \frac{1}{t^{2/3}}\langle x/t^{1/3}\rangle^{1/4}\|v(t)\|_{\mathscr{E}(t)}.$$

It remains to prove the pointwise spatial decay for

$$(e^{-t\partial_x^3}z)(x) = \frac{1}{t^{1/3}}\int \mathrm{Ai}\left(\frac{x-y}{t^{1/3}}\right) z(y)dy.$$

Recall the well-known estimate for the Airy-Fock function,

$$|\mathrm{Ai}(x)| \lesssim \langle x\rangle^{-1/4}.$$

If $|x-y| \leqslant |x|$, then

$$\begin{aligned}
|(e^{-t\partial_x^3}z)(x)| &\lesssim \left(\frac{1}{t^{1/3}}\int_{|x-y|\leqslant|x|}\left\langle\frac{x-y}{t^{1/3}}\right\rangle^{-1/4}\langle x\rangle^{-1}dy\right)\|\langle x\rangle z\|_{L^\infty} \\
&\lesssim \left(\frac{|x|}{t^{1/3}}\right)^{3/4}\langle x\rangle^{-1}|||z||| \lesssim \frac{1}{t^{1/3}\langle x/t^{1/3}\rangle^{1/4}}|||z|||.
\end{aligned}$$

If $|x-y| \geqslant |x|$, then

$$|(e^{-t\partial_x^3}z)(x)| \lesssim \frac{1}{t^{1/3}}\int\left\langle\frac{x-y}{t^{1/3}}\right\rangle^{-1/4}|z(y)|dy \lesssim \frac{1}{t^{1/3}\langle x/t^{1/3}\rangle^{1/4}}\|z\|_{L^1}.$$

For the uu_x estimate, one notices that

$$uu_x = vv_x + v_x e^{-t\partial_x^3}z + ve^{-t\partial_x^3}z_x + (e^{-t\partial_x^3}z)(e^{-t\partial_x^3}z_x).$$

The first two terms are estimated by observing the cancellation of the $\langle x/t^{1/3}\rangle^{1/4}$ factor. For the third term, the estimate is even better:

$$\|ve^{-t\partial_x^3}z_x\|_{L^\infty} \leqslant \|v\|_{L^\infty}\|e^{-t\partial_x^3}z_x\|_{L^\infty} \lesssim \frac{1}{t^{1/3}}\|v(t)\|_{\mathscr{E}(t)}|||z|||.$$

The quadratic term in z is bounded by a constant. The other estimates on u are easy consequences. □

3. Construction of an approximation sequence

Now that we possess the necessary estimates, let us begin the construction of the error function w in the same spirit as [7]. Fix $\chi \in \mathcal{S}(\mathbb{R})$ such that $0 < \chi \leqslant 1$ and $\chi \equiv 1$ on $[-1,1]$ and let $\chi_n(\xi) = \chi^2(\xi/n)$. Given $u \in \mathcal{S}'(\mathbb{R})$, set

$$\widehat{\Pi_n u} := \chi_n\hat{u}$$

and define the approximation space at time $t > 0$

$$X_n(t) := \left\{u \in \mathcal{S}'(\mathbb{R}) : \|u\|_{X_n(t)} < \infty\right\},$$

where

$$\|u\|_{X_n(t)} := \left\|e^{it\xi^3}\hat{u}\chi_n^{-1}\right\|_{L^\infty} + \left\|\partial_\xi\left(e^{it\xi^3}\hat{u}\right)\chi_n^{-1/2}\right\|_{L^2}.$$

If $I \subset [0,+\infty)$ is an interval,

$$\|u\|_{X_n(I)} := \sup_{t\in I}\|u(t)\|_{X_n(t)} = \sup_{t\in I}\|\tilde{u}(t)\chi_n^{-1}\|_{L^\infty} + \|\partial_\xi\tilde{u}(t)\chi_n^{-1/2}\|_{L^2},$$

and

$$X_n(I) := \left\{u \in \mathscr{C}(I,\mathcal{S}'(\mathbb{R})) : \tilde{u}\chi_n^{-1} \in \mathscr{C}(I,\mathscr{C}_b(\mathbb{R})),\ \partial_\xi\tilde{u}\chi_n^{-1/2} \in \mathscr{C}(I,L^2(\mathbb{R}))\right\}.$$

Keeping in mind we are looking for a solution that decomposes as $u(t) = S(t) + e^{-t\partial_x^3}z + w(t)$, we consider a suitable approximation of the error w by cutting off the nonlinearity at large frequencies. However, as we expand the nonlinearity, we need to address the linear term in w, $(SSw)_x$: for this, we also truncate the self-similar solution. In order to keep the self-similar structure, we set

$$\tilde{S}_n(t,\xi) = \chi_n(t^{1/3}\xi)\tilde{S}(t^{1/3}\xi).$$

This cut-off is well-behaved in $\mathscr{E}$:

$$\begin{aligned}\|S_n\|_{\mathscr{E}((0,+\infty))} &= \|S_n\|_{\mathscr{E}(1)} = \|\tilde{\chi}_n\tilde{S}\|_{L^\infty} + \|\partial_\xi(\chi_n\tilde{S})\|_{L^2}\\ &\lesssim \|\tilde{S}\|_{L^\infty} + \|\partial_x\tilde{S}\|_{L^2} + \|\partial_x\chi_n\|_{L^2}\|\tilde{S}\|_{L^\infty}\\ &\lesssim \|S\|_{\mathscr{E}(1)}.\end{aligned}$$

The resulting approximate problem is

$$\partial_t u_n + \partial_x^3 u_n = \Pi_n\partial_x(u_n^3), \quad u_n(t) - S_n(t) \to z \quad \text{as } t \to 0^+.$$

Writing

$$u_n(t) =: S_n(t) + e^{-t\partial_x^3}z + w_n(t),$$

one arrives at the (equivalent) problem for the error w_n

$$\partial_t w_n + \partial_x^3 w_n = \Pi_n\partial_x(u_n^3 - S_n^3), \quad w_n(0) = 0. \tag{3.1}$$

Observe that, in frequency space, this equation reads as

$$\partial_t\tilde{w}_n = \chi_n(N[\tilde{u}_n] - N[\tilde{S}_n]).$$

Proposition 13. *Given $z \in \mathcal{Z}$ and a self-similar solution $S \in \mathscr{E}((0,+\infty))$, there exists $T_n = T_n(z,S) > 0$ and a unique maximal solution $w_n \in X_n([0,T_n))$ of (3.1), in the sense that if $T_n < +\infty$, then $\|w_n\|_{X_n(t)} \to +\infty$ as $t \to T_n^-$. Moreover, there exists $0 < T_n^0 < T_n$ such that*

$$\forall t \in [0,T_n^0], \quad \|w_n\|_{X_n(t)} \lesssim_n t^{1/3}.$$

Remark 14. Since $w_n(0) = 0$, $\tilde{w}_n$ will not have any jump discontinuity at $\xi = 0$. Therefore $\partial_\xi\tilde{w}_n$ will be bounded in $L^2(\mathbb{R})$ (see Proposition 15).

Sketch of the proof. In this proof only, the implicit constants in the notation $\lesssim$ are allowed to depend on n (without further mention). The proof follows from a fixed-point argument in $X_n([0,T])$ for the map Ψ defined as

$$\widetilde{\Psi[w_n]}(t,\xi) = \chi_n(\xi)\int_0^t (N[\tilde{u}_n] - N[\tilde{S}_n])(s,\xi)ds.$$

for some $0 < T \leqslant 1$ to be determined later. Let us consider the source term

$$N[\tilde{S}_n,\tilde{S}_n,\tilde{z}](t,\xi) = i\frac{\xi}{4\pi^2}\iint_{\xi_1+\xi_2+\xi_3=\xi} e^{it\Phi}\tilde{S}_n(t,\xi_1)\tilde{S}_n(t,\xi_2)\tilde{z}(\xi_3)d\xi_1 d\xi_2.$$

The idea is to place the z factor in L^1 based spaces. To bound the L^∞ term in the X_n norm, we estimate

$$\begin{aligned}&|N[\tilde{S}_n,\tilde{S}_n,\tilde{z}](t,\xi)|\\ &\quad\lesssim \|\tilde{S}\|_{L^\infty}^2 \iint_{\xi_1+\xi_2+\xi_3=\xi} (|\xi_1|+|\xi_2|+|\xi_3|)\chi_n(t^{1/3}\xi_1)\chi_n(t^{1/3}\xi_2)|\hat{z}(\xi_3)|d\xi_1 d\xi_2\\ &\quad\lesssim \frac{\|\tilde{S}\|_{L^\infty}^2}{t^{2/3}}\|\xi\chi_n\|_{L^\infty}\|\chi_n\|_{L^1}\|\hat{z}\|_{L^1} + \frac{\|\tilde{S}\|_{L^\infty}^2}{t^{1/3}}\|\chi_n\|_{L^\infty}\|\chi_n\|_{L^1}\|\langle\xi\rangle\hat{z}\|_{L^1}\\ &\quad\lesssim \frac{1}{t^{2/3}}\|\langle\xi\rangle\chi_n\|_{L^\infty}\|\chi_n\|_{L^1}\|\langle\xi\rangle\hat{z}\|_{L^1}\|\tilde{S}\|_{L^\infty}^2.\end{aligned}$$

Therefore, for $t \in [0,1]$,

$$\left\| \int_0^t N[\tilde S_n, \tilde S_n, \tilde z](s,\xi)ds \right\|_{L^\infty} \lesssim \int_0^t \frac{\|\tilde S\|_{L^\infty}^2}{s^{2/3}} \|\langle\xi\rangle\chi_n\|_{L^\infty}\|\chi_n\|_{L^1}\|\langle\xi\rangle\hat z\|_{L^1}ds$$

$$\lesssim t^{1/3}\|\langle\xi\rangle\chi_n\|_{L^\infty}\|\chi_n\|_{L^1}\|\langle\xi\rangle\hat z\|_{L^1}\|\tilde S\|_{L^\infty}^2. \tag{3.2}$$

For the estimate of the L^2 term in X_n, we have to bound in L^2

$$\chi_n^{-1/2}(\xi)\partial_\xi\left(\chi_n(\xi)\int_0^t (N[\tilde S_n, \tilde S_n, \tilde z])(s,\xi)ds\right).$$

This requires us to consider the following three terms:

$$\left|\chi_n^{-1/2}\partial_\xi(\xi\chi_n(\xi))\frac{1}{\xi}N[\tilde S_n, \tilde S_n, \tilde z](t,\xi)\right|$$
$$\lesssim |\partial_\xi(\xi\chi_n^{1/2}(\xi))|\|\tilde S\|_{L^\infty}^2 \iint_{\xi_1+\xi_2+\xi_3=\xi} \chi_n(t^{1/3}\xi_1)\chi_n(t^{1/3}\xi_2)|\hat z(\xi_3)|d\xi_1 d\xi_2$$
$$\lesssim |\partial_\xi(\xi\chi_n^{1/2}(\xi))|\frac{1}{t^{1/3}}\|\chi_n\|_{L^1}\|\hat z\|_{L^1}\|\chi_n\|_{L^\infty}\|\tilde S\|_{L^\infty}^2,$$

$$\left|\xi\chi_n^{1/2}(\xi)\iint_{\xi_1+\xi_2+\xi_3=\xi} t(\partial_\xi\Phi)\ e^{it\Phi}\tilde S_n(t,\xi_1)\tilde S_n(t,\xi_2)\tilde z(t,\xi_3)d\xi_1 d\xi_2\right|$$
$$\lesssim |\xi\chi_n^{1/2}(\xi)|\|\tilde S\|_{L^\infty}^2 \iint_{\xi_1+\xi_2+\xi_3=\xi} t(\xi_1^2+\xi_2^2+\xi_3^2)\chi_n(t^{1/3}\xi_1)\chi_n(t^{1/3}\xi_2)|\hat z(\xi_3)|d\xi_1 d\xi_2$$
$$\lesssim |\xi\chi_n^{1/2}(\xi)|\|\langle\xi\rangle^2\chi_n\|_{L^1}\|\langle\xi\rangle^2\hat z\|_{L^1}\|\langle\xi\rangle^2\chi_n\|_{L^\infty}\|\tilde S\|_{L^\infty}^2,$$

$$\left|\chi_n^{1/2}(\xi)N[\tilde S_n, \tilde S_n, \partial_\xi\tilde z](t,\xi)\right|$$
$$\lesssim |\xi\chi_n^{1/2}(\xi)|\|\tilde S\|_{L^\infty}^2 \iint_{\xi_1+\xi_2+\xi_3=\xi} \chi_n(t^{1/3}\xi_1)\chi_n(t^{1/3}\xi_2)|\partial_\xi\hat z(\xi_3)|d\xi_1 d\xi_2$$
$$\lesssim |\xi\chi_n^{1/2}(\xi)|\frac{1}{t^{1/3}}\|\chi_n\|_{L^1}\|\hat z\|_{W^{1,1}}\|\chi_n\|_{L^\infty}\|\tilde S\|_{L^\infty}^2.$$

Since $\chi_n^{1/2} \in \mathcal S(\mathbb R)$, these terms are bounded in L^2 and

$$\left\|\chi_n^{-1/2}(\xi)\partial_\xi\left(\chi_n N[\tilde S_n, \tilde S_n, \tilde z](s,\xi)\right)\right\|_{L^2} \lesssim t^{-1/3}\|\tilde S\|_{L^\infty}^2 \|\!|z|\!\|.$$

After integration in time, we get for $t \in [0,1]$

$$\left\|\chi_n^{-1/2}(\xi)\partial_\xi\left(\chi_n \int_0^t N[\tilde S_n, \tilde S_n, \tilde z](s,\xi)ds\right)\right\|_{L^2} \lesssim t^{2/3}\|\tilde S\|_{L^\infty}^2 \|\!|z|\!\|,$$

and so, together with (3.2), we conclude

$$\left\|\chi_n\int_0^t N[\tilde S_n, \tilde S_n, \tilde z](s)ds\right\|_{X_n(t)} \lesssim t^{1/3}\|\tilde S\|_{L^\infty}^2 \|\!|z|\!\|.$$

The others source terms (where z is quadratic or cubic) can be treated in a similar fashion (and are actually better behaved).
Similarly, let us consider the linear term $N[\tilde S_n, \tilde S_n, \tilde w_n]$:

$$|N[\tilde S_n, \tilde S_n, \tilde w_n](t,\xi)|$$
$$\lesssim \|\tilde S\|_{L^\infty}^2\|\tilde w_n(t)\chi_n^{-1}\|_{L^\infty}\iint(|\xi_1|+|\xi_2|+|\xi_3|)\chi_n(t^{1/3}\xi_1)\chi_n(t^{1/3}\xi_2)\chi_n(\xi_3)d\xi_1 d\xi_2$$

$$\lesssim \|\tilde{S}\|_{L^\infty}^2 \|w_n\|_{X_n(t)} \left(\frac{1}{t^{2/3}} \|\xi\chi_n\|_{L^\infty} \|\chi_n\|_{L^1}^2 + \frac{1}{t^{1/3}} \|\chi_n\|_{L^1} \|\chi_n\|_{L^\infty} \|\xi\chi_n\|_{L^1} \right)$$
$$\lesssim \frac{1}{t^{2/3}} \|\langle\xi\rangle\chi_n\|_{L^\infty} \|\chi_n\|_{L^1} \|\tilde{S}\|_{L^\infty}^2 \|w_n\|_{X_n(t)}.$$

Hence, after integration in time,

$$\left\| \int_0^t N[\tilde{S}_n, \tilde{S}_n, \tilde{w}_n](s,\xi) ds \right\|_{L^\infty} \lesssim t^{1/3} \|w_n\|_{X_n([0,t])}. \tag{3.3}$$

For the L^2 term, we have

$$\left| \chi_n^{-1/2} \partial_\xi(\xi\chi_n(\xi)) \frac{1}{\xi} N[\tilde{S}_n, \tilde{S}_n, \tilde{w}_n](t,\xi) \right|$$
$$\lesssim |\partial_\xi(\xi\chi_n^{1/2}(\xi))| \|\tilde{S}\|_{L^\infty}^2 \|\tilde{w}_n(t)\chi_n^{-1}\|_{L^\infty} \iint \chi_n(t^{1/3}\xi_1)\chi_n(t^{1/3}\xi_2)\chi_n(\xi_3) d\xi_1 d\xi_2$$
$$\lesssim |\partial_\xi(\xi\chi_n^{1/2}(\xi))| \frac{1}{t^{1/3}} \|\chi_n\|_{L^\infty} \|\chi_n\|_{L^1}^2 \|\tilde{S}\|_{L^\infty}^2 \|w_n\|_{X_n(t)},$$

$$\left| \xi\chi_n^{1/2}(\xi) \iint_{\xi_1+\xi_2+\xi_3=\xi} t(\partial_\xi\Phi)\, e^{it\Phi} \tilde{S}_n(t,\xi_1)\tilde{S}_n(t,\xi_2)\tilde{w}_n(t,\xi_3) d\xi_1 d\xi_2 \right|$$
$$\lesssim \|\tilde{S}\|_{L^\infty}^2 \|\tilde{w}_n(t)\chi_n^{-1}\|_{L^\infty} |\xi\chi_n^{1/2}(\xi)|$$
$$\times \iint t(\xi_1^2 + \xi_2^2 + \xi_3^2)\chi_n(t^{1/3}\xi_1)\chi_n(t^{1/3}\xi_2)\chi(\xi_3) d\xi_1 d\xi_2$$
$$\lesssim |\xi\chi_n^{1/2}(\xi)| \|\langle\xi\rangle^2\chi_n\|_{L^\infty} \|\chi_n\|_{L^1}^2 \|\tilde{S}\|_{L^\infty}^2 \|w_n\|_{X_n(t)},$$

$$\left| \chi_n^{1/2}(\xi) N[\tilde{S}_n, \tilde{S}_n, \partial_\xi\tilde{w}_n](t,\xi) \right|$$
$$\lesssim |\xi\chi_n^{1/2}(\xi)| \|\tilde{S}\|_{L^\infty}^2 \iint \chi_n(t^{1/3}\xi_1)\chi_n(t^{1/3}\xi_2) |\partial_\xi\tilde{w}_n(t,\xi_3)| d\xi_1 d\xi_2$$
$$\lesssim |\xi\chi_n^{1/2}(\xi)| \|\tilde{S}\|_{L^\infty}^2 \|\partial_\xi\tilde{w}_n\chi_n^{-1/2}\|_{L^2}$$
$$\times \int \left(\int \chi_n^2(t^{1/3}\xi_1)\chi_n(\xi_3) d\xi_1 \right)^{1/2} \chi_n(t^{1/3}\xi_2) d\xi_2$$
$$\lesssim |\xi\chi_n^{1/2}(\xi)| \|\tilde{S}\|_{L^\infty}^2 \|w_n\|_{X_n(t)} \frac{1}{t^{1/3}} \|\chi_n\|_{L^\infty} \|\chi_n\|_{L^1}^{1/2} \|\chi_n\|_{L^1}.$$

Therefore, taking the L^2 norm in ξ gives

$$\left\| \chi_n^{-1/2} \partial_\xi \left(\chi_n N[\tilde{S}_n, \tilde{S}_n, \tilde{w}_n](t) \right) \right\|_{L^2} \lesssim t^{-1/3} \|\tilde{S}\|_{L^\infty}^2 \|w_n\|_{X_n(t)}.$$

Integrating in time, we get, for $t \in [0,1]$,

$$\left\| \chi_n^{-1/2} \partial_\xi \left(\chi_n \int_0^t N[\tilde{S}_n, \tilde{S}_n, \tilde{z}](s) ds \right) \right\|_{L^2} \lesssim t^{2/3} \|\tilde{S}\|_{L^\infty}^2 \|w_n\|_{X_n([0,t])},$$

and with (3.3), we obtain

$$\left\| \chi_n \int_0^t N[\tilde{S}_n, \tilde{S}_n, \tilde{w}_n](s,\xi) ds \right\|_{X_n(t)} \lesssim t^{1/3} \|\tilde{S}\|_{L^\infty}^2 \|w_n\|_{X_n([0,t])}.$$

All the remaining terms can be estimated similarly (and enjoy in fact better bounds) and the difference estimates can be performed in the same way as well. Choosing T suffcently small, Φ becomes a contraction over $X_n([0,T])$. This concludes the proof. □

In order to take the limit $n \to \infty$, one must prove that the maximal existence time T_n does not tend to 0 and also that the approximations remain bounded in $\mathscr{E}$. To do this, we actually prove *a priori* bounds in the stronger spaces X_n, thus tackling both problems at once. The methodology to prove this follows the heuristics presented in the introduction, formalized in a suitable bootstrap argument; we emphasize that the specific structure of the X_n spaces will be used throughout the proof of Proposition 15 below.
In the remainder of this work, we now assume that, for some small δ to be fixed later on, (1.9) holds:

$$|||z||| + \|S\|_{\mathscr{E}((0,+\infty))} < \delta.$$

In order to bound $\partial_\xi \tilde{w}_n$, the scaling operator I comes into play: we recall its definition given in (1.5)

$$\widehat{Iu}(t,\xi) = ie^{it\xi^3}\left(\partial_\xi \tilde{u} - \frac{3t}{\xi}\partial_t \tilde{u}\right).$$

Proposition 15. *There exist $\delta_0 > 0$ such that, given $\delta \in (0,\delta_0]$, the solution w_n of Proposition 13 satisfies $T_n > 1$ and*

$$\forall t \in (0,1], \quad \|w_n(t)\|_{\mathscr{E}(t)} \lesssim \delta^3 t^{1/9} \quad \text{and} \quad \|w_n(t)\|_{L^2} \lesssim \delta^3 t^{1/18}. \tag{3.4}$$

Moreover, $\partial_\xi \tilde{w}_n(t) \in L^2(\mathbb{R})$, $\partial_t \tilde{w}_n(t) \in L^\infty(\mathbb{R})$ and

$$\forall t \in (0,1], \quad \|\partial_\xi \tilde{w}_n(t)\|_{L^2} \lesssim \delta^3 t^{5/18} \quad \text{and} \quad \|\partial_t \tilde{w}_n(t)\|_{L^\infty} \lesssim \frac{\delta^3}{t^{8/9}}. \tag{3.5}$$

Proof. We perform a bootstrap argument. Fix $\delta_0 \in (0,1)$ and $A = A(\delta_0) \geqslant 1$ to be chosen later on. For now, we only require $A\delta_0^2 \leqslant 1$. We let $\delta \in (0,\delta_0]$.
Define, for any $t \in (0,T_n)$,

$$f_n(t) = t^{-1/9}\left(\|\tilde{w}_n(t)\chi_n^{-1}\|_{L^\infty} + t^{-1/6}\|\partial_\xi \tilde{w}_n(t)\chi_n^{-1/2}\|_{L^2}\right)$$

and

$$\tau_n = \sup\left\{t \in [0,\min(1,T_n)) : \forall s \in (0,t],\ f_n(s) \leqslant A\delta^3\right\}.$$

In the following argument, the implicit constants in $\lesssim$ do not depend on δ or A.
From Proposition 13, f_n is continuous on $(0,T_n)$ and $f_n(t) \lesssim t^{1/18}$ for $t \in [0,T_n^0]$, so that $f_n(t) \to 0$ as $t \to 0^+$. In particular, $\tau_n > 0$.

Step 1. Improved estimates on the nonlinear term.
First observe that

$$\forall t \in (0,\tau_n), \quad \|w_n(t)\|_{\mathscr{E}(t)} \leqslant t^{1/9} f_n(t) \leqslant A\delta^3 t^{1/9}. \tag{3.6}$$

Using the estimates from Lemma 12, for all $t \in [0,\tau_n]$,

$$\begin{aligned}
\|u_n^3 - S_n^3\|_{L^2} &\lesssim \left(\|w_n\|_{L^6}^2 + \|S_n\|_{L^6}^2 + \|e^{-t\partial_x^3}z\|_{L^6}^2\right)\left(\|w_n\|_{L^6} + \|e^{-t\partial_x^3}z\|_{L^6}\right) \\
&\lesssim \left(\frac{1}{t^{5/9}}\|w_n(t)\|_{\mathscr{E}(t)}^2 + \frac{1}{t^{5/9}}\|S_n(t)\|_{\mathscr{E}(t)}^2 + |||z|||^2\right)\left(\frac{1}{t^{5/18}}\|w_n(t)\|_{\mathscr{E}(t)} + |||z|||\right) \\
&\lesssim \frac{1}{t^{5/9}}\left((A\delta^3)^2 t^{2/9} + \delta^2\right)\left(\frac{A\delta^3 t^{1/9}}{t^{5/18}} + \delta\right).
\end{aligned}$$

Recall that $A\delta^2 \leqslant 1$ so that $A\delta^3 t^{1/9} \leqslant \delta$, and so

$$\forall t \in [0,\tau_n), \quad \|u_n^3 - S_n^3\|_{L^2} \lesssim \frac{A\delta^5}{t^{13/18}} + \frac{\delta^3}{t^{5/9}}. \tag{3.7}$$

Let emphasize the gain of $t^{1/9}$ with respect to the rough estimate of Lemma 12

$$\|v^3\|_{L^2} = \|v\|_{L^6}^3 \lesssim \frac{\|v\|_{\mathscr{E}(t)}^3}{t^{5/6}}.$$

Step 2. A priori estimate for Iw_n. In this step, we prove that

$$\forall t \in [0,\tau_n), \quad \left\| \widehat{Iw_n}(t)\chi_n^{-1/2} \right\|_{L^2} \lesssim \delta^3 t^{5/18}\left(t^{1/18} + A\delta^2\right). \tag{3.8}$$

Let us notice that from the equation for $\tilde{w}_n$, Lemma 12 and Proposition 13, given $t < T_n^0$,

$$\begin{aligned}
\left\| \widehat{Iw_n}(t)\chi_n^{-1/2} \right\|_{L^2} &\leqslant \|\partial_\xi \tilde{w}_n(t)\chi_n^{-1/2}\|_{L^2} + 3t\|\chi_n^{1/2}\mathcal{F}(u_n^3 - S_n^3)\|_{L^2} \\
&\lesssim \|\partial_\xi \tilde{w}_n(t)\chi_n^{-1/2}\|_{L^2} + 3t\|u_n^3 - S_n^3\|_{L^2} \\
&\lesssim \|w_n(t)\|_{X_n(t)} + t^{1/6}(\delta^3 + \|w_n(t)\|_{X_n(t)}^3) \lesssim t^{1/3} + t^{1/6}(\delta^3 + t).
\end{aligned}$$

We conclude that

$$\forall \gamma < 1/6, \quad \left\| \widehat{Iw_n}(t)\chi_n^{-1/2} \right\|_{L^2} = o(t^\gamma) \quad \text{as} \quad t \to 0^+. \tag{3.9}$$

Now that we have a control near $t = 0$, we aim to control Iw_n for $0 < t < 1$. Denote

$$\widehat{\Pi_n' u}(\xi) = \chi_n'(\xi)\hat{u}(\xi).$$

A simple computation yields

$$\begin{aligned}
(\partial_t + \partial_x^3)Iw_n &= 3\Pi_n(u_n^2(Iu_n)_x - S_n^2(IS_n)_x) + \Pi_n'(u_n^3 - S_n^3)_x \\
&= 3\Pi_n(u_n^2(Iw_n)_x) + 3\Pi_n(u_n^2(Ie^{-t\partial_x^3}z)_x) + \Pi_n'(u_n^3 - S_n^3)_x
\end{aligned}$$

where we used the decisive property $(IS_n)_x \equiv 0$. Equivalently, on Fourier side it writes

$$\partial_t \widehat{Iw_n} - i\xi^3\widehat{Iw_n} = 3\chi_n\mathcal{F}\left(u_n^2(Iw_n)_x\right) + 3\chi_n\mathcal{F}\left(u_n^2(Ie^{-t\partial_x^3}z)_x\right) + i\xi\chi_n'\mathcal{F}\left(u_n^3 - S_n^3\right)$$

We now multiply by $\overline{\widehat{Iw_n}}\chi_n^{-1}$, integrate on $\mathbb{R}$ and take the real part. Due to the jump of $\tilde{u}_n$ at $\xi = 0$, an extra care should be taken in the computations: one should first integrate over $\mathbb{R}\backslash(-\varepsilon,\varepsilon)$ and let $\epsilon \to 0$. This procedure shows that no unexpected term occur, we refer to [7, Lemma 13] for full details. This justifies the following computations:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int |\widehat{Iw_n}|^2\chi_n^{-1}d\xi = 3\int \mathcal{F}(u_n^2(Iw_n)_x)\overline{\widehat{Iw_n}}d\xi + 3\int \mathcal{F}(u_n^2(Ie^{-t\partial_x^3}z)_x)\overline{\widehat{Iw_n}}d\xi \\
- \operatorname{Im}\int \xi\frac{\chi_n'}{\chi_n}\mathcal{F}(u_n^3 - S_n^3)\overline{\widehat{Iw_n}}d\xi.
\end{aligned}$$

Now by Plancherel and integration by parts,

$$\begin{aligned}
&\int \mathcal{F}(u_n^2(Iw_n)_x)\overline{\widehat{Iw_n}}d\xi = \frac{1}{2\pi}\int u_n^2(Iw_n)_x(Iw_n)dx = -\frac{1}{2\pi}\int (u_n)_x u_n(Iw_n)^2dx \\
&\left|\int \mathcal{F}(u_n^2(Iw_n)_x)\overline{\widehat{Iw_n}}d\xi\right| \lesssim \frac{\delta^2}{t}\|Iw_n\|_{L^2}^2 \lesssim \frac{\delta^2}{t}\|\widehat{Iw_n}\|_{L^2}^2, \\
&\left|\int \mathcal{F}(u_n^2(Ie^{-t\partial_x^3}z)_x)\overline{\widehat{Iw_n}}d\xi\right| \leqslant \|u_n^2(Ie^{-t\partial_x^3}z)_x\|_{L^2}\|\widehat{Iw_n}\|_{L^2} \\
&\qquad \lesssim \|u_n\|_{L^\infty}^2\|(Ie^{-t\partial_x^3}z)_x\|_{L^2}\|\widehat{Iw_n}\|_{L^2} \\
&\qquad \lesssim \frac{\delta^2}{t^{2/3}}\|\xi\partial_\xi\hat{z}\|_{L^2}\|\widehat{Iw_n}\|_{L^2} \lesssim \frac{\delta^3}{t^{2/3}}\|\widehat{Iw_n}\|_{L^2}, \\
&\left|\int \xi\frac{\chi_n'}{\chi_n}\mathcal{F}(u_n^3 - S_n^3)\overline{\widehat{Iw_n}}d\xi\right| = 2\left|\int \xi(\chi_n^{1/2})'\mathcal{F}(u_n^3 - S_n^3)\overline{\widehat{Iw_n}}\chi_n^{-1/2}d\xi\right|, \\
&\qquad \leqslant \|\xi(\chi_n^{1/2})'\|_{L^\infty}\|u_n^3 - S_n^3\|_{L^2}\|\widehat{Iw_n}\chi_n^{-1/2}\|_{L^2} \\
&\qquad \lesssim \left(\frac{A\delta^5}{t^{13/18}} + \frac{\delta^3}{t^{5/9}}\right)\|\widehat{Iw_n}\chi_n^{-1/2}\|_{L^2}.
\end{aligned}$$

(We also used (3.7) for the last estimate). As $0 < \chi_n < 1$, $\|\widehat{Iw_n}\|_{L^2} \leqslant \|\widehat{Iw_n}\chi_n^{-1/2}\|_{L^2}$. Therefore, we obtain, for $t < \tau_n$,

$$\frac{1}{2}\frac{d}{dt}\left\|\widehat{Iw_n}(t)\chi_n^{-1/2}\right\|_{L^2}^2 \lesssim \frac{\delta^2}{t}\left\|\widehat{Iw_n}(t)\chi_n^{-1/2}\right\|_{L^2}^2 + \left(\frac{\delta^3}{t^{2/3}} + \left(\frac{A\delta^5}{t^{13/18}} + \frac{\delta^3}{t^{5/9}}\right)\right)\left\|\widehat{Iw_n}(t)\chi_n^{-1/2}\right\|_{L^2}.$$

This implies that, for some universal constant C,

$$\left|\frac{d}{dt}\left(t^{-2C\delta^2}\left\|\widehat{Iw_n}(t)\chi_n^{-1/2}\right\|_{L^2}\right)\right| \leqslant 2Ct^{-2C\delta^2}\left(\frac{\delta^3}{t^{2/3}} + \frac{A\delta^5}{t^{13/18}}\right).$$

Now, for $\delta < \delta_0$ small enough, $2C\delta^2 < 5/18$ so that the right hand side is integrable in time and due to (3.9), $t^{-2C\delta^2}\left\|\widehat{Iw_n}(t)\chi_n^{-1/2}\right\|_{L^2} \to 0$ as $t \to 0^+$. Hence, we can integrate the above estimate on $[0, t]$ and get

$$\forall t \in [0, \tau_n), \quad \left\|\widehat{Iw_n}(t)\chi_n^{-1/2}\right\|_{L^2} \lesssim \delta^3 t^{5/18}\left(t^{1/18} + A\delta^2\right),$$

which is (3.8).

Step 3. A priori estimate for $\partial_t w_n$. We claim that for all $n \in \mathbb{N}$,

$$\begin{aligned}\forall t \in (0, T_n), \quad \left\|\partial_t \tilde{w}_n(t)\chi_n^{-1}\right\|_{L^\infty} &\lesssim \frac{1}{t}\|w_n(t)\|_{\mathscr{E}(t)}\left(\|w_n(t)\|_{\mathscr{E}(t)}^2 + \|S\|_{\mathscr{E}(1)}^2\right) \\ &+ \frac{1}{t^{8/9}}|||z|||\left(\|w_n(t)\|_{\mathscr{E}(t)}^2 + \|S\|_{\mathscr{E}(1)}^2 + |||z|||^2\right).\end{aligned} \tag{3.10}$$

Indeed, we have

$$\partial_t \tilde{w}_n(t)\chi_n^{-1} = N[\tilde{S}_n + \tilde{z} + \tilde{w}_n](t) - N[\tilde{S}_n](t).$$

For the nonlinear terms with at least one z, we use Lemma 11 which gives the pointwise (in ξ) bound

$$\frac{1}{t^{8/9}}(\|S\|_{\mathscr{E}(1)} + \|w_n(t)\|_{\mathscr{E}(t)})^2|||z||| + \frac{1}{t^{2/3}}(\|S\|_{\mathscr{E}(1)} + \|w_n(t)\|_{\mathscr{E}(t)})|||z|||^2 + |||z|||^3.$$

The remaining terms are $N[\tilde{S}_n, \tilde{S}_n, \tilde{w}_n]$, $N[\tilde{S}_n, \tilde{w}_n, \tilde{w}_n]$ and $N[\tilde{w}_n, \tilde{w}_n, \tilde{w}_n]$ (they all have at least one w_n). Using Lemma 10, they are bounded pointwise by

$$\frac{1}{t}(\|S\|_{\mathscr{E}(1)} + \|w_n(t)\|_{\mathscr{E}(t)})^2\|w_n\|_{\mathscr{E}(t)}.$$

This gives (3.10). If we restrict to the interval $(0, \tau_n)$, using (3.6), and $A\delta^2 \leqslant 1$, this rewrites simply

$$\forall t \in (0, t_n), \quad \left\|\partial_t \tilde{w}_n(t)\chi_n^{-1}\right\|_{L^\infty} \lesssim \frac{\delta^3}{t^{8/9}}. \tag{3.11}$$

Step 4. Bound on $f_n(t)$. Let us bound separately the two terms of f_n. First, after integration in time of (3.11) (recall that $\|\tilde{w}_n(t)\chi_n^{-1}\|_{L^\infty} \to 0$ as $t \to 0^+$), we get

$$\forall t \in [0, \tau_n], \quad \|\tilde{w}_n(t)\chi_n^{-1}\|_{L^\infty} \lesssim \delta^3 t^{1/9}. \tag{3.12}$$

Second, in view of the definition of I, we can write

$$\partial_\xi \tilde{w}_n = -ie^{it\xi^3}\widehat{Iw_n} + \frac{3t}{\xi}\partial_t \tilde{w}_n = -ie^{it\xi^3}\widehat{Iw_n} + 3te^{-it\xi^3}\chi_n\mathcal{F}(u_n^3 - S_n^3),$$

and so, using (3.8) and (3.7), we infer

$$\begin{aligned}\left\|\partial_\xi \tilde{w}_n\chi_n^{-1/2}\right\|_{L^2} &\leqslant \left\|\widehat{Iw_n}(t)\chi_n^{-1/2}\right\|_{L^2} + 3t\|u_n^3 - S_n^3\|_{L^2} \\ &\lesssim \delta^3 t^{5/18}(t^{1/18} + A\delta^2) + A\delta^5 t^{5/18} + \delta^3 t^{4/9}\end{aligned}$$

$$\lesssim \delta^3 t^{1/3} + A\delta^5 t^{5/18}. \tag{3.13}$$

Estimates (3.12) and (3.13) give that, for all $t \in [0, \tau_n]$,

$$\begin{aligned} f_n(t) &= t^{-1/9}\left(\|\tilde{w}_n(t)\chi_n^{-1}\|_{L^\infty} + t^{-1/6}\|\partial_\xi \tilde{w}_n \chi_n^{-1/2}\|_{L^2}\right) \\ &\lesssim t^{-1/9}\left(\delta^3 t^{1/9} + t^{-1/6}(\delta^3 t^{1/3} + A\delta^5 t^{5/18})\right) \lesssim \delta^3 + A\delta^5. \end{aligned}$$

Let us remark that both terms in f_n contribute (the leading powers of t cancel for both).

Since $A\delta^2 \leqslant 1$, there exists a universal constant M such that

$$\forall t \in [0, \tau_n], \quad f_n(t) \leqslant M\delta^3. \tag{3.14}$$

Step 5. Closing the bootstrap. Choosing

$$\delta_0 = \min\left\{\frac{1}{2\sqrt{M}}, \sqrt{\frac{5}{36C}}\right\} \quad \text{and then} \quad A = 1/\delta_0^2,$$

Step 2 and (3.14) imply that

$$\forall t \in [0, \tau_n], \quad f_n(t) \leqslant \frac{A}{2}\delta^3.$$

From a continuity argument, we necessarily have $\tau_n = \min(1, T_n)$. But if $T_n \leqslant 1$, then we also have

$$\sup_{t \in [0, T_n)} \|w_n(t)\|_{X_n(t)} \leqslant \sup_{t \in [0, T_n)} f_n(t) \leqslant A\delta^3,$$

which contradicts the maximality of T_n in Proposition 13. Hence $T_n > 1$ and $\tau_n = 1$. As $\|w_n(t)\|_{\mathscr{E}(t)} \leqslant t^{1/9} f_n(t)$, this gives the first part of (3.4). Also notice that $\|\partial_x \tilde{w}_n(t)\|_{L^2} \leqslant t^{5/18} f_n(t)$, and in view of (3.11), we also obtain both estimates of (3.5).

Step 6. L^2 bound on w_n. Finally, we will prove that

$$\forall t \in [0, 1], \quad \|\hat{w}_n(t)\chi_n^{-1/2}\|_{L^2} \lesssim \sqrt{A}\delta^3 t^{1/18}. \tag{3.15}$$

(3.12) gives that, for each n, $\|\hat{w}_n \chi_n^{-1}\|_{L^\infty} \lesssim A\delta^3 t^{1/9}$, so that one has for free that $w_n \chi_n^{-1/2} \in L^\infty([0,1], L^2)$ and for all $t \in [0,1]$

$$\|\hat{w}_n(t)\chi_n^{-1/2}\|_{L^2} \lesssim \|\hat{w}_n(t)\chi_n^{-1}\|_{L^\infty} \|\chi_n^{1/2}\|_{L^2} \lesssim_n A\delta^3 t^{1/9}. \tag{3.16}$$

Since

$$\partial_t \hat{w}_n - i\xi^3 \hat{w}_n = -i\xi\chi_n \mathcal{F}(u_n^3 - S_n^3),$$

multiplying by $\overline{\hat{w}_n}\chi_n^{-1}$, integrating in ξ and taking the real part

$$\frac{1}{2}\frac{d}{dt}\int |\hat{w}_n(t)|^2 \chi_n^{-1} d\xi = -\operatorname{Re}\int i\xi \mathcal{F}((u_n^3 - S_n^3)\overline{\hat{w}_n} d\xi = \int (u_n^3 - S_n^3)\partial_x w_n dx.$$

We claim that

$$\begin{aligned} \left|\int (u_n^3 - S_n^3)\partial_x w_n dx\right| \lesssim \frac{1}{t}&\left(\|w_n(t)\|_{\mathscr{E}(t)} + \|S_n\|_{\mathscr{E}(1)}^2 + \|e^{-t\partial_x^3} z\|_{\mathscr{E}(t)}^2\right) \\ &\cdot \left(\|w_n\|_{L^2}^2 + \|z\|_{L^1}\|w_n(t)\|_{\mathscr{E}(t)}\right). \end{aligned} \tag{3.17}$$

To see this, we expand the terms (made of 4 factors) and split them depending on whether w_n occurs at most once or at least twice. In the following discussion, the factors v_1, v_2 stand for either one of w_n, S_n or $e^{-t\partial_x^3} z$.

For terms where w_n occur at most once, $e^{-t\partial_x^3} z$ also appear, and they all can take the form

$$\int v_1 v_2 (e^{-t\partial_x^3} z)\partial_x w_n dx = \int \mathcal{F}(v_1 v_2 \partial_x w_n)\overline{\mathcal{F}(e^{-t\partial_x^3} z)} d\xi,$$

which is bounded by

$$\begin{aligned}\|\mathcal{F}(v_1v_2\partial_x w_n)\|_{L^\infty}\|\mathcal{F}(e^{-t\partial_x^3}z)\|_{L^1} &= \|N[\tilde{v}_1,\tilde{v}_2,i\xi\tilde{w}_n](t)\|_{L^\infty}\|\hat{z}\|_{L^1}\\ &\lesssim \frac{1}{t}\|v_1(t)\|_{\mathscr{E}(t)}\|v_2(t)\|_{\mathscr{E}(t)}\|w_n(t)\|_{\mathscr{E}(t)}\|\hat{z}\|_{L^1},\end{aligned}$$

where we used Lemma 9 in a crucial way.
For terms where w_n occur at least twice, by integration by parts, we see that they can all take the form

$$\int \partial_x(v_1v_2)(w_n)^2dx$$

and so, they are bounded by

$$\|\partial_x(v_1v_2)(t)\|_{L^\infty}\|w_n(t)\|_{L^2}^2 \lesssim \frac{1}{t}\|v_1(t)\|_{\mathscr{E}(t)}\|v_2(t)\|_{\mathscr{E}(t)}\|w_n(t)\|_{L^2}^2.$$

This proves (3.17).
Since $\|w_n\|_{L^2} \lesssim \|\hat{w}_n\chi_n^{-1/2}\|_{L^2}$, and, by Step 5, $\|w_n(t)\|_{\mathscr{E}(t)} \lesssim A\delta^3t^{1/9}$ (valid for $t\in[0,1]$), (3.17) implies that

$$\left|\frac{d}{dt}\|\hat{w}_n(t)\chi_n^{-1/2}\|_{L^2}^2\right| \lesssim \frac{\delta^2}{t}\left(\|\hat{w}_n(t)\chi_n^{-1/2}\|_{L^2}^2 + A\delta^4t^{1/9}\right).$$

Recalling that from (3.16), $\|\hat{w}_n(t)\chi_n^{-1/2}\|_{L^2} = O(t^{1/9})$, a Gronwall argument as in Step 2 gives (3.15). By Plancherel, the L^2 bound in (3.4) follows. □

4. Proof of the main result

Proposition 16 (Uniqueness)**.** *There exists a universal constant $K>0$ such that, given $\eta>\delta^2/K^2$ and $T>0$, there exists at most one solution to* (1.8) *satisfying*

$$\forall t\in(0,T],\quad \|w(t)\|_{\mathscr{E}(t)}\leqslant K\sqrt{\eta}\quad \textit{and}\quad \|w(t)\|_{L^2}\leqslant t^\eta.$$

Proof. Suppose w_1, w_2 are two solutions in the above conditions and set $w = w_1 - w_2$. Then

$$\partial_t w + \partial_x^3 w = (u_1^3-u_2^3)_x,\quad u_j(t) = S(t) + e^{-t\partial_x^3}z + w_j(t),\ j=1,2.$$

Observe that

$$\|u_1(t)\|_{\mathscr{E}(t)},\ \|u_2(t)\|_{\mathscr{E}(t)} \lesssim \delta \lesssim K\sqrt{\eta}.$$

Define $A = \sup_{t\in(0,T]} t^{-\eta}\|w(t)\|_{L^2}$ (which is finite by assumption). Direct integration and Lemma 12 give that for all $t\in[0,T]$,

$$\begin{aligned}\|w(t)\|_{L^2}^2 &\leqslant \int_0^t\int(u_1^3-u_2^3)_x w dxds \leqslant \frac{1}{2}\int_0^t \|(u_1^2+u_1u_2+u_2^2)_x\|_{L^\infty}\|w(s)\|_{L^2}^2ds\\ &\leqslant C\int_0^t \frac{K^2\eta}{s}As^{2\eta}ds \leqslant \frac{CK^2}{2}A^2t^{2\eta}.\end{aligned}$$

Dividing by $t^{2\eta}$ and taking the supremum in $t\in[0,T]$, we get $A^2 \leqslant \frac{CK^2}{2}A^2$, which implies $A=0$ for $K^2<2/C$. □

Remark 17. In [7], forward uniqueness of solutions in $\mathscr{E}$ was obtained for strictly positive times. The argument goes through an estimate for the L^2 norm on positive half-lines (which is finite for elements in $\mathscr{E}$). The bounds given by Lemma 12 give a behavior of $1/t$, which must then be integrated in $(0,T)$. This can be compensated if one assumes the polynomial growth $\|w(t)\|_{L^2_x}\leqslant t^\eta$. In conclusion, this strategy can be used to provide an alternate proof of Proposition 16, but not to further improve it.

Proof of Theorem 7. Consider the approximations w_n defined in Proposition 13. By Proposition 15, these solutions are defined on $[0,1]$ and

$$\|w_n(t)\|_{\mathscr{E}(t)} \lesssim \delta^3 t^{1/9}, \quad \|w_n(t)\|_{L^2} \lesssim \delta^3 t^{1/18}. \tag{4.1}$$

Notice that we also have, for $t \in (0,1]$,

$$\|\partial_\xi \tilde{w}_n(t)\|_{L^2} \lesssim \delta^3 t^{5/18} \quad \text{and} \quad \|\partial_t \tilde{w}_n(t)\|_{L^\infty} \lesssim \frac{\delta^3}{t^{8/9}},$$

so that, by Sobolev embedding,

$$\forall t \in [0,1],\ \forall \xi_1, \xi_2 \in \mathbb{R}, \quad |\tilde{w}_n(t,\xi_1) - \tilde{w}_n(t,\xi_2)| \lesssim \delta^3 t^{5/18} |\xi_1 - \xi_2|^{1/2}, \tag{4.2}$$

and,

$$\forall t_1, t_2 \in [0,1],\ \forall \xi \in \mathbb{R}, \quad |\tilde{w}_n(t_1,\xi) - \tilde{w}_n(t_2,\xi)| \lesssim \delta^3 |t_1^{1/9} - t_2^{1/9}|. \tag{4.3}$$

Consequently, for any $R > 0$, $(\tilde{w}_n)_{n\in\mathbb{N}}$ is equibounded and equicontinuous on $[0,1] \times [-R,R]$. By Ascoli-Àrzela theorem,

$$\tilde{w}_n \to \tilde{w} \quad \text{uniformly in } [0,T] \times [-R,R]$$

and $\tilde{w}$ satisfies (4.1), (4.2) and (4.3). In particular, $w \in \mathscr{E}((0,1)) \cap L^\infty((0,1), L^2(\mathbb{R}))$ and bound (1.10) holds.
We now prove that w solves (1.8) in the sense of distributions. Since w_n is uniformly bounded in $\mathscr{E}((0,1))$, Lemma 12 implies that w_n is also equibounded and equicontinuous on $[\varepsilon, 1] \times [-R,R]$ for any $\varepsilon > 0$ and $R > 0$. Thus

$$w_n \to w \quad \text{uniformly in } [\varepsilon,1] \times [-R,R].$$

Since

$$|w_n(t,x)| \lesssim \frac{1}{t^{1/3} \langle x/t^{1/3} \rangle^{1/4}},$$

the uniform convergence implies that $w_n \to w$ in $L^\infty((\varepsilon,1), L^6(\mathbb{R}))$. The exact same reasoning also yields $S_n \to S$ in $L^\infty((\varepsilon,1), L^6(\mathbb{R}))$. These convergences can now be used to conclude that

$$(u_n^3 - S_n^3)_x \to (u^3 - S^3)_x \text{ in } \mathscr{D}'((\varepsilon,1) \times \mathbb{R})$$

and that w satisfies (1.8) in the distributional sense on $(0,1)$.
To extend the solution up to $t = +\infty$, observe that

$$\forall t \in (0,1], \quad \|u(t)\|_{\mathscr{E}(t)} \leqslant \|S(t)\|_{\mathscr{E}(t)} + \|e^{-t\partial_x^3} z\|_{\mathscr{E}(t)} + C\delta^3 t^{1/9} \leqslant 3\delta.$$

Thus the global existence result of [7, Theorem 2] can be applied (at $t = 1/2$) to extend u for all positive times. Finally, Proposition 16 with $\eta = 1/18$ gives the uniqueness property (decreasing the value of δ_0 further, if necessary). $\square$

Appendix A. Proof of Estimate (2.1)

Proof of Lemma 9. As we are to prove a trilinear estimate, we can rescale and we will assume, without loss of generality, that

$$\|f\|_{\mathscr{E}(t)} = \|g\|_{\mathscr{E}(t)} = \|h\|_{\mathscr{E}(t)} = 1,$$

so that

$$\|\tilde{f}\|_{L^\infty} \leqslant 1 \quad \text{and} \quad \|\partial_\xi \tilde{f}\|_{L^2} \leqslant t^{1/6}, \tag{A.1}$$

and the same for $\tilde{g}$ and $\tilde{h}$.
As in Lemma 11, given a domain $\mathscr{D} \subset \mathbb{R}^2$, we write

$$J(\mathscr{D}) = \iint_{\mathscr{D}} e^{it\Phi} \tilde{f}(\xi_1) \tilde{g}(\xi_2) \xi_3 \tilde{h}(\xi_3) d\xi_1 d\xi_2.$$

We will only do the proof when the integral is restricted to the domain

$$\mathscr{A} := \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_3| \geqslant |\xi_2| \geqslant |\xi_1|\},$$

where ξ_3 is the largest frequency: this is actually the worst-case scenario, and it is clear from the computations below how to adapt the estimates to the others domains.

1) For small frequencies, we have a crude bound: let $\mathscr{A}_0 = \{(\xi_1, \xi_2) \in \mathscr{A} : |\xi_3| \leqslant t^{-1/3}\}$. Then

$$|J(\mathscr{A}_0)| \leqslant \iint\limits_{\mathscr{A}_0} |\xi_3| d\xi_1 d\xi_2 \|\tilde{f}\|_{L^\infty} \|\tilde{g}\|_{L^\infty} \|\tilde{h}\|_{L^\infty} \leqslant \frac{1}{t}.$$

2) We now consider the case when $|\xi_2|$ is significantly smaller than $|\xi_3|$ (and so $|\xi_1|$ as well):

$$\mathscr{A}_1 = \left\{(\xi_1, \xi_2) \in \mathscr{A} \backslash \mathscr{A}_0 : |\xi_3| - |\xi_2| \geqslant \frac{|\xi_3|}{10}\right\}.$$

Notice that, on $\mathscr{A}_1$,

$$|\partial_{\xi_1} \Phi| = 3(\xi_3^2 - \xi_1^2) \gtrsim |\xi_3|^2 \text{ and } |\partial^2_{\xi_1} \Phi| \lesssim |\xi_3, |$$

which allows us to perform an IBP in ξ_1 using that $e^{it\Phi} = \frac{1}{it\partial_1 \Phi} \partial_{\xi_1}(e^{it\Phi})$:

$$J(\mathscr{A}_1) = \iint\limits_{\mathscr{A}_1} e^{it\Phi} \partial_{\xi_1} \left(\frac{\xi_3}{it\partial_1 \Phi} \tilde{f}(\xi_1) \tilde{g}(\xi_2) \tilde{h}(\xi_3) \right) d\xi_1 d\xi_2 + \int_{\partial \mathscr{A}_1} \frac{\xi_3 e^{it\Phi}}{it\partial_1 \Phi} \tilde{f}(\xi_1) \tilde{g}(\xi_2) \tilde{h}(\xi_3) d\sigma(\xi_1, \xi_2). \tag{A.2}$$

Unfortunately, one can check that a direct bound on these integrals (after distributing the derivative ∂_{ξ_1} on all factors of the first integral) lead to at least a logarithmic divergence.

For the terms in (A.2) where the derivative in ξ_1 does not fall on $\tilde{h}(\xi_3)$, we perform a second IBP, this time in ξ_2, writing this time $e^{it\Phi} = \frac{1}{it\partial_2 \Phi} \partial_{\xi_2}(e^{it\Phi})$. Notice that on $\mathscr{A}_1$, $|\partial_2 \Phi| \gtrsim |\xi_3|^2$ and $|\nabla^2 \Phi| \lesssim |\xi_3|$, $|\nabla^3 \Phi| \lesssim 1$. For example, when the derivative in (A.2) falls on the phase $\partial_1 \Phi$:

$$\begin{aligned}
J_1 &:= \left| \iint\limits_{\mathscr{A}_1} e^{it\Phi} \left(\frac{\xi_3 \partial^2_{11} \Phi}{it(\partial_1 \Phi)^2} \tilde{f}(\xi_1) \tilde{g}(\xi_2) \tilde{h}(\xi_3) \right) d\xi_1 d\xi_2 \right| \\
&\leqslant \left| \iint\limits_{\mathscr{A}_1} e^{it\Phi} \partial_{\xi_2} \left(\frac{\xi_3 \partial^2_{11} \Phi}{t^2 (\partial_1 \Phi)^2 \partial_2 \Phi} \tilde{f}(\xi_1) \tilde{g}(\xi_2) \tilde{h}(\xi_3) \right) d\xi_1 d\xi_2 \right| \\
&\qquad + \left| \int_{\partial \mathscr{A}_1} \frac{e^{it\Phi} \xi_3 \partial^2_{11} \Phi}{t^2 (\partial_1 \Phi)^2 \partial_2 \Phi} \tilde{f}(\xi_1) \tilde{g}(\xi_2) \tilde{h}(\xi_3) d\sigma(\xi_1, \xi_2) \right| \\
&\lesssim \frac{1}{t^2} \iint\limits_{\mathscr{A}_1} \frac{1}{|\xi_3|^5} d\xi_1 d\xi_2 \|\tilde{f}\|_{L^\infty} \|\tilde{g}\|_{L^\infty} \|\tilde{h}\|_{L^\infty} \\
&\qquad + \frac{1}{t^2} \iint\limits_{\mathscr{A}_1} \|\tilde{f}\|_{L^\infty} (|\partial_\xi \tilde{g}(\xi_2)| \|\tilde{h}\|_{L^\infty} + \|\tilde{g}\|_{L^\infty} |\partial_\xi \tilde{h}(\xi_3)|) \frac{d\xi_1 d\xi_2}{|\xi_3|^4} \\
&\qquad + \frac{1}{t^2} \int_{\partial \mathscr{A}_1} \frac{d\sigma(\xi_1, \xi_2)}{|\xi_3|^4} \|\tilde{f}\|_{L^\infty} \|\tilde{g}\|_{L^\infty} \|\tilde{h}\|_{L^\infty}.
\end{aligned}$$

Now with (A.1) in mind, and with the change of variable in ξ_2 defined by $\xi_2' = \xi_3 = \xi - \xi_1 - \xi_2$ and $\mathscr{B}_1 = \{(\xi_1, \xi_2') : (\xi_1, \xi - \xi_1 - \xi_2') \in \mathscr{A}_1\}$ we estimate

$$\iint_{\mathscr{A}_1} \frac{d\xi_1 d\xi_2}{|\xi_3|^5} = \iint_{\mathscr{B}_1} \frac{d\xi_1 d\xi_2'}{|\xi_2'|^5} \leqslant \int_{|\xi_2'| \geqslant t^{-1/3}} \left(\int_{|\xi_1| \leqslant |\xi_2'|} \frac{d\xi_1}{|\xi_2'|^5} \right) d\xi_2' = 2 \int_{|\xi_2'| \geqslant t^{-1/3}} \frac{d\xi_2'}{|\xi_2'|^4} \lesssim t.$$

Similarly, using the Cauchy-Schwarz inequality,

$$\begin{aligned} \iint_{\mathscr{A}_1} |\partial_\xi \tilde{g}(\xi_2)| \frac{d\xi_1 d\xi_2}{|\xi|^4} &\leqslant \int_{|\xi_1| \geqslant t^{-1/3}} \|\partial_\xi \tilde{g}\|_{L^2} \left(\int_{|\xi_2| \geqslant |\xi_1|} \frac{d\xi_2}{|\xi_2|^8} \right)^{1/2} d\xi_1 \\ &\leqslant \int_{|\xi_1| \geqslant t^{-1/3}} \frac{d\xi_1}{|\xi_1|^{7/2}} \|\partial_\xi \tilde{g}\|_{L^2} \lesssim t^{5/2 \cdot 1/3 + 1/6} \lesssim t, \end{aligned}$$

which allows to take care of the two terms the second last line. As for the boundary term,

$$\int_{\partial \mathscr{A}_1} \frac{d\sigma(\xi_1, \xi_2)}{|\xi_3|^4} = \int_{\partial \mathscr{B}_1} \frac{d\sigma(\xi_1, \xi_2')}{|\xi_2'|^4} \lesssim t.$$

We hence obtained

$$J_1 \lesssim \frac{1}{t}.$$

As mentioned, similar computations can be done for all terms, except for the term with a derivative on $\tilde{h}$:

$$J_1' := \iint_{\mathscr{A}_1} \left(\frac{e^{it\Phi} \xi_3}{it\partial_1 \Phi} \tilde{f}(\xi_1) \tilde{g}(\xi_2) \partial_\xi \tilde{h}(\xi_3) \right) d\xi_1 d\xi_2$$

Here, we first perform the change of variable in ξ_1 defined by $\xi_1' := \xi - \xi_1 - \xi_2 = \xi_3$ so that $\xi_3' := \xi - \xi_1' - \xi_2 = \xi_1$, with $\mathscr{B}_1' = \{(\xi_1', \xi_2) : (\xi - \xi_1' - \xi_2, \xi_2) \in \mathscr{A}_1\}$:

$$J_1' = \iint_{\mathscr{B}_1'} \left(\frac{e^{it\Phi} \xi_2'}{it\partial_1 \Phi(\xi_3', \xi_2)} \tilde{f}(\xi_3') \tilde{g}(\xi_2) \partial_\xi \tilde{h}(\xi_1') \right) d\xi_1' d\xi_2.$$

Then we are in a position to perform once again an IBP in ξ_2 (because there will be no derivative falling on $\partial_\xi \tilde{h}$). On $\mathscr{B}_1'$,

$$\partial_1 \Phi(\xi_3', \xi_2) = 3|\xi_1'^2 - \xi_3'| \gtrsim |\xi_1'|^2, \quad |\partial_2 \Phi(\xi_3', \xi_2)| = 3|\xi_1'^2 - \xi_2^2| \gtrsim |\xi_1'|^2,$$

$|\xi_1'| \geqslant t^{-1/3}$ and $|\xi_1'| \geqslant |\xi_2| \geqslant |\xi_3'|$. These estimates allow us to argue as for J_1 and derive

$$|J(\mathscr{A}_1)| \lesssim \frac{1}{t}.$$

3) We then consider the case when only ξ_1 is significantly smaller:

$$\mathscr{A}_2 = \left\{ (\xi_1, \xi_2) \in \mathscr{A} \backslash \mathscr{A}_0 : ||\xi_3| - |\xi_2|| \leqslant \frac{|\xi_3|}{10} \text{ and } |\xi_3| - |\xi_1| \geqslant \frac{|\xi_3|}{10} \right\}.$$

We start as in 2), first performing an IBP in ξ_1:

$$\begin{aligned} J(\mathscr{A}_2) &= \iint_{\mathscr{A}_2} e^{it\Phi} \partial_{\xi_1} \left(\frac{\xi_3}{it\partial_1 \Phi} \tilde{f}(\xi_1) \tilde{g}(\xi_2) \tilde{h}(\xi_3) \right) d\xi_1 d\xi_2 \\ &\quad + \int_{\partial \mathscr{A}_2} \frac{\xi_3 e^{it\Phi}}{it\partial_1 \Phi} \tilde{f}(\xi_1) \tilde{g}(\xi_2) \tilde{h}(\xi_3) d\sigma(\xi_1, \xi_2). \end{aligned} \tag{A.3}$$

Except on the term where the derivative ∂_{ξ_1} falls on $\tilde{h}$ (see step 3.3 below), we perform on (A.3) an IBP in ξ_2, using the slighlty different identity

$$e^{it\Phi} = \frac{1}{1+it\xi_2\partial_2\Phi}\partial_{\xi_2}(\xi_2 e^{it\Phi}).$$

For example, if in (A.3) the ∂_{ξ_1} derivative fell on $\tilde{f}$, we are to bound

$$\begin{aligned} J_2 &:= \left| \iint_{\mathscr{A}_2} e^{it\Phi}\left(\frac{\xi_3}{it\partial_1\Phi}\partial_\xi \tilde{f}(\xi_1)\tilde{g}(\xi_2)\tilde{h}(\xi_3)\right)d\xi_1 d\xi_2 \right| \\ &\leqslant \left| \iint_{\mathscr{A}_2} e^{it\Phi}\partial_{\xi_2}\left(\frac{\xi_2\xi_3}{(1+it\xi_2\partial_2\Phi)(it\partial_1\Phi)}\partial_\xi \tilde{f}(\xi_1)\tilde{g}(\xi_2)\tilde{h}(\xi_3)\right)d\xi_1 d\xi_2 \right| \\ &\quad + \left| \iint_{\partial\mathscr{A}_2} e^{it\Phi}\frac{\xi_2\xi_3}{(1+it\xi_2\partial_2\Phi)(it\partial_1\Phi)}\partial_\xi \tilde{f}(\xi_1)\tilde{g}(\xi_2)\tilde{h}(\xi_3)d\sigma(\xi_1,\xi_2) \right| \end{aligned}$$

We now split into two cases.
3.1) Let $\mathscr{B}_2 = \{(\xi_1,\xi_2) \in \mathscr{A}_2 : |\xi_3 - \xi_2| \leqslant \frac{|\xi_3|}{100}\}$.
On $\mathscr{B}_2$, $|\partial_1\Phi| = 3|\xi_3^2 - \xi_1^2| \gtrsim |\xi_3|^2$ and

$$|\partial_2\Phi| = 3|\xi_3^2 - \xi_2^2| \gtrsim |\xi_3||\xi_3 - \xi_2| = |\xi_3||\xi - \xi_1 - 2\xi_2|.$$

Also using that $|\xi_2| \gtrsim |\xi_3| \geqslant |\xi_2|$ and $|\nabla^2\Phi| \lesssim |\xi_3|$, we infer

$$J_2 \lesssim \iint_{\mathscr{A}_1} \frac{|\xi_2||\partial_\xi \tilde{f}(\xi_1)|}{(1+t|\xi_2|^2|\xi_3-\xi_2|)t|\xi_2|^2}d\xi_1 d\xi_2 \|\tilde{g}\|_{L^\infty}\|\tilde{h}\|_{L^\infty} \tag{A.4}$$

$$+ \iint_{\mathscr{A}_1} \frac{|\xi_2|^3|\partial_\xi \tilde{f}(\xi_1)|}{(1+t|\xi_2|^2|\xi_3-\xi_2|)^2 t|\xi_2|^2}d\xi_1 d\xi_2 \|\tilde{g}\|_{L^\infty}\|\tilde{h}\|_{L^\infty} \tag{A.5}$$

$$+ \iint_{\mathscr{A}_1} \frac{|\xi_2|^2|\partial_\xi \tilde{f}(\xi_1)|(|\partial_\xi \tilde{g}(\xi_2)|\|\tilde{h}\|_{L^\infty} + \|\tilde{g}\|_{L^\infty}|\partial_\xi \tilde{h}(\xi_3)|)}{(1+t|\xi_2|^2||\xi_3-\xi_2||)t|\xi_2|^2}d\xi_1 d\xi_2$$

$$+ \iint_{\partial\mathscr{A}_1} \frac{|\xi_2|^2|\partial_\xi \tilde{f}(\xi_1)|}{(1+t|\xi_2|^2||\xi_3-\xi_2||)t|\xi_2|^2}d\sigma(\xi_1,\xi_2)\|\tilde{g}\|_{L^\infty}\|\tilde{h}\|_{L^\infty}$$

Then for fixed ξ_2, we can bound

$$\begin{aligned} \int_{\xi_1} \frac{|\partial_\xi \tilde{f}(\xi_1)|}{(1+t|\xi_2|^2|\xi_3-\xi_2|)}d\xi_1 &\leqslant \left(\int \frac{d\xi_1}{(1+t|\xi_2|^2|\xi-\xi_1-2\xi_2|)^2}\right)^{1/2}\|\partial_\xi \tilde{f}\|_{L^2} \\ &\lesssim \frac{1}{t^{1/2}|\xi_2|}\|\partial_\xi \tilde{f}\|_{L^2} \end{aligned}$$

so that for the term in (A.4)

$$\begin{aligned} \iint_{\mathscr{A}_1} \frac{|\xi_2||\partial_\xi \tilde{f}(\xi_1)|}{(1+t|\xi_2|^2|\xi_3-\xi_2|)t|\xi_2|^2}d\xi_1 d\xi_2 &\lesssim \frac{1}{t^{3/2}}\int_{|\xi_2|\geqslant t^{-1/3}} \frac{d\xi_2}{|\xi_2|^2}\|\partial_\xi \tilde{f}\|_{L^2} \\ &\lesssim t^{-3/2+1/3+1/6} \leqslant \frac{1}{t}. \end{aligned}$$

Similarly, for (A.5)

$$\iint_{\mathscr{A}_1} \frac{|\xi_2|^2|\partial_\xi \tilde{f}(\xi_1)||\partial_\xi \tilde{g}(\xi_2)|}{(1+t|\xi_2|^2|\xi_3-\xi_2|)t|\xi_2|^2}d\xi_1 d\xi_2$$

$$\lesssim \frac{1}{t}\int_{|\xi_2|\geqslant t^{-1/3}} |\partial_\xi \tilde{g}(\xi_2)| \left(\int_{\xi_1} \frac{|\partial_\xi \tilde{f}(\xi_1)|d\xi_1}{(1+t|\xi_2|^2|\xi-\xi_1-2\xi_2|)}\right) d\xi_2$$
$$\lesssim \frac{1}{t^{3/2}}\int_{|\xi_2|\geqslant t^{-1/3}} |\partial_\xi \tilde{g}(\xi_2)| \frac{d\xi_2}{|\xi_2|}\|\partial_\xi \tilde{f}\|_{L^2} \lesssim t^{-3/2+1/3\cdot 1/2}\|\partial_\xi \tilde{g}\|_{L^2}\|\partial_\xi \tilde{f}\|_{L^2} \lesssim \frac{1}{t}.$$

One can bound the other terms in J_2 in the same fashion.
3.2) Let $\mathscr{C}_2 = \{(\xi_1,\xi_2) \in \mathscr{A}_2 : |\xi_3+\xi_2| \leqslant \frac{|\xi_3|}{100}\}$. Then on $\mathscr{C}_2$, $|\partial_1 \Phi| = 3|\xi_3^2 - \xi_1^2| \gtrsim |\xi_3|^2$ and

$$|\partial_2 \Phi| = 3|\xi_3^2 - \xi_2^2| \gtrsim |\xi_3||\xi_3+\xi_2| = |\xi_3||\xi-\xi_1|.$$

We still have $|\xi_2| \gtrsim |\xi_3| \geqslant |\xi_2| \gtrsim |\xi|, |\xi_1|$ and $|\nabla^2\Phi| \lesssim |\xi_3|$.
We can then proceed as in 3.1), integrating first in ξ_1 for ξ_2 fixed.
Summing up, as $\mathscr{A}_2 = \mathscr{B}_2 \cup \mathscr{C}_2$, we get that

$$J_2 \lesssim \frac{1}{t}.$$

3.3) To complete the bound on $J(\mathscr{A}_2)$, it remains to consider the term in (A.3) when the derivative ∂_{ξ_1} of the (first) IBP falls on $\tilde{h}$:

$$J_2' := \left| \iint_{\mathscr{A}_2} e^{it\Phi}\left(\frac{\xi_3}{it\partial_1\Phi}\tilde{f}(\xi_1)\tilde{g}(\xi_2)\partial_\xi \tilde{h}(\xi_3)\right) d\xi_1 d\xi_2 \right|$$

As in case 2), we perform the change of variable in ξ_1 defined by $\xi_1' := \xi-\xi_1-\xi_2 = \xi_3$ so that $\xi_3' := \xi - \xi_1' - \xi_2 = \xi_1$, with $\mathscr{B}_2' = \{(\xi_1',\xi_2) : (\xi-\xi_1'-\xi_2,\xi_2) \in \mathscr{A}_2\}$:

$$J_2' = \iint_{\mathscr{B}_2'} \left(\frac{e^{it\Phi}\xi_2'}{it\partial_1\Phi(\xi_3',\xi_2)}\tilde{f}(\xi_3')\tilde{g}(\xi_2)\partial_\xi \tilde{h}(\xi_1')\right) d\xi_1' d\xi_2,$$

and then we are in a position for an IBP in ξ_2. We conclude by doing the same computations as in 3.1) and 3.2), and we obtain, as desired,

$$J(\mathscr{A}_2) \lesssim \frac{1}{t}.$$

4) We are now in the case when $|\xi_1|, |\xi_2|, |\xi_3|$ are similar, which corresponds to the stationary points. There are two cases: either they all have same signs (corresponding to $(\xi/3,\xi/3,\xi/3)$), or they don't (corresponding to $(\xi,\xi,-\xi)$ and its two symmetric). We thus consider

$$\mathscr{A}_3 := \left\{(\xi_1,\xi_2) \in \mathscr{A} : |\xi_3-\xi_2|, |\xi_3-\xi_1| \leqslant \frac{|\xi_3|}{10}\right\}$$
$$\mathscr{A}_4 := \left\{(\xi_1,\xi_2) \in \mathscr{A} : |\xi_3+\xi_2|, |\xi_3+\xi_1| \leqslant \frac{|\xi_3|}{10}\right\}.$$

The other two stationary points can be treated as for $\mathscr{A}_4$.

In these stationary regions, we must exploit as much as possible the phase oscillations. To that end, we adapt our reference frame to the directions of maximal oscillation (see (A.6)). This analysis was crucial to derive the refined estimates in [18] and [7]. The computations below follow closely the arguments therein.
As ξ_1, ξ_2, ξ_3 are all of the order of ξ, it is convenient to rescale and denote $q_i = \xi_i/\xi$ so that $q_1+q_2+q_3 = 1$, and the phase

$$\Phi(\xi_1,\xi_2) = \xi^3 Q(q_1,q_2) = 3\xi^3(1-q_1)(1-q_2)(1-q_3).$$

By symmetry, we can also assume without loss of generality that $\xi > 0$, and we denote the rescaling factor

$$\tau := t\xi^3 \geqslant 1 \quad \text{on } \mathscr{A}_3 \cup \mathscr{A}_4.$$

($|\tau| \leqslant 1$ corresponds to $\mathscr{A}_0$). Denote the nonlinear function without the ξ_3 factor

$$F := \tilde{f}(\xi q_1)\tilde{g}(\xi q_2)\tilde{h}(\xi q_3).$$

We now introduce the change of variable:

$$1 - q_1 = \lambda - \mu, \quad 1 - q_2 = \lambda + \mu, \quad 1 - q_3 = 2(1 - \lambda),$$

so that

$$Q(q_1, q_2) = 6(1 - \lambda)(\lambda - \mu)(\lambda + \mu).$$

The stationary point correspond to $(\lambda, \mu) = (2/3, 0)$ and $(0, 0)$ respectively. We integrate by parts using the relation

$$e^{i\tau Q} = \frac{1}{1 + 12i\tau\mu^2(1 - \lambda)}\partial_\mu(\mu e^{i\tau Q})$$

so that, with $\mathscr{D}' = \{(\lambda, \mu) : (1 - (\lambda - \mu), 1 - (\lambda + \mu) \in \mathscr{D}\}$ where $\mathscr{D}$ is an integration domain and

$$A = 1 + 12i\tau\mu^2(1 - \lambda), \quad \partial_\mu A = 24i\tau\mu(1 - \lambda)$$

and depending whether the derivative falls on the phase or not:

$$\begin{aligned}
|J(\mathscr{D})| &= 2\xi^3 \iint_{\mathscr{D}} e^{i\tau Q} F(2\lambda - 1) d\lambda d\mu = -2\xi^3 \iint_{\mathscr{D}} \frac{1}{A}\partial_\mu(\mu e^{i\tau Q}) F(2\lambda - 1) d\lambda d\mu \\
&= 2\xi^3 \iint_{\mathscr{D}'} e^{i\tau Q} F \frac{\mu \partial_\mu A(2\lambda - 1)}{A^2} d\lambda d\mu + 2\xi^3 \iint_{\mathscr{D}'} e^{i\tau Q} \partial_1 F \frac{\mu(2\lambda - 1)}{A} d\lambda d\mu \\
&\quad - 2\xi^3 \iint_{\mathscr{D}'} e^{i\tau Q} \partial_2 F \frac{\mu(2\lambda - 1)}{A} d\lambda d\mu \\
&=: 2\xi^3 (J_\sharp(\mathscr{D}') + J_\flat(\mathscr{D}') + J_\natural(\mathscr{D}')).
\end{aligned}$$

(observe that there is no derivative on $\tilde{h}$ in this computation). By symmetry between $\partial_1 F$ and $\partial_2 F$, it suffices to bound $J_\sharp(\mathscr{D}')$ and $J_\flat(\mathscr{D}')$.

5) Here we bound $J_\sharp$. We do yet another change of variable of μ defined by $\nu = \mu\sqrt{1 - \lambda}$ so that Q has separate variables:

$$Q = 6(1 - \lambda)\lambda^2 - 6\nu^2, \quad \partial_\lambda Q = 6\lambda(2 - 3\lambda). \tag{A.6}$$

We perform an integration by parts using for fixed $\lambda_0 \in \{0, 2/3\}$

$$e^{i\tau Q} = \frac{1}{1 + 6i\tau(\lambda - \lambda_0)\lambda(2 - 3\lambda)}\partial_\lambda((\lambda - \lambda_0)e^{i\tau Q}).$$

Denoting

$$B = 1 + 6i\tau(\lambda - \lambda_0)\lambda(2 - 3\lambda), \quad \partial_\lambda B = 6i\tau(\lambda(2 - 3\lambda) + 2(\lambda - \lambda_0)(1 - 3\lambda)),$$

we have

$$\begin{aligned}
J_\sharp(\mathscr{D}') &= \iint_{\mathscr{D}'} e^{i\tau Q} F \frac{24i\tau(1 - \lambda)\mu^2}{A^2} d\lambda d\mu \\
&= \iint_{\mathscr{D}'} \partial_\lambda (\lambda - \lambda_0)e^{i\tau Q}) F \frac{24i\tau\nu^2}{A^2 B} \frac{2\lambda - 1}{\sqrt{1 - \lambda}} d\lambda d\nu \\
&= -\iint_{\mathscr{D}'} e^{i\tau Q} \left[-\partial_1 F - \partial_2 F + 2\partial_3 F\right] \times \frac{24i\tau(\lambda - \lambda_0)\nu^2}{A^2 B} \frac{2\lambda - 1}{\sqrt{1 - \lambda}} d\lambda d\nu
\end{aligned}$$

$$+\iint_{\mathscr{D}'} e^{i\tau Q} F \frac{24i\tau(\lambda-\lambda_0)\nu^2}{A^2B}\left(2+\frac{1}{2(1-\lambda)}+\frac{\partial_\lambda B}{B}\right)\frac{2\lambda-1}{\sqrt{1-\lambda}}d\lambda d\nu$$
$$+\int_{\partial\mathscr{D}'}(\lambda-\lambda_0)e^{i\tau Q}F\frac{24i\tau\nu^2}{A^2B}\frac{2\lambda-1}{\sqrt{1-\lambda}}d\sigma(\tilde\lambda,\nu).$$

5.1) On $\mathscr{A}_3'$, we choose $\lambda_0=2/3$. Then we have $|\lambda-2/3|\leqslant 1/10$ and $|\nu|\leqslant 1/10$ so that

$$\lambda, 1-\lambda, 1-3\lambda, 2\lambda-1$$

can be bounded (above and below) by uniform constants. In particular, letting $\tilde\lambda := \lambda - 2/3$,

$$|A|\gtrsim 1+\tau\nu^2, \quad |B|\gtrsim 1+\tau\tilde\lambda^2,$$

and we can bound

$$\text{(A.7)} \qquad |J_\sharp(\mathscr{A}_3')| \lesssim \iint_{\mathscr{A}_3^\sharp}\frac{\tau\nu^2|\tilde\lambda|}{(1+\tau\nu^2)^2(1+\tau\tilde\lambda^2)}(|\partial_1F|+|\partial_2F|+|\partial_3F|)d\tilde\lambda d\nu$$
$$+\iint_{\mathscr{A}_3^\sharp}\frac{\tau\nu^2|\tilde\lambda|}{(1+\tau\tilde\lambda^2)(1+\tau\nu^2)^2}\left(1+\frac{\tau|\tilde\lambda|}{1+\tau\tilde\lambda^2}\right)|F|d\tilde\lambda d\nu$$
$$+\int_{\partial\mathscr{A}_3^\sharp}\frac{\tau\nu^2\tilde|\lambda|}{(1+\tau\tilde\lambda^2)(1+\tau\nu^2)^2}|F|d\sigma(\tilde\lambda,\nu).$$

First consider the first term. Recall that $\|F\|_{L^\infty}\lesssim\|\tilde f\|_{L^\infty}\|\tilde g\|_{L^\infty}\|\tilde h\|_{L^\infty}\lesssim 1$ and

$$\left(\int|\partial_1F|^2d\tilde\lambda\right)^{1/2}\lesssim|\xi|^{1/2}\|\partial_\xi\tilde f\|_{L^2}\|\tilde g\|_{L^\infty}\|\tilde h\|_{L^\infty}\lesssim\tau^{1/6},$$

and similarly for ∂_2F and ∂_3F. Keeping in mind that $\tilde\lambda\in[-1,1]$ uniformly on $\mathscr{A}_3^\sharp=\{(\tilde\lambda,\nu):(\lambda,\mu)\in\mathscr{A}_3'\}$, we bound

$$\int_{-1}^1\frac{|\tilde\lambda|}{1+\tau\tilde\lambda^2}|\partial_jF|d\tilde\lambda\lesssim\|\partial_jF\|_{L^2}\left(\int_{-1}^1\frac{\tilde\lambda^2d\tilde\lambda}{(1+\tau\tilde\lambda^2)^2}\right)^{1/2}\lesssim\tau^{1/6-3/4}, \quad j=1,2,3.$$

Hence the first term in (A.7) is bounded by

$$\tau^{1/6-3/4}\int\frac{\tau\nu^2}{(1+\tau\nu^2)^2}d\nu\lesssim\tau^{1/6-3/4-1/2}\lesssim\tau^{-13/12}$$

(we used that $\tau\geqslant 1$).
Similarly, for the second term, observe that

$$\int_{-1}^1\frac{|\tilde\lambda|}{1+\tau\tilde\lambda^2}\left(1+\frac{\tau|\tilde\lambda|}{1+\tau\tilde\lambda^2}\right)d\tilde\lambda\lesssim\tau^{-1}\ln(1+\tau)+\tau^{-1/2}\lesssim\tau^{-1/2},$$

so that the second term is bounded by

$$\tau^{-1/2}\int\frac{\tau\nu^2}{(1+\tau\nu^2)^2}d\nu\lesssim\tau^{-1}.$$

The boundary term can be bounded accordingly by

$$\int_{-1}^1\frac{\tau|\rho^3|d\rho}{(1+\tau\rho^2)^3}\lesssim\tau^{-1}.$$

Hence, as $\tau\geqslant 1$, we obtain

$$|J_\sharp(\mathscr{A}_3')|\lesssim\tau^{-1}.$$

5.2) On $\mathscr{A}_4'$, we choose $\lambda_0=0$. Then we have $|\lambda|\leqslant 1/10$ and $|\nu|\leqslant 1/10$ so that

$$1-\lambda, 1-3\lambda, 2-3\lambda$$

can be bounded (above and below) by uniform constants and

$$|A| \gtrsim 1 + \tau\nu^2, \quad |B| \gtrsim 1 + \tau\lambda^2.$$

Hence, we can bound (with $\mathscr{A}_4^\sharp = \{(\lambda, \nu) : (\lambda, \mu) \in \mathscr{A}_4'\}$)

$$\begin{aligned}
|J_\sharp(\mathscr{A}_4')| \lesssim & \iint_{\mathscr{A}_4^\sharp} \frac{\tau\nu^2|\lambda|}{(1 + (\tau\nu^2)^2)(1 + \tau\lambda^2)} (|\partial_1 F| + |\partial_2 F| + |\partial_3 F|) d\lambda d\nu \\
& + \iint_{\mathscr{A}_4^\sharp} \frac{\tau\nu^2|\lambda|}{(1 + \tau\lambda^2)(1 + \tau\nu^2)^2} \left(1 + \frac{|\lambda|}{1 + \tau\lambda^2}\right) |F| d\lambda d\nu \\
& + \int_{\partial\mathscr{A}_4^\sharp} \frac{\tau\nu^2|\lambda|}{(1 + \tau\lambda^2)(1 + \tau\nu^2)^2} d\sigma(\lambda, \nu).
\end{aligned}$$

The exact same computations as in 5.1) give the bound

$$|J_\sharp(\mathscr{A}_4')| \lesssim \tau^{-1}.$$

6) Here we bound $J_\flat$. We consider the change of variables

$$\mu = \frac{3\zeta + \chi - 2}{2}, \quad \lambda = \frac{2 - \zeta + \chi}{2}.$$

One may obtain this transformation by going back to the ξ variables, switching q_1 with q_3 and then redoing the λ, μ transformation. In this way, q_1 depends on a single variable ζ, and more precisely

$$q_1 = 2\zeta, \quad 1 - q_2 = \zeta + \chi, \quad 1 - q_3 = \zeta - \chi.$$

This permits the integration by parts in χ without the introduction of second-order derivatives in f. In this coordinate system, the stationary points are

$$(\chi, \zeta) = (0, 2/3) \text{ and } (-1, 1).$$

Also

$$Q = 6(\zeta + \chi)(\zeta - \chi)(1 - \zeta), \quad \partial_\chi Q = 12\chi(\zeta - 1).$$

6.1) On $\mathscr{A}_3'$, we use the relation

$$e^{i\tau Q} = \frac{1}{1 + 12i\tau\chi^2(1 - \zeta)} \partial_\chi(\chi e^{i\tau Q}).$$

Define

$$A = 1 + 4i\tau\mu^2(1 - \lambda) = 1 + i\tau(3\zeta + \chi - 2)^2(\zeta - \chi)/2, \quad C = 1 + 12i\tau\chi^2(1 - \zeta).$$

We now integrate by parts:

$$\begin{aligned}
J_\flat(\mathscr{A}_3') = & \iint_{\mathscr{A}_3'} e^{i\tau Q} \frac{\mu\partial_1 F}{A} d\lambda d\mu = \iint_{\mathscr{A}_3^\flat} \frac{1}{C} \partial_\chi(\chi e^{i\tau Q}) \frac{\mu}{2A} \partial_1 F d\zeta d\chi \\
= & - \iint_{\mathscr{A}_3^\flat} \chi e^{i\tau Q} \left[-\partial_{12} F + \partial_{13} F\right] \frac{\mu}{2AC} d\chi d\zeta \\
& - \iint_{\mathscr{A}_3^\flat} \chi e^{i\tau Q} \frac{\partial_1 F}{2AC} \left(1 - \mu\left(\frac{\partial_\chi A}{A} + \frac{\partial_\chi C}{C}\right)\right) d\chi d\zeta \\
& - \iint_{\partial\mathscr{A}_3^\flat} e^{i\tau Q} \frac{\chi\mu}{2AC} \partial_1 F d\sigma(\zeta, \chi).
\end{aligned}$$

On $\mathscr{A}_3^\flat$, $1 - \zeta$, $\zeta - \chi$ are bounded above and below and

$$A \gtrsim 1 + \tau\mu^2, \quad C \gtrsim 1 + \tau\chi^2, \quad |\partial_\chi A| \lesssim \tau|\mu|, \quad |\partial_\chi C| \lesssim \tau|\chi|.$$

Therefore, after changing again the variable ζ back for μ (and $\mathscr{A}_3^\flat = \{(\chi,\mu) : (\lambda,\mu) \in \mathscr{A}_3'\}$),

$$\text{(A.8)} \qquad |J_\flat(\mathscr{A}_3')| \lesssim \iint\limits_{\mathscr{A}_3^\flat} \frac{|\chi\mu|}{(1+\tau\mu^2)(1+\tau\chi^2)} |\partial_{12}F| + |\partial_{13}F|) d\chi d\mu$$

$$\text{(A.9)} \qquad + \iint\limits_{\mathscr{A}_3^\flat} \frac{|\chi|}{(1+\tau\mu^2)(1+\tau\chi^2)} \left(1 + \frac{|\tau|\mu^2}{1+\tau\mu^2} + \frac{\tau|\mu\chi|}{1+\tau\chi^2}\right) |\partial_1 F| d\chi d\mu$$

$$\text{(A.10)} \qquad + \iint\limits_{\partial\mathscr{A}_3^\flat} \frac{|\chi\mu|}{(1+\tau\mu^2)(1+\tau\chi^2)} |\partial_1 F| d\sigma(\chi,\mu).$$

By the Cauchy-Schwarz inequality,

$$\left(\iint |\partial_{12}F|^2 d\chi d\mu\right)^{1/2} \lesssim |\xi| \|\partial_\xi \tilde f\|_{L^2} \|\partial_\xi \tilde g\|_{L^2} \|\tilde h\|_{L^\infty} \lesssim \tau^{1/3},$$

$$\left(\int |\partial_1 F|^2 d\chi\right)^{1/2} \lesssim |\xi|^{1/2} \|\partial_\xi \tilde f\|_{L^2} \|\tilde g\|_{L^\infty} \|\tilde h\|_{L^\infty} \lesssim \tau^{1/6}, \quad \text{so that}$$

$$\|\partial_1 F\|_{L^\infty_\mu L^2_\chi}, \|\partial_1 F\|_{L^\infty_\chi L^2_\mu}, \|\partial_1 F\|_{L^2(\partial\mathscr{A}_3^\flat)} \lesssim \tau^{1/6}.$$

and the first bound also holds with $\partial_{13}F$. Hence the integral in (A.8) is bounded by

$$\left(\iint \frac{|\mu|^2}{(1+\tau\mu^2)^2} \frac{|\chi|^2}{(1+\tau\chi^2)^2} d\mu d\chi\right)^{1/2} \left(\iint (|\partial_{12}F| + |\partial_{13}F|)^2 d\chi d\mu\right)^{1/2} \lesssim \tau^{-3/2+1/3} \lesssim \tau^{-7/6}.$$

The first two terms of the integral in (A.9) can be controlled as

$$\int \frac{d\mu}{1+\tau\mu^2} \left(\int \frac{\chi^2 d\chi}{1+(\tau\chi^2)^2}\right)^{1/2} \|\partial_1 F\|_{L^\infty_\mu L^2_\chi} \lesssim \tau^{-1/2-3/4+1/6} \lesssim \tau^{-13/12},$$

while the third term in (A.9) is bounded by

$$\int \frac{\tau\chi^2 d\chi}{(1+\tau\chi^2)^2} \left(\int \frac{|\mu|^2 d\mu}{(1+\tau\mu^2)^2}\right)^{1/2} \|\partial_1 F\|_{L^\infty_\chi L^2_\mu} \lesssim \tau^{-1/2-3/4+1/6} \lesssim \tau^{-13/12}.$$

Finally, the boundary term (A.10) can be bounded by

$$\|\partial_1 F\|_{L^2(\partial\mathscr{A}_3^\flat)} \left(\int \left(\frac{\rho^2}{(1+\tau\rho^2)^2}\right)^2 d\rho\right)^{1/2} \lesssim \tau^{1/6-5/4} \lesssim \tau^{-13/12}$$

and we obtain

$$J_\flat(\mathscr{A}_3') \lesssim \tau^{-13/12}.$$

6.2) On $\mathscr{A}_4'$, we use the relation

$$e^{i\tau Q} = \frac{1}{1+12i\tau\chi(\chi+1)(1-\zeta)} \partial_\chi((\chi+1)e^{i\tau Q}).$$

Let

$$A = 1 + 4i\tau\mu^2(1-\lambda) = 1 + i\tau(3\zeta+\chi-2)^2(\zeta-\chi)/2, \quad D = 1 + 12i\tau\chi(\chi+1)(1-\zeta).$$

The integration by parts yields

$$J_\flat(\mathscr{A}_4') = \iint\limits_{\mathscr{A}_4'} e^{i\tau Q} \frac{\mu\partial_1 F}{A} d\lambda d\mu = \iint\limits_{\mathscr{A}_4^\flat} \frac{1}{D} \partial_\chi((\chi+1)e^{i\tau Q}) \frac{\mu}{2A} \partial_1 F d\zeta d\chi$$

$$= -\iint_{\mathscr{A}_4^\flat} (\chi+1)e^{i\tau Q}\left[-\partial_{12}F + \partial_{13}F\right]\frac{\mu}{2AD}d\chi d\zeta$$
$$- \iint_{\mathscr{A}_4^\flat} (\chi+1)e^{i\tau Q}\frac{\partial_1 F}{2AD}\left(1-\mu\left(\frac{\partial_\chi A}{A}+\frac{\partial_\chi D}{D}\right)\right)d\chi d\zeta$$
$$- \iint_{\partial\mathscr{A}_4^\flat} (\chi+1)e^{i\tau Q}\frac{\chi\mu}{2AD}\partial_1 F d\sigma(\zeta,\chi).$$

On $\mathscr{A}_4^\flat$, $1-\zeta$, $\zeta-\chi$ are bounded above and below. Moreover, denoting $\tilde{\chi}=\chi+1$, $\tilde{\zeta}=1-\zeta$, there holds

$$A \gtrsim 1+\tau\mu^2, \quad D \gtrsim 1+\tau|\tilde{\chi}\tilde{\zeta}|, \quad |\partial_\chi A| \lesssim \tau|\mu|, \quad |\partial_\chi D| \lesssim \tau|\tilde{\zeta}|.$$

After we perform once again the change of variable $(\mu,\chi)\to(\tilde{\zeta},\tilde{\chi})$ with $\tilde{\zeta}=1-\zeta=\frac{2\mu-\tilde{\chi}}{3}$), we find

$$\text{(A.11)} \quad |J_\flat(\mathscr{A}_4')| \lesssim \iint_{\mathscr{A}_4^\flat} \frac{|\tilde{\chi}\mu|}{(1+\tau\mu^2)(1+\tau|\tilde{\chi}\tilde{\zeta}|)}(|\partial_{12}F| + |\partial_{13}F)d\tilde{\chi}d\mu$$

$$\text{(A.12)} \quad + \iint_{\mathscr{A}_4^\flat} \frac{|\tilde{\chi}|}{(1+\tau\mu^2)(1+\tau|\tilde{\chi}\tilde{\zeta}|)}\left(1+\frac{\tau\mu^2}{1+\tau\mu^2}+\frac{\tau|\mu\tilde{\zeta}|}{1+\tau|\tilde{\chi}\tilde{\zeta}|}\right)|\partial_1 F|d\tilde{\chi}d\mu$$

$$\text{(A.13)} \quad + \int_{\partial\mathscr{A}_4^\flat} \frac{|\tilde{\chi}\mu|}{(1+\tau\mu^2)(1+\tau|\tilde{\chi}\tilde{\zeta}|)}|\partial_1 F|d\sigma(\tilde{\chi},\mu).$$

As in 6.1),

$$\left(\iint (|\partial_{12}F| + |\partial_{13}F)^2 d\tilde{\chi}d\mu\right)^{1/2} \lesssim \tau^{1/3},$$
$$\|\partial_1 F\|_{L^\infty_{\tilde{\chi}}L^2_\mu}, \|\partial_1 F\|_{L^\infty_\mu L^2_{\tilde{\chi}}}, \|\partial_1 F\|_{L^2(\mathscr{A}_4^\flat)} \lesssim \tau^{1/6}.$$

Hence, rescaling $\check{\chi}=\sqrt{\tau}\tilde{\chi}$, $\check{\mu}=\sqrt{\tau}\mu$ and using

$$\text{(A.14)} \quad \int \frac{\check{\chi}^2 d\check{\chi}}{(1+|\check{\chi}(2\check{\mu}-\check{\chi})|)^2} \lesssim \int_{|2\check{\mu}-\check{\chi}|\leqslant 1/|\check{\mu}|} \check{\chi}^2 d\check{\chi} + \int_{|2\check{\mu}-\check{\chi}|\geqslant 1/|\check{\mu}|} \frac{d\check{\chi}}{(2\check{\mu}-\check{\chi})^2} \lesssim |\check{\mu}|,$$

we can bound the integral in (A.11) by

$$(\|\partial_{12}F\|_{L^2} + \|\partial_{13}F\|_{L^2})|\tau|^{-3/2}\left(\iint_{|\check{\mu}|,|\check{\chi}|\leqslant 10\sqrt{\tau}} \left(\frac{|\check{\chi}\check{\mu}|}{(1+\check{\mu}^2)(1+|\check{\chi}(2\check{\mu}-\check{\chi})|)}\right)^2 d\check{\chi}d\check{\mu}\right)^{1/2}$$
$$\lesssim \tau^{1/3-3/2}\left(\int_{|\check{\mu}|\leqslant 10\sqrt{\tau}} \frac{|\check{\mu}|^3}{(1+\check{\mu}^2)^2}d\check{\mu}\right)^{1/2} \lesssim \tau^{-7/6}\ln(1+\tau) \lesssim \tau^{-1}.$$

We now consider the integral in (A.12). For the first two terms, we do Cauchy-Schwarz in $\tilde{\chi}$ and rescale as before $\check{\chi}=\sqrt{\tau}\tilde{\chi}$ and $\check{\mu}=\sqrt{\tau}\mu$, so as to get the bound

$$\|\partial_1 F\|_{L^\infty_\mu L^2_{\tilde{\chi}}} \int_{|\mu|\leqslant 1} \left(\int_{|\tilde{\chi}|\leqslant 1} \frac{\tilde{\chi}^2 d\tilde{\chi}}{(1+|\tau\tilde{\chi}\tilde{\zeta}|)^2}\right)^{1/2} \frac{d\mu}{(1+\tau\mu^2)}$$
$$\lesssim \tau^{1/6-1/2-3/4}\int_{|\check{\mu}|\leqslant 10\sqrt{\tau}} \left(\int_{|\check{\chi}|\leqslant 10\sqrt{\tau}} \frac{\check{\chi}^2 d\check{\chi}}{(1+|\check{\chi}\check{\zeta}|)^2}\right)^{1/2} \frac{d\check{\mu}}{(1+\check{\mu}^2)} \lesssim \tau^{-13/12},$$

where we used once again (A.14).

For the third term of the integral in (A.12), we do Cauchy-Schwarz in $\tilde{\chi}$ and rescale again $\check{\chi} = \sqrt{\tau}\tilde{\chi}$, $\check{\mu} = \sqrt{\tau}\mu$, and get the bound

$$\|\partial_1 F\|_{L^\infty_\mu L^2_{\tilde{\chi}}} \int_{|\mu|\leqslant 1} \left(\int_{|\tilde{\chi}|\leqslant 1} \frac{d\tilde{\chi}}{(1+|\tau\tilde{\chi}\tilde{\zeta}|)^2} \right)^{1/2} \frac{\mu d\mu}{(1+\tau\mu^2)}$$
$$\lesssim \tau^{1/6-1-1/4} \int_{|\check{\mu}|\leqslant 10\sqrt{\tau}} \left(\int_{|\check{\chi}|\leqslant 10\sqrt{\tau}} \frac{d\check{\chi}}{(1+|\check{\chi}\check{\zeta}|)^2} \right)^{1/2} \frac{|\check{\mu}|d\check{\mu}}{(1+\check{\mu}^2)} \lesssim \tau^{-13/12}.$$

Indeed, we show that the integral $\lesssim 1$ as follows. We can assume $|\check{\mu}| \geqslant 10$. Then, for fixed $\check{\mu}$,

$$\int_{10\leqslant|\check{\chi}|} \frac{d\check{\chi}}{(1+|\check{\chi}(2\check{\mu}-\check{\chi})|)^2} \lesssim \int_{|2\check{\mu}-\check{\chi}|\leqslant 1/|\check{\mu}|} d\check{\chi} + \int_{1/|\check{\mu}|\leqslant|2\check{\mu}-\check{\chi}|\leqslant|\check{\mu}|/100} \frac{d\check{\chi}}{\check{\mu}^2(2\check{\mu}-\check{\chi})^2}$$
$$+ \int_{|2\check{\mu}-\check{\chi}|\geqslant|\check{\mu}|/100} \frac{d\check{\chi}}{(2\check{\mu}-\check{\chi})^2} \lesssim \frac{1}{|\check{\mu}|}.$$

Hence

$$\int_{|\check{\mu}|\leqslant 10\sqrt{\tau}} \left(\int_{|\check{\chi}|\leqslant 10\sqrt{\tau}} \frac{d\check{\chi}}{(1+|\check{\chi}\check{\zeta}|)^2} \right)^{1/2} \frac{\check{\mu}d\check{\mu}}{(1+\check{\mu}^2)} \lesssim \int \frac{|\check{\mu}|^{1/2}d\check{\mu}}{(1+\check{\mu}^2)} \lesssim 1.$$

Finally, we consider the boundary term (A.13): there, $\tilde{\chi}$, μ and $\tilde{\zeta}$ are proportional; by Cauchy-Schwarz inequality, this term is bounded up to a constant by

$$\|\partial_1 F\|_{L^2(\mathscr{A}_4^\flat)} \left(\int_{[-1,1]} \frac{\rho^4 d\rho}{(1+\tau\rho^2)^4} \right)^{1/2} \lesssim \tau^{1/6-5/4} \lesssim \tau^{-13/12}.$$

7) In summary, (recalling $\tau \geqslant 1$), we proved in 5) and 6) that

$$|J_\sharp(\mathscr{A}_3)| + |J_\flat(\mathscr{A}_3)| + |J_\sharp(\mathscr{A}_4)| + |J_\flat(\mathscr{A}_4)| \lesssim \tau^{-1}.$$

Hence $|J(\mathscr{A}_3)| + |J(\mathscr{A}_4)| \lesssim \tau^{-1}\xi^3 \lesssim 1/t$. Summing up with the bounds in 1)-3), we infer that $|J(\mathscr{A})| \lesssim 1/t$, and the same estimate holds for $J(\mathbb{R}^2)$, as claimed. □

Statements

There is no data associated to this manuscript.
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