Scattering below critical energy for the radial 4D Yang-Mills equation and for the 2D corotational wave map system

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Abstract

Given g and $f = gg'$, we consider solutions to the following non linear wave equation :

$$\begin{cases} u_{tt} - u_{rr} - \frac{1}{r}u_r = -\frac{f(u)}{r^2}, \\ (u, u_t)|_{t=0} = (u_0, u_1). \end{cases}$$

Under suitable assumptions on g, this equation admits non-constant stationary solutions : we denote Q one with least energy. We caracterize completely the behavior as time goes to $\pm\infty$ of solutions (u, u_t) corresponding to data with energy less than or equal to the energy of Q : either it is $(Q, 0)$ up to scaling, or it scatters in the energy space.

Our results include the cases of the 2 dimensional corotational wave map system, with target $\mathbb{S}^2$, in the critical energy space, as well as the 4 dimensional, radially symmetric Yang-Mills fields on Minkowski space, in the critical energy space.

1 Introduction

In this paper we study the asymptotic behavior of solutions to a class of non-linear wave equations in $\mathbb{R} \times \mathbb{R}$, with data in the natural energy space. The equations covered by our results include the 2 dimensional corotational wave map system, with target $\mathbb{S}^2$, in the critical energy space, as well as the 4 dimensional, radially symmetric Yang-Mills fields on Minkowski space, in the critical energy space.

The equations under consideration admit non-constant solutions that are independent of time, of minimal energy, the so-called harmonic maps Q (see [3] and the discussion below). It is known, from the work of Struwe [13], that if the data has energy smaller than or equal to the energy of Q, then the corresponding solution exists globally in time (see Proposition 1 below). (A recent result [8] shows that large energy data may lead to a finite time blow up solution for the 2 dimensional corotational wave map system, with target $\mathbb{S}^2$ – see also [9]). In this paper, we show that, for this class of solutions, an alternative holds : either the data is $(Q, 0)$ (or $(-Q, 0)$ if $-Q$ is also a harmonic map), modulo the natural symmetries of the problem, and the solution is independent of time, or a (suitable) space-time norm is finite, which results in the scattering at times $\pm\infty$. Thus the asymptotic behavior as $t \to \pm\infty$ for solutions of energy

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smaller than or equal to that of Q, is completely described. Because of the existence of Q, the result is clearly sharp.

The result is inspired by the recent works [6, 5] of the last two authors, who developed a method to attack such problems, reducing them, by a concentration-compactness approach, to a rigidity theorem. An important element in the proof of the rigidity theorem in [6, 5] is the use of a virial identity. This is also the case in this work, where the virial identity we use in the proof of Lemma 8 is very close to the one used in Lemma 5.4 of [5]. Lemma 8 in turn follows from Lemma 7, which has its origin in the work of the first author [3]. The concentration-compactness approach we use here is the same as the one in [5], with an important proviso. The results in [5] are established for dimension $N = 3, 4, 5$, while here, in order to include the case of radial Yang-Mills in $\mathbb{R}^4$, we need to deal with a case similar to $N = 6$; it also establishes the result in [5] for $N = 6$. This is carried out in Theorem 2 below.

It is conjectured that similar results will hold without the restriction to data with symmetry (for wave maps or Yang-Mills fields). These are extremely challenging problems for future research.

We now turn to a more detailed description of our results. Let $g : \mathbb{R} \to \mathbb{R}$ be C^3 such that $g(0) = 0$, $g'(0) = k \in \mathbb{N}^*$, denote $f = gg'$, and N be the surface of revolution with polar coordinates $(\rho, \theta) \in [0, \infty) \times \mathbb{S}^1$, and metric $ds^2 = d\rho^2 + g^2(\rho)d\theta^2$ (hence N is fully determined by g).

We consider u, an equivariant wave map in dimension 2 with target N, or a radial solution to the critical Yang-Mills equations in dimension 4, that is, a solution to the following problem (see [10] for the derivation of the equation).

$$\begin{cases} u_{tt} - u_{rr} - \dfrac{1}{r} u_r = -\dfrac{f(u)}{r^2}, \\ (u, u_t)|_{t=0} = (u_0, u_1). \end{cases} \tag{1}$$

At least formally, the energy is conserved by such wave maps :

$$E(u, u_t) = \int \left(u_t^2 + u_r^2 + \frac{g^2(u)}{r^2} \right) r dr = E(u_0, u_1).$$

Shatah and Tahvildar-Zadeh [11] proved that (1) is locally well posed in the energy space

$$\mathcal{H} \times L^2 = \{(u_0, u_1) | E(u_0, u_1) < \infty.\}.$$

For such wave maps, energy is preserved.

From Struwe [13] we have the following dichotomy regarding long time existence of solutions to (1), depending on the geometry of the target manifold N, and thus on g :

- If $g(\rho) > 0$ for all $\rho > 0$ (and $\int_0^\infty g(\rho) d\rho = \infty$, to prevent a sphere at infinity), then any finite energy wave map is global in time.
- Otherwise there exists a non-constant harmonic map Q, and one may have blow up (cf. [9, 8]).

Our goal in this paper is to study the latter case, and to describe the dynamics of equivariant wave maps and of radial solutions to the critical Yang-Mills equations in dimension 4, with energy smaller or equal to $E(Q)$.

1.1 Statement of the result

Notations and Assumptions :

Denote by $v = W(t)(u_0, u_1)$ the solution to

$$\begin{cases} u_{tt} - u_{rr} - \dfrac{1}{r}u_r - \dfrac{k^2}{r^2}u = 0, \\ (u, u_t)|_{t=0} = (u_0, u_1). \end{cases} \tag{2}$$

$W(t)$ is the linear operator associated with the wave equation with a quadratic potential.

For a single function u, we use $E(u)$ for $E(u, 0)$, with a slight abuse of notation, and we also use

$$E_a^b(u) = \int_a^b \left(u_r^2 + \frac{g^2(u)}{r^2} \right) rdr.$$

To avoid degeneracy (existence of infinitely small spheres), we assume that the set of points where g vanishes is discrete. Denote $G(\rho) = \int_0^\rho |g|$. G is an increasing function. We make the following assumptions on g (that is on N, the wave map target) :

(A1) g vanishes at some point other than 0, and we denote $C^* > 0$ the smallest positive real satisfying $g(C^*) = 0$.

(A2) $g'(0) = k \in \{1, 2\}$ and if $k = 1$, we also have $g''(0) = 0$.

(A3) $g'(-\rho) \geq g'(\rho)$ for $\rho \in [0, C^*]$ and $g'(\rho) \geq 0$ for all $\rho \in [0, D^*]$, where we denote by D^* the point in $[0, C^*]$ such that $G(D^*) = G(C^*)/2$.

The first assumption is a necessary and sufficient condition on g for the existence of stationary solutions to (1), that is, non-constant harmonic maps. Hence denote $Q \in \mathcal{H}$ the solution to $rQ_r = g(Q)$, with $Q(0) = 0$, $Q(\infty) = C^*$ and $Q(1) = C^*/2$, so that $(Q, 0)$ is a stationary wave map (see [3] for more details). Note that

$$E(Q) = 2G(C^*).$$

The second assumption is a technical one : the restriction on the range of k should be removable using harmonic analysis. Recall that $k \in \mathbb{N}^*$, and for equivariant wave maps, one usually assumes g odd. To remain at a lower level of technicality, we stick to the two assumptions in (A2) which encompass the cases of greater interest (see below).

The first part of third assumption is a way to ensure that Q is a non-constant harmonic map (with $Q(0) = 0$) with least energy. The second part arises crucially in the proof of some positivity estimates. This assumption could be somehow relaxed, but as such encompasses the two cases below, avoiding technicalities which are beside the point. We conjecture that this assumption is removable.

These assumptions encompass

- corotational equivariant wave maps to the sphere $\mathbb{S}^2$ in energy critical dimension $n = 2$ ($g(u) = \sin u$, $f(u) = \sin(2u)/2$), $k = 1$ – we refer to [10] for more details).
- the critical (4-dimensional) radial Yang-Mills equation ($f(u) = 2u(1-u^2)$, $g(u) = (1-u^2)$, notice that to enter our setting we should consider $\tilde{g}(u) = g(u-1) = u(2-u)$, $k = 2$ – we refer to [2] for more details).

Recall that if $u \in \mathcal{H}$, then u has finite limits at $r \to 0$ and $r \to \infty$, which are zeroes of g : we denote them by $u(0)$ and $u(\infty)$ (see [3, Lemma 1]). We can now introduce

$$\mathcal{V}(\delta) = \{(u_0, u_1) \in \mathcal{H} \times L^2 | E(u_0, u_1) < E(Q) + \delta,\ u_0(0) = u_0(\infty) = 0\}. \tag{3}$$

Denote $H = \left\{ u | \|u\|_H^2 = \int \left(u_r^2 + \frac{u^2}{r^2} \right) r dr < \infty \right\}$. As we shall see below (Lemma 2), for $\delta \leq E(Q)$, $\mathcal{V}(\delta)$ is naturally endowed with the Hilbert norm

$$\|(u_0, u_1)\|_{H \times L^2}^2 = \|u_0\|_H^2 + \|u_1\|_{L^2}^2 = \int \left(u_1^2 + {u_0}_r^2 + \frac{u_0^2}{r^2} \right) r dr. \tag{4}$$

Finally, for I an interval of time, introduce the Strichartz space $S(I) = L_{t \in I}^{\frac{2k+3}{k}}(dt) L^{\frac{2k+3}{k}}(r^{-2} dr)$ and

$$\|u\|_{S(I)} = \|u\|_{L_{t \in I}^{2+3/k}(dt) L_r^{2+3/k}(r^{-2} dr)}.$$

Notice that $S(I)$ is simply the Strichartz space $L_{t,x}^{2+3/k}$ adapted to the energy critical wave equation in dimension $2k+2$ (see [5]), under the conjugation by the map $u \mapsto u/r^k$. This space appears naturally, see Section 3 for further details.

Theorem 1. *Assume $k=1$ or $k=2$, and g satisfies* (A1), (A2) *and* (A3). *There exists $\delta = \delta(g) > 0$ such that the following holds. Let $(u_0, u_1) \in \mathcal{V}(\delta)$ and denote by $u(t)$ the corresponding wave map. Then $u(t)$ is global in time, and scatters, in the sense that $\|u\|_{S(\mathbb{R})} < \infty$. As a consequence, there exist $(u_0^\pm, u_1^\pm) \in H \times L^2$ such that*

$$\|u(t) - W(t)(u_0^\pm, u_1^\pm)\|_{H \times L^2} \to 0 \quad \text{as} \quad t \to \pm\infty.$$

As a direct consequence, we have the following

Corollary 1. *Let (u_0, u_1) be such that $E(u_0, u_1) \leq E(Q, 0)$, and denote by $u(t)$ the corresponding wave map. Then $u(t)$ is global and we have the following dichotomy :*

- *If $u_0 = Q$ (or $u_0 = -Q$ if $-Q$ is a harmonic map) up to scaling, then $u(t)$ is a constant harmonic map ($u_t(t) = 0$).*
- *Otherwise $u(t)$ scatters, in the sense that there exist $(u_0^\pm, u_1^\pm) \in H \times L^2$ such that*

$$\|u(t) - W(t)(u_0^\pm, u_1^\pm)\|_{H \times L^2} \to 0 \quad \text{as} \quad t \to \pm\infty.$$

***Remark* 1.** *The fact that $u(t)$ is global in time is a direct corollary of [13] (in fact one has global well posedness in $\mathcal{V}(E(Q))$ as recalled in Proposition 1). The new point in our result is linear scattering.*

***Remark* 2.** *We conjecture that $\delta = E(Q)$. The only point missing for this is to improve Lemma 7 to $\delta = E(Q)$.*

***Remark* 3.** *This result corresponds to what is expected in a "focusing" setting. Similarly, there is a defocusing setting, in the case $g(\rho) > 0$ for $\rho > 0$. Arguing in the same way as in Theorem 1, we can prove that if g satifies (A2), (A3) and $g'(\rho) \geq 0$ for all $\rho \in \mathbb{R}$, then any wave map is global and scatters in the sense of Theorem 1. Again, we conjecture that the correct assumptions for this result are $g(\rho) > 0$ for $\rho > 0$ and $G(\rho) \to \pm\infty$ as $\rho \to \pm\infty$ (to prevent a sphere at infinity).*

2 Variational results and global well posedness in $\mathcal{V}(E(Q))$

First recall the pointwise bound derived from the energy

$$\forall r, r' \in \mathbb{R}^+, \quad |G(u(r)) - G(u(r'))| \leq \frac{1}{2} E_r^{r'}(u), \tag{5}$$

with equality at points r, r' if and only if there exist $\lambda > 0$ and $\varepsilon \in \{-1, 1\}$ such that

$$\forall \rho \in [r, r'], \quad u(\rho) = \varepsilon Q(\lambda \rho).$$

(See [3, Proposition 1].)

Lemma 1 ($\mathcal{V}(\delta)$ is stable through the wave map flow)**.** *If $u \in \mathcal{H}$, u is continuous and has limits at 0 and ∞ which are points where g vanishes : we denote them $u(0)$ and $u(\infty)$. Furthermore if $u(t)$ is a finite energy wave map defined on some interval I containing 0, then for all $t \in I$,*

$$\forall t \in I, \quad u(t, 0) = u(0, 0) \quad \text{and} \quad u(t, \infty) = u(0, \infty).$$

In particular, for all $\delta \geq 0$, $\mathcal{V}(\delta)$ is preserved under the wave map flow.

Proof. The properties of u are well known : see [10] or [3]. Let us prove that the $u(t, 0)$ is constant in time by a continuity argument.

For all y such that $g(y) = 0$, denote $I_y = \{t \in I | u(t, 0) = y\}$. Let $t \in I$.

As g vanishes on a discrete set, denote $\varepsilon > 0$ such that if $g(\rho) = 0$, $|G(\rho) - G(u(t, 0))| \geq 2\varepsilon$. Since u is defined in I, it does not concentrate energy in a neighbourhood of $(t, 0)$: there exists $\delta_0, \delta_1 > 0$ such that

$$\forall \tau \in [t - \delta_0, t + \delta_0], \quad E_0^{\delta_1}(u(\tau)) \leq \varepsilon.$$

From this and the pointwise bound, we deduce

$$\forall \tau \in [t - \delta_0, t + \delta_0], \forall r \in [0, \delta_1], \quad |G(u(\tau), 0) - G(u(\tau, r)| \leq \varepsilon/2.$$

Now compute for $t' \in [t - \delta_0, t + \delta_0]$:

$$\begin{aligned} \left| \int_0^{\delta_1} G(u)(t, \rho) d\rho - \int_0^{\delta_1} G(u)(t', \rho) d\rho \right| &\leq \int_0^{\delta_1} \int_t^{t'} g(u(\tau, \rho) |u_t(\tau, \rho)| d\tau d\rho \\ &\leq \frac{1}{2} \int_t^{t'} E(u) d\tau \leq \frac{1}{2} E(u) |t - t'|. \end{aligned}$$

Suppose t' is such that $u(t, 0) \neq u(t', 0)$, and then $|G(u)(t, 0) - G(u)(t', 0)| \geq 2\varepsilon$. Then

$$\begin{aligned} &\left| \int_0^{\delta_1} G(u)(t, \rho) d\rho - \int_0^{\delta_1} G(u)(t', \rho) d\rho \right| \\ &\geq \left| \int_0^{\delta_1} ((G(u)(t, \rho) - G(u)(t, 0)) + (G(u)(t, 0) - G(u)(t', 0)) + G(u)(t', 0) - G(u)(t', \rho))) d\rho \right| \\ &\geq \delta_1 (2\varepsilon - \varepsilon/2 - \varepsilon/2) \geq \delta_1 \varepsilon. \end{aligned}$$

We just proved that

$$\frac{1}{2} E(u) |t' - t| \geq \varepsilon \delta_1.$$

This means that $I_{u(t,0)}$ is open in I. In the same way, $I \setminus I_{u(t,0)} = \bigcup_{y,\ y \neq u(t,0)} I_y$ is also open in I, so that $I_{u(t,0)}$ is closed in I. As I is connected, $I = I_{u(t,0)}$.

Similarly, one can prove that $u(t, \infty)$ is constant in time. The rest of the Lemma follows from conservation of energy. □

Lemma 2. *There exists an increasing function $K : [0, 2E(Q)) \to [0, C^*)$, and a decreasing function $\delta : [0, 2E(Q)) \to (0, 1]$ such that the following holds. For all $u \in \mathcal{H}$ such that $E(u) < 2E(Q)$, and $u(0) = u(\infty) = 0$, one has the pointwise bound*

$$\forall r, \quad |u(r)| \leq K(E(u)) < C^*.$$

Moreover, one has

$$\delta(E(u))\|u\|_H \leq E(u) \leq \|g'\|_{L^\infty}\|u\|_H.$$

Proof. From the pointwise bound (5), we have

$$|G(u)(r)| = |G(u)(r) - G(u)(0)| \leq \frac{1}{2}E_0^r(u), \quad |G(u)(r)| \leq \frac{1}{2}E_r^\infty(u).$$

So that $2|G(u)(r)| \leq E(u) < 2E(Q)$. As G is an increasing function on $[-E(Q), E(Q)]$, and $|G(-\rho)| \geq G(\rho)$ for $\rho \in [0, C^*]$, we obtain

$$|u(r)| \leq G^{-1}(E(u)/2) < G^{-1}(E(Q)) = C^*.$$

Then $K(\rho) = G^{-1}(\rho/2)$ fits.

We now turn to the second line. For the upper bound, notice that $g(0) = 0$ so that $g^2(\rho) \leq \|g'\|_{L^\infty}^2\rho^2$, and $\|g'\|_{L^\infty} \geq |g'(0)| \geq 1$.

For the lower bound, notice that as $|u| \leq K(E(u)) < C^*$, then $g^2(u) \geq \delta(E(u))u^2$ for some positive continuous function $\delta : (-C^*, C^*) \to (0, 1]$ ($g(\rho)/\rho$ is a continuous positive function on $(-C^*, C^*)$, $\delta(\rho) = \min(1, \inf\{g(r)/r \mid |r| \leq \rho\})$). □

Proposition 1 (Struwe [13]). *Let $(u_0, u_1) \in \mathcal{V}(E(Q))$. Then the corresponding wave map is global in time, and satisfies the bound*

$$\forall t, r \quad |u(t, r)| \leq K(E(u_0, u_1)).$$

Proof. Indeed suppose that u blows-up, say at time T. By Struwe [13], there exists a non-constant harmonic map $\tilde{Q}$, and two sequences $t_n \uparrow T$ and $\lambda(t_n)$ such that $\lambda(t_n)|T - t_n| \to \infty$ and

$$u_n(t, r) = u\left(t_n + \frac{t}{\lambda(t_n)}, \frac{r}{\lambda(t_n)}\right) \to \tilde{Q}(r) \quad H_{\text{loc}}(]-1, 1[_t \times \mathbb{R}_r).$$

From Lemma 1, one deduces $\tilde{Q}(0) = 0$, and hence (with assumption (A3)) $|\tilde{Q}(\infty)| \geq C^*$.

However, as $(u, u_t) \in \mathcal{V}(E(Q))$, from Lemma 2, $|u(t, r)| \leq K(E(u)) < C^*$ (uniformly in t). Now $\{r \geq 0 | |\tilde{Q}(r)| \geq (K(E(u)) + C^*)/2\}$ is an interval of the form $[A_{E(u)}, \infty)$ ($\tilde{Q}$ is monotone) so that

$$\int_{t\in[-1/2,1/2]} \int_{[A_{E(u)}, A_{E(u)}+1]} |u_n(t, r) - \tilde{Q}(r)|^2 r dr dt \geq (C^* - K(E(u)))^2/4 \nrightarrow 0.$$

This is in contradiction with the H_{loc} convergence : hence u is global. □

3 Local Cauchy problem revisited

Denote $\Delta = \partial_{rr} + \frac{2k+1}{r}\partial_r = \frac{1}{r^{2k+1}}\partial_r(r^{2k+1}\partial_r)$ the radial Laplacian in dimension $\mathbb{R}^{2k+2}$ and $U(t)$ the linear wave operator in $\mathbb{R}^{2k+2}$:

$$U(t)(v_0, v_1) = \cos(t\sqrt{-\Delta})v_0 + \sqrt{-\Delta}\sin(t\sqrt{-\Delta})v_1.$$

Notice that

$$W(t)(u_0,u_1) = r^k U(t)(u_0/r^k, u_1/r^k), \tag{6}$$

as v solves $v_{tt} - \Delta v = 0$ if and only if $r^k v$ solves (2).

Given an interval I of $\mathbb{R}$, denote

$$\begin{aligned} \|v\|_{N(I)} &= \|v(t,x)\|_{N(t\in I)} \\ &= \|v\|_{L^\infty_{t\in I}\dot{H}^1_x} + \|v\|_{L^{\frac{2k+3}{k}}_{t\in I,x}} + \|v\|_{L^{\frac{2(2k+3)}{2k+1}}_{t\in I}\dot{W}^{1/2,\frac{2(2k+3)}{2k+1}}_x} + \|v\|_{W^{1,\infty}_{t\in I}L^2_x}, \end{aligned} \tag{7}$$

where the space variable x belongs to $\mathbb{R}^{2k+2}$. This norm appears in the Strichartz estimate (Lemma 6).

Theorem 2. *Assume $k = 1$ or 2. Problem (1) is locally well-posed in the space H in the sense that there exist two functions $\delta_0, C : [0,\infty) \to (0,\infty)$ such that the following holds. Let $(u_0,u_1) \in H \times L^2$ be such that $\|u_0,u_1\|_{H\times L^2} \leq A$, and let I be an open interval containing 0 such that*

$$\|W(t)(u_0,u_1)\|_{S(I)} = \eta \leq \delta_0(A).$$

Then there exist a unique solution $u \in C(I,H) \cap S(I)$ to Problem (1) and $\|u\|_{S(I)} \leq C(A)\eta$, (and we also have $\|u/r^k\|_{N(I)} \leq C(A)$ and $E(u,u_t) = E(u_0,u_1)$).

As a consequence, if u is such a solution defined on $I = \mathbb{R}^+$, satisfying $\|u\|_{S(\mathbb{R}^+)} < \infty$, there exist $(u_0^+, u_1^+) \in H \times L^2$ such that

$$\|u(t) - W(t)(u_0^+,u_1^+)\|_{H\times L^2} \to 0 \quad \textit{as} \quad t \to +\infty.$$

3.1 Preliminary lemmas

Let us first recall some useful lemmas. We consider $D^s = (-\Delta)^{s/2}$ the fractional derivative operator and the homogeneous Sobolev space

$$\dot{W}^{s,p} = \dot{W}^{s,p}(\mathbb{R}^n) = \left\{\varphi \in \mathcal{S}'(\mathbb{R}^n) \;\middle|\; \|\varphi\|_{\dot{W}^{s,p}} \stackrel{\text{def}}{=} \|D^s\varphi\|_{L^p} < \infty\right\}.$$

For integer s, it is well known that $\|\cdot\|_{\dot{W}^{s,p}}$ is equivalent to the Sobolev semi-norm :

$$\|\varphi\|_{\dot{W}^{s,p}} \sim \|\nabla^s\varphi\|_{L^p}.$$

Lemma 3 (Hardy-Sobolev embedding)**.** *Let $n \geq 3$, and $p,q,\alpha,\beta \geq 0$ be such that $1 \leq q \leq p \leq \infty$, and $0 < (\beta-\alpha)q < n$. There exist $C = C(n,p,q,\alpha,\beta)$ such that for all φ radial in $\mathbb{R}^n$,*

$$\|r^{\frac{n}{q}-\frac{n}{p}-\beta+\alpha}\varphi\|_{\dot{W}^{\alpha,p}} \leq C\|\varphi\|_{\dot{W}^{\beta,q}}.$$

Proof. Given n,p,q and β, we show the estimate for α in the suitable range.

The case $\alpha = 0$ is the standard Hardy inequality in L^p combined with the Sobolev embedding (see [11] and the references therein - where the conditions $n \geq 3$, $1 \leq q \leq p \leq \infty$ and $0 < \beta < n$ are required). If α is an integer, we use the Sobolev semi-norm : as

$$\partial_r^\alpha(r^\gamma v) = \sum_{k=0}^{\alpha} c_k r^{\gamma-k}\partial_r^{\alpha-\gamma} v,$$

the inequality follows from the case $\alpha = 0$.

In the general case, let $\alpha = k+\theta$ for $k \in \mathbb{N}$ and $\theta \in]0,1[$, and $\gamma = \frac{n}{q} - \frac{n}{p} - \beta + \alpha$. We define ℓ so that $\beta = \ell + \theta$, hence $\frac{n}{q} - \frac{n}{p} - \ell + k = \gamma$. We consider the operator $T : \varphi \mapsto D^k(r^\gamma D^{-\ell}\varphi)$:

T maps L^q to L^p and $\dot W^{1,q}$ to $\dot W^{1,p}$ (integer case). By complex interpolation (see [12]), T maps $[L^q, \dot W^{1,q}]_\theta = \dot W^{\theta,q}$ to $[L^p, \dot W^{1,p}]_\theta = \dot W^{\theta,p}$. This means that

$$\|r^\gamma \varphi\|_{\dot W^{k+\theta,p}} \le C\|\varphi\|_{\dot W^{\ell+\theta,q}},$$

which is what we needed to prove. □

Lemma 4. *If $v = u/r^k$, then*

$$\frac{1}{3}\int v_r^2 r^{2k+1} dr \le \int \left(u_r^2 + \frac{u^2}{r^2}\right) r dr \le (k^2+1)\int v_r^2 r^{2k+1} dr.$$

Proof. First notice that $v_r = -ku/r^{k+1} + u_r/r^k$, hence $v_r^2 \le (k^2+1)(u^2/r^{2k+2} + u_r^2/r^{2k})$ and

$$\int v_r^2 r^{2k+1} dr \le (k^2+1)\int \left(u_r^2 + \frac{u^2}{r^2}\right) r dr.$$

Then from the Hardy-Sobolev inequality in dimension $2k+2 \ge 3$ (optimal constant is $1/k^2$),

$$\int \frac{u^2}{r^2} r dr = \int \frac{v^2}{r^2} r^{2k+1} dr \le \frac{1}{k^2}\int v_r^2 r^{2k+1} dr.$$

As $u_r = r^k v_r + ku/r$, $u_r^2 \le 2r^{2k} v_r^2 + 2k^2 u^2/r^2$ and

$$\int \left(u_r^2 + \frac{u^2}{r^2}\right) r dr \le \left(2 + \frac{1}{k^2}\right)\int v_r^2 r^{2k+1} r dr. \qquad \square$$

Lemma 5 (Derivation rules)**.** *Let $1 < p < \infty$, $0 < \alpha < 1$. Then*

$$\begin{aligned}
\|D^\alpha(\varphi\psi)\|_{L^p} &\le C\|\varphi\|_{L^{p_1}}\|D^\alpha\psi\|_{L^{p_2}} + \|D^\alpha\varphi\|_{L^{p_3}}\|\psi\|_{L^{p_4}},\\
\|D^\alpha(h(\varphi))\|_{L^p} &\le C\|h'(\varphi)\|_{L^{p_1}}\|D^\alpha\varphi\|_{L^{p_2}}.\\
\|D^\alpha(h(\varphi)-h(\psi))\|_{L^p} &\le C(\|h'(\varphi)\|_{L^{p_1}} + \|h'(\psi)\|_{L^{p_1}})\|D^\alpha(\varphi-\psi)\|_{L^{p_2}}\\
&\quad + C(\|h''(\varphi)\|_{L^{r_1}} + h''(\psi)\|_{L^{r_1}})(\|D^\alpha\varphi\|_{L^{r_2}} + \|D^\alpha\psi\|_{L^{r_2}})\|\varphi-\psi\|_{L^{r_3}},
\end{aligned}$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$, *and* $1 < p_2, p_3, r_1, r_2, r_3 < \infty$.

Proof. See [7, Theorem A.6 and A.8] with functions which do not depend on times, [7, Theorem A.7 and A.12] and [5, Lemma 2.5]. □

From now on, we work in dimension $2k+2$ (radial), and the underlying measure is $r^{2k+1}dr$ unless otherwise stated. In particular, notice that from Lemma 5, we have :

$$\|D^{1/2}(\varphi\psi)\|_{L^{\frac{2(2k+3)}{2k+5}}} \le \|D^{1/2}\varphi\|_{L^{\frac{2(2k+3)}{2k+5}}}\|\psi\|_{L^\infty} + \|\varphi\|_{L^{\frac{4(2k^2+5k+3)}{4k^2+12k+7}}}\|D^{1/2}\psi\|_{L^{4(k+1)}}. \tag{8}$$

Recall

$$w = \cos(t\sqrt{-\Delta})v_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}v_1 + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}\chi(s)ds$$

solves the problem

$$\begin{cases} w_{tt} - \Delta w = \chi, \\ (w, w_t)|_{t=0} = (v_0, v_1), \end{cases}$$

Lemma 6 (Strichartz estimate). *Let I be an interval. There exist a constant C (not depending on I) such that (in dimension $2k+2$),*

$$\|\cos(t\sqrt{-\Delta})v_0\|_{N(\mathbb{R})} \leq C\|v_0\|_{\dot{H}^1_x},$$
$$\|\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}v_1\|_{N(\mathbb{R})} \leq \|v_1\|_{L^2_x},$$
$$\|\int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}\chi(s)ds\|_{N(I)} \leq \|D_x^{1/2}\chi\|_{L_{t\in I}^{\frac{2(2k+3)}{2k+5}} L_x^{\frac{2(2k+3)}{2k+5}}}.$$

Proof. This result is well-known : see [5] and the references therein. □

3.2 Proofs of Theorem 2 in the case $k=1$ and $k=2$

Proof of Theorem 2. Denote $v = u/r^k$. Then $v_r = u_r/r^k - ku/r^{k+1}$, $v_{rr} = \frac{u_{rr}}{r^k} - \frac{2ku_r}{r^{k+1}} + k(k+1)\frac{u}{r^{k+2}}$, so that

$$\begin{cases} v_{tt} - v_{rr} - (2k+1)\dfrac{v_r}{r} = -\dfrac{f(r^k v) - k^2 r^k v}{(r^k v)^{1+2/k}} v^{1+2/k}, \\ (v, v_t)|_{t=0} = (v_0, v_1) = (u_0/r^k, u_1/r^k). \end{cases} \tag{9}$$

This is something like the energy critical wave equation in dimension $2k+2$. For the rest of this section, the underlying dimension will always be $2k+2$, in particular, Lebesgue and Sobolev space will be defined with respect to the measure $r^{2k+1}dr$ (as we are in a radial setting).

Denote

$$h(\rho) = \frac{f(\rho) - k^2\rho}{\rho^{1+2/k}}.$$

Assume that h, h' and h'' are bounded on compact sets : this is automatic if g is C^3 and satifies (A2). Indeed if $k=2$, $1+2/k=2$ and it is a direct application of Taylor's expansion, and if $k=1$, $1+2/k=3$, and it suffices to notice additionally that $f''(0) = 3kg''(0) = 0$).

Our assumptions on (u_0, u_1) translate to :

$$\|v_0\|_{\dot{H}^1} + \|v_1\|_{L^2} \leq CA, \qquad \|U(t)(v_0,v_1)\|_{L_{t\in I}^{2+3/k} L_r^{2+3/k}} \leq C\eta.$$

Consider the map Φ :

$$\Phi : v \mapsto \cos(t\sqrt{-\Delta})v_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}v_1 + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}(v^{1+2/k}(s)h(r^k v)(s))ds,$$

that is $\Phi(v)$ solves the (linear in $\Phi(v)$) equation

$$\begin{cases} \Phi(v)_{tt} - \Phi(v)_{rr} - (2k+1)\dfrac{\Phi(v)_r}{r} = -h(r^k v)v^{1+2/k}, \\ (v, v_t)|_{t=0} = (v_0, v_1) = (u_0/r^k, u_1/r^k). \end{cases} \tag{10}$$

We will find a fixed point for Φ, related to smallness in the norm :

$$\|v\|_{L_{t\in I,r}^{2+3/k}} \quad \text{and} \quad \|D^{1/2}v\|_{L_{t\in I,r}^{2(2k+3)/(2k+1)}}.$$

The Strichartz estimate shows that we are to control $\|D_r^{1/2}(v^{1+2/k}h(r^{1+2/k}v))\|_{L_{t\in I,r}^{\frac{2(2k+3)}{2k+5}}}$. For convenience in the following, denote :

$$p = \frac{4(k+2)(2k^2+5k+3)}{4k(k^2+12k+7)}.$$

Now, we use (8) together with Lemma 3 and Lemma 5 :

$$
\begin{aligned}
&\|D^{1/2}(v^{1+2/k}h(r^k v))\|_{L^{\frac{2(2k+3)}{2k+5}}} \\
&\leq \|D^{1/2}(v^{1+2/k})\|_{L^{\frac{2(2k+3)}{2k+5}}}\|h(r^k v)\|_{L^\infty} + \|v^{1+2/k}\|_{L^{\frac{4(2k^2+5k+3)}{4k^2+12k+7}}}\|D^{1/2}h(r^k v)\|_{L^{4(k+1)}} \\
&\leq C\|v^{2/k}\|_{L^{k+3/2}}\|D^{1/2}v\|_{L^{\frac{2(2k+3)}{2k+1}}}\|h(r^k v)\|_{L^\infty} + C\|v\|_{L^p}^{1+2/k}\|h'(r^k v)\|_{L^\infty}\|r^k v\|_{\dot{W}^{1/2,4(k+1)}} \\
&\leq C\|v\|_{L^{2+3/k}}^{2/k}\|D^{1/2}v\|_{L^{\frac{2(2k+3)}{2k+1}}}\|h(r^k v)\|_{L^\infty} + C\|v\|_{L^p}^{1+2/k}\|h'(r^k v)\|_{L^\infty}\|v_r\|_{L^2}.
\end{aligned}
$$

From interpolation of Lebesgue spaces and Hölder inequality,

$$
\begin{aligned}
\left\|\|v\|_{L^p}^{1+2/k}\right\|_{L_t^{\frac{2(2k+3)}{2k+5}}} &= \left\|\|v\|_{L_r^{\frac{4(k+2)(2k^2+5k+3)}{k(4k^2+12k+7)}}}\right\|_{L_t^{(1+2/k)(2(2k+3)/(2k+5)}}^{1+2/k} \\
&\leq \|v\|_{L_{t,r}^{2+3/k}}^{2/k}\|v\|_{L_t^{2(2k+3)/(2k+1)}L_r^{\frac{4(2k+3)(k+1)}{4k^2+4k-1}}} \\
&\leq \|v\|_{L_{t,r}^{2+3/k}}^{2/k}\|D_r^{1/2}v\|_{L_{t,r}^{\frac{2(2k+3)}{2k+1}}} \quad \text{and} \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
\left\|\|v\|_{L_r^{2+3/k}}^{2/k}\|D^{1/2}v\|_{L_r^{\frac{2(2k+3)}{2k+1}}}\right\|_{L_t^{\frac{2(2k+3)}{2k+5}}} &\leq \left\|\|v\|_{L_r^{2+3/k}}^{2/k}\right\|_{L_t^{k+3/2}}\|D^{1/2}v\|_{L_{t,r}^{\frac{2(2k+3)}{2k+1}}} \\
&\leq \|v\|_{L_{t,r}^{2+3/k}}^{2/k}\|D_r^{1/2}v\|_{L_{t,r}^{\frac{2(2k+3)}{2k+1}}}. \qquad (12)
\end{aligned}
$$

Using again Lemma 3 to show $\|r^k v\|_{L^\infty} \leq C\|v_r\|_{L^2}$, we hence get our main estimate, for some increasing function ω (ω is a function of h, h' and essentially the constant in the Strichartz estimate, and does not depend on I or v) :

$$
\|D_x^{1/2}(v^{1+2/k}h(r^k v))\|_{L_{t\in I,r}^{\frac{2(2k+3)}{2k+5}}} \leq \omega(\|v_r\|_{L_{t\in I}^\infty L_r^2})\|v\|_{L_{t\in I,r}^{2+3/k}}^{2/k}\|D^{1/2}v\|_{L_{t\in I,r}^{\frac{2(2k+3)}{2k+1}}}. \qquad (13)
$$

We now turn to difference estimates. Using the same inequalities, we get :

$$
\begin{aligned}
&\|D^{1/2}(v^{1+2/k}h(r^k v) - w^{1+2/k}h(r^k w))\|_{L_r^{\frac{2(2k+3)}{2k+5}}} \\
&\leq C\|D^{1/2}\left((v^{1+2/k} - w^{1+2/k})h(r^k v)\right)\|_{L^{\frac{2(2k+3)}{2k+5}}} \\
&\quad + C\left\|D^{1/2}\left(r^k w^{1+2/k}(v-w)\int_0^1 h'(\theta r^k(v-w) + r^k w)d\theta\right)\right\|_{L^{\frac{2(2k+3)}{2k+5}}} \\
&\leq \|D^{1/2}(v^{1+2/k} - w^{1+2/k})\|_{L^{\frac{2(2k+3)}{2k+5}}}\|h(r^k v)\|_{L^\infty} \\
&\quad + \|v^{1+2/k} - w^{1+2/k}\|_{L^{\frac{4(2k^2+5k+3)}{4k^2+12k+7}}}\|D^{1/2}h(r^k v)\|_{L^{4(k+1)}} \\
&\quad + \|D^{1/2}(r^k w^{1+2/k}(v-w))\|_{L^{\frac{2(2k+3)}{2k+5}}}\left\|\int_0^1 h'(\theta r^k(v-w) + r^k w)d\theta\right\|_{L^\infty} \\
&\quad + \|r^k w^{1+2/k}(v-w)\|_{L^{\frac{4(2k^2+5k+3)}{4k^2+12k+7}}}\left\|\int_0^1 D^{1/2}(h'(\theta r^k(v-w) + r^k w))d\theta\right\|_{L^{4(k+1)}} \\
&\leq \|D^{1/2}(v^{1+2/k} - w^{1+2/k})\|_{L^{\frac{2(2k+3)}{2k+5}}}\|h(r^k v)\|_{L^\infty} \\
&\quad + \|v-w\|_{L^p}(\|v\|_{L^p}^{2/k} + \|w\|_{L^p}^{2/k})\|h'(r^k v)\|_{L^\infty}\|v_r\|_{L^2}
\end{aligned}
$$

$$+\left(\|D^{1/2}(w^{2/k}(v-w))\|_{L^{\frac{2(2k+3)}{2k+5}}}\|r^k w\|_{L^\infty}+\|w^{2/k}(v-w)\|_{L^{\frac{4(2k^2+5k+3)}{4k^2+12k+7}}}\|D^{1/2}(r^k w)\|_{L^{4(k+1)}}\right)$$
$$\times\sup_{\theta\in[0,1]}\|h'(r^k v+\theta r^k(w-v))\|_{L^\infty}+\|r^k w\|_{L^\infty}\|w^{2/k}(v-w)\|_{L^{\frac{4(2k^2+5k+3)}{4k^2+12k+7}}}$$
$$\times\sup_{\theta\in[0,1]}\left(\|h''(\theta r^k(v-w)+r^k w)\|_{L^\infty}\|D^{1/2}(r^k(\theta v+(1-\theta)w))\|_{L^{4(k+1)}}\right).$$

Then we have as previously :

$$\begin{aligned}
&\|D^{1/2}(w^{2/k}(v-w))\|_{L^{\frac{2(2k+3)}{2k+5}}}\\
&\quad\le C\|D^{1/2}(v-w)\|_{L^{\frac{(2(2k+3)}{2k+1}}}\|w^{2/k}\|_{L^{k+3/2}}+C\|v-w\|_{L^{2+3/k}}\|D^{1/2}(w^{2/k})\|_{L^{\frac{2(2k+3)}{5}}}\\
&\quad\le C\|w\|_{L^{2+3/k}}^{2/k}\|D^{1/2}(v-w)\|_{L^{\frac{(2(2k+3)}{2k+1}}}+\|w\|_{L^{2+3/k}}^{2/k-1}\|D^{1/2}w\|_{L^{\frac{(2(2k+3)}{2k+1}}}\|v-w\|_{L^{2+3/k}},\\
&\|w^{2/k}(v-w)\|_{L^{\frac{4(2k^2+5k+3)}{4k^2+12k+7}}}\le\|w\|_{L^p}^{2/k}\|v-w\|_{L^p}.
\end{aligned}$$

Doing the computations in each case $k=1$ or $k=2$, we have that

$$\begin{aligned}
\|D^{1/2}(v^3-w^3)\|_{L^{10/7}}&\le\|D^{1/2}v-w\|_{L^{10/3}}(\|v\|_{L^5}^2+\|w^2\|_{L^5}^2)\\
&\quad+\|v-w\|_{L^5}(\|D^{1/2}v\|_{L^{10/3}}+\|D^{1/2}w\|_{L^{10/3}})(\|v\|_{L^5}+\|w\|_{L^5})\quad\text{and}\\
\|D^{1/2}(v^2-w^2)\|_{L^{14/9}}&=\|D^{1/2}((v-w)(v+w))\|_{L^{14/9}}\\
&\le C\|D^{1/2}(v-w)\|_{L^{14/5}}(\|v\|_{L^{7/2}}+\|w\|_{L^{7/2}})\\
&\quad+C\|v-w\|_{L^{7/2}}(\|D^{1/2}v\|_{L^{14/5}}+\|D^{1/2}w\|_{L^{14/5}}).
\end{aligned}$$

so that in both cases

$$\begin{aligned}
&\|D^{1/2}(v^{1+2/k}-w^{1+2/k})\|_{L^{\frac{2(2k+3)}{2k+5}}}\\
&\quad\le C(\|v\|_{L^{2+3/k}}+\|w\|_{L^{2+3/k}})^{2/k-1}\Bigg((\|v\|_{L^{2+3/k}}+\|w\|_{L^{2+3/k}})\|D^{1/2}(v-w)\|_{L^{\frac{2(2k+3)}{2k+1}}}\\
&\qquad+(\|D^{1/2}v\|_{L^{\frac{2(2k+3)}{2k+1}}}+\|D^{1/2}w\|_{L^{\frac{2(2k+3)}{2k+1}}}\Bigg)\|v-w\|_{L^{2+3/k}}).
\end{aligned}$$

Here, the assumption $k\le 2$ is crucially needed. Finally observe that

$$|\theta v+(1-\theta)w|\le|v|+|w|,\quad |D^{1/2}(\theta v+(1-\theta)w)\le|D^{1/2}v|+|D^{1/2}w|.$$

We can now summarize these computations, and using (11) and (12), we obtain the space time difference estimate (up to a change in the function ω, which now depends on h, h' and h'', but not on I or v) :

$$\begin{aligned}
&\|D^{1/2}(v^{1+2/k}h(r^k v)-w^{1+2/k}h(r^k w))\|_{L_{t\in I,r}^{\frac{2(2k+3)}{2k+5}}}\le(\omega(\|v\|_{L^\infty_{t\in I}\dot H^1_r})+\omega(\|w\|_{L^\infty_{t\in I}\dot H^1_r}))\\
&\quad\times(\|v\|_{L^{2+3/k}_{t\in I,r}}^{2/k-1}+\|w\|_{L^{2+3/k}_{t\in I,r}}^{2/k-1})\Bigg((\|v\|_{L^{2+3/k}_{t\in I,r}}+\|w\|_{L^{2+3/k}_{t\in I,r}})\|D^{1/2}(v-w)\|_{L_{t\in I,r}^{\frac{2(2k+3)}{2k+1}}}\\
&\qquad+(\|D^{1/2}v\|_{L_{t\in I,r}^{\frac{2(2k+3)}{2k+1}}}+\|D^{1/2}w\|_{L_{t\in I,r}^{\frac{2(2k+3)}{2k+1}}})\|v-w\|_{L^{2+3/k}_{t\in I,r}}\Bigg).
\end{aligned}$$

Given $a, b, A \in \mathbb{R}^+$, I a time interval, introduce

$$B(a,b,A,I) = \left\{ v | \, \|v\|_{L^{2+3/k}_{t\in I,r}} \le a, \quad \|D^{1/2}v\|_{L^{\frac{2(2k+3)}{2k+1}}_{t\in I,r}} \le b, \quad \|v\|_{C(t\in I,\dot{H}^1_r)} \le 2CA \right\}.$$

Hence for $v \in B(a,b,A,I)$, we have

$$\begin{aligned}
\|\Phi(v)\|_{L^{2+3/k}_{t\in I,r}} &\le \|U(t)(v_0,v_1)\|_{L^{2+3/k}_{t\in I,r}} + \omega(2CA)a^{2/k}b \\
\|D^{1/2}\Phi(v)\|_{L^{\frac{2(2k+3)}{2k+1}}_{t\in I,r}} &\le \|D^{1/2}U(t)(v_0,v_1)\|_{L^{\frac{2(2k+3)}{2k+1}}_{t\in I,r}} + \omega(2CA)a^{2/k}b \\
\|\Phi(v)\|_{C(t\in I,\dot{H}^1)} &\le \|(v_0,v_1)\|_{\dot{H}^1\times L^2} + \omega(2CA)a^{2/k}b, \\
\|\Phi(v)-\Phi(w)\|_{N(I)} &\le 2\omega(2CA)a^{2/k-1}b(\|D^{1/2}(v-w)\|_{L^{\frac{2(2k+3)}{2k+1}}_{t\in I,r}} + \|v-w\|_{L^{2+3/k}_{t\in I,r}})
\end{aligned}$$

Case $k=1$

We compute $2+3/k = 5$ and $\frac{2(2k+3)}{2k+1} = 10/3$.

Given A, set $b = 2CA$ and $\delta_0(A) = \min(1, 1/C, \frac{1}{8CA\omega(2CA)})$. Then for (v_0,v_1) such that $\|(v_0,v_1)\|_{\dot{H}^1\times L^2} \le A$ and $\|U(t)(v_0,v_1)\|_{L^5_{t\in I,r}} = \eta \le \delta_0(A)$, set $a = 2\eta$. Notice that the Strichartz estimate gives

$$\|D^{1/2}U(t)(v_0,v_1)\|_{L^{10/3}_{t\in I,r}} \le CA.$$

Our relations now write (the main point is $2/k - 1 = 1 > 0$) :

$$\begin{aligned}
\|\Phi(v)\|_{L^5_{t\in I,r}} &\le \frac{a}{2} + \omega(2CA)(2\delta_0 a)(2CA) \le a \\
\|D^{1/2}\Phi(v)\|_{L^{10/3}_{t\in I,r}} &\le CA + \omega(2CA)(2\delta_0 a)(2CA) \le 2CA \\
\|\Phi(v)\|_{C(t\in I,\dot{H}^1)} &\le A + \omega(2CA)(2\delta_0 a)(2CA) \le 2A, \\
\|\Phi(v)-\Phi(w)\|_{N(I)} &\le \frac{1}{2}(\|D^{1/2}(v-w)\|_{L^{10/3}_{t\in I,r}} + \|v-w\|_{L^5_{t\in I,r}})
\end{aligned}$$

Hence $\Phi : B(a,2CA,A,I) \to B(a,2CA,A,I)$ is a well defined 1/2-Lipschitz map, so that Φ has a unique fixed point, which is our solution.

Case $k=2$

We compute $2+3/k = 7/2$, $\frac{2(2k+3)}{2k+1} = 14/5$ and $\frac{2(2k+3)}{2k+5} = 14/9$.

In this case $2/k - 1 = 0$, so that the procedure used in the case $k=1$ no longer applies (it is the same problem as for the energy critical wave equation in dimension 6).

However, we still have a solution on an interval I where both quantities $\|U(t)(v_0,v_1)\|_{L^{7/2}_{t\in I,r}}$ and $\|D^{1/2}U(t)(v_0,v_1)\|_{L^{14/5}_{t\in I,r}}$ are small.

Indeed, given A, set $\delta_1(A) = \min(1, \frac{1}{C}, \frac{1}{8\omega(2CA)})$. For (v_0,v_1) such that $\|(v_0,v_1)\|_{\dot{H}^1\times L^2} \le A$, $\|U(t)(v_0,v_1)\|_{L^{7/2}_{t\in I,r}} = \eta \le \delta_1(A)$, and $\|D^{1/2}U(t)(v_0,v_1)\|_{L^{14/5}_{t\in I,r}} = \eta' \le \delta_1(A)$, we set $a = 2\eta$ and $b = 2\eta'$. Then we have

$$\begin{aligned}
\|\Phi(v)\|_{L^{7/2}_{t\in I,r}} &\le \frac{a}{2} + \omega(2CA)a)(2\delta_0) \le a \\
\|D^{1/2}\Phi(v)\|_{L^{14/5}_{t\in I,r}} &\le \frac{b}{2} + \omega(2CA)(2\delta_1(A))b \le b \\
\|\Phi(v)\|_{C(t\in I,\dot{H}^1)} &\le A + \omega(2CA)(2\delta_1(A))^2 \le 2A,
\end{aligned}$$

$$\|\Phi(v)-\Phi(w)\|_{N(I)} \leq \frac{1}{2}(\|D^{1/2}(v-w)\|_{L^{14/5}_{t\in I,r}} + \|v-w\|_{L^{7/2}_{t\in I,r}})$$

Hence $\Phi\ : B(a,b,A,I) \to B(a,b,A,I)$ has a unique fixed point. We just proved the following

Claim : Let $A>0$. There exist $\delta_1(A)>0$ such that for (v_0,v_1) with $\|(v_0,v_1)\|_{\dot H^1\times L^2} \leq A$, and I such that

$$\|U(t)(v_0,v_1)\|_{L^{7/2}_{t\in I,r}} = \eta \leq \delta_1(A), \quad \text{and} \quad \|D^{1/2}U(t)(v_0,v_1)\|_{L^{14/5}_{t\in I,r}} = \eta' \leq \delta_1(A),$$

Then there exist a unique solution $v(t)$ to (9) satisfying

$$\|(v,v_t)\|_{L^\infty_{t\in I}(\dot H^1\times L^2)} \leq 2A, \quad \|v\|_{L^{7/2}_{t\in I,r}} \leq 2\eta, \quad \|D^{1/2}v\|_{L^{14/5}_{t\in I,r}} \leq 2\eta'.$$

Let us now do a small computation.

Given $h,\, n \in \mathbb{N}$ and $0=t_0<t_1<\ldots<t_n=T$ (with $T\in(0,\infty]$), we have for $i=0,\ldots,n$,

$$\begin{aligned}
&\|\int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}\chi(s)ds\|_{N(t_i,t_{i+1})} \\
&\leq \sum_{j=0}^{i-1}\|\int_{t_j}^{t_{j+1}} \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}\chi(s)ds\|_{N(t_i,t_{i+1})} + \|\int_{t_i}^{t} \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}\chi(s)ds\|_{N(t_i,t_{i+1})} \\
&\leq \sum_{j=0}^{i-1}\|\int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}(\chi(s)\mathbb{1}_{s\in[t_j,t_{j+1}]})ds\|_{N(t_i,t_{i+1})} \\
&\qquad + \|\int_{t_i}^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}(\chi(s)\mathbb{1}_{s\in[t_i,t_{i+1}]})ds\|_{N(t_i,t_{i+1})} \\
&\leq \sum_{j=0}^{i-1}\|\int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}(\chi(s)\mathbb{1}_{s\in[t_j,t_{j+1}]})ds\|_{N(\mathbb{R})} \\
&\qquad + \|\int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}(\chi(s)\mathbb{1}_{s\in[t_i,t_{i+1}]})ds\|_{N(\mathbb{R})} \\
&\leq C\sum_{j=0}^{i}\|D_x^{1/2}\chi(s)\mathbb{1}_{s\in[t_j,t_{j+1}]}\|_{L^{14/9}_{s,x}} \leq C\sum_{j=0}^{i}\|D_x^{1/2}\chi\|_{L^{14/9}_{t\in[t_j,t_j+1]}L^{14/9}_x} \qquad (14)
\end{aligned}$$

Let us now complete the case $k=2$. Let $A>0$, define $n=n(A)$ such that $n=n(A)=1/(4CA\omega(2CA))$, so that $2CA\omega(2CA)/n \leq 1/2$ and $\delta_0(A)=\delta_1(A)/2^{n+2}$ (recall $\delta_1(A)=\min(1,\frac{1}{C},\frac{1}{8CA\omega(2CA)})$).

Let (v_0,v_1) be such that $\|v_0,v_1\|_{\dot H^1\times L^2}\leq A$ and for $I=(T_0,T_1)$ an interval (possibly with infinite endpoints), $\|U(t)(v_0,v_1)\|_{L^{7/2}_{t\in I,r}} = \eta \leq \delta_0(A)$.

From the Strichartz estimate, we also have

$$\|D^{1/2}U(t)(v_0,v_1)\|_{L^{14/5}_{t\in I,r}} \leq CA.$$

From (v_0,v_1), we have a solution v defined on a interval $\tilde I=[0,T)$. We choose $J=(T_0',T_1')\subset\tilde I$ to be maximal such that

$$\|v\|_{L^{7/2}_{t\in J,r}} \leq \delta_1(A), \quad \|D^{1/2}v\|_{L^{14/5}_{t\in J,r}} \leq 2CA, \quad \|v\|_{C(J,\dot H^1)} \leq 2CA.$$

From the claim, we can choose J non empty. Let $T_0' = t_0 < t_1 < \dots t_n = T_1'$ be such that

$$\forall i \in [\![0, n-1]\!], \quad \|D^{1/2}v\|_{L^{14,5}_{t\in[t_i,t_{i+1}],r}} \leq \frac{2CA}{n} \leq \frac{1}{2}\frac{1}{\omega(2CA)}.$$

From (13) and (14), we obtain

$$\|v\|_{N(J)} \leq CA + \omega(2CA)\|v\|_{L^{7/2}_{t\in J,r}}\|v\|_{N(J)},$$

$$\|v\|_{L^{7/2}_{t\in[t_i,t_{i+1}],r}} \leq \|U(t)(v_0,v_1)\|_{L^{7/2}_{t\in[t_i,t_{i+1}],r}} + \omega(2CA)\sum_{j=0}^{i}\|v\|_{L^{7/2}_{t\in[t_j,t_{j+1}],r}}\|D^{1/2}v\|_{L^{14/5}_{t\in[t_j,t_{j+1}],r}}.$$

Let us denote $a_i = \|v\|_{L^{7/2}_{t\in[t_i,t_{i+1}],r}}$ for $i \in [\![0, n-1]\!]$. Then we have

$$\|v\|_{N(J)} \leq CA + \frac{1}{4}\|D^{1/2}v\|_{L^{14/5}_{t\in I,r}} \leq 3/2CA < 2CA, \tag{15}$$

$$a_i \leq \eta + \omega(2CA)\sum_{j=0}^{i}\frac{a_j}{2\omega(2CA)} \quad \text{or equivalently} \quad a_i \leq 2\eta + \sum_{j=0}^{i-1}a_j.$$

By recurrence, we deduce that

$$a_i \leq 2^{i+1}\eta.$$

In particular,

$$\|v\|_{L^{7/2}_{t\in J}L^{7/2}_r} = \sum_{i=0}^{n-1}a_i \leq 2^{n+1}\eta \leq 2^{n+1}\delta_0(A) < \delta_1(A). \tag{16}$$

Hence, from with (15) and (16) and a standard continuity argument, we deduce that $J = \tilde{I} = I$, $\|v\|_{N(I)} \leq 2CA$ and $\|v\|_{L^{7/2}_{t\in I,r}} \leq 2^{n+1}\eta = c(A)\eta$.

Going back to u, we obtain the first part of Theorem 2, in both cases $k=1$ and $k=2$ (conservation of energy is clear from the construction).

Let us now prove the consequence mentioned in Theorem 2. Given u, we associate $v(t,r) = u(t,r)/r^k$: v is defined on $\mathbb{R}^+$, and satisfies (9).

If we denote $A = \|(u,u_t)\|_{L^\infty_t(H\times L^2)}$, then there exist T large enough such that $\|u\|_{S([T,\infty))} \leq \delta_0(A)$. From the previous part, we have that

$$\|v\|_{N[T,\infty)} \leq 2CA, \quad \|v\|_{L^{2+3/k}_{t\in[T,\infty),r}} \leq \delta_0(A).$$

Denote $\nu(t) = U(-t)v(t)$. Then

$$\nu(t) - \nu(s) = \int_s^t U(-\tau)v^{1+2/k}(\tau)h(r^k v)(\tau)d\tau.$$

Hence, for $t \geq s \geq T$, from the Strichartz estimate and (13), we have

$$\begin{aligned}\|\nu(t)-\nu(s)\|_{\dot{H}^1} + \|\nu_t(t)-\nu_t(s)\|_{L^2} &\leq \|\nu(\tau)-\nu(s)\|_{N(\tau\in[s,t])}\\ &\leq \|v^{1+2/k}(\tau)h(r^k v)(\tau)\|_{L^{\frac{2(2k+1)}{2k+5}}_{\tau\in[s,t],r}}\\ &\leq \omega(2CA)\|v\|^{2/k}_{L^{2+3/k}_{\tau\in[s,t],r}}(2CA) \to 0 \quad \text{as} \quad s,t\to+\infty.\end{aligned}$$

This means that $(\nu(t),\nu_t(t))$ is a Cauchy sequence in $\dot{H}^1 \times L^2$, hence converges to some $(v^+, v_t^+) \in \dot{H}^1 \times L^2$.

Going back to u, using Lemma 4 and remark (6), we obtain the second part of Theorem 2. □

4 Rigidity property

Recall that g is such that $g(0) = 0$, $g'(0) = k \in \mathbb{N}^*$, with C^* the smallest positive real such that $g(C^*) = 0$, $f = g'g$ and $G(\rho) = \int_0^\rho |g|(\rho')d\rho'$; $D^* \in [0, C^*]$ is such that $G(D^*) = G(C^*)/2$.

Introduce the energy density $e(u,v) = v^2 + u_r^2 + \frac{g^2(u)}{r^2}$ and $p(u) = u_r^2 + \frac{g^2(u)}{r^2}$. Denote

$$E(u,v) = \int e(u,v) r dr, \quad E_a^b(u,v) = \int_a^b e(u,v) r dr,$$

and similarly for a single function u

$$E(u) = \int p(u) r dr, \qquad E_a^b(u) = \int_a^b p(u) r dr.$$

We will also need the function $d(\rho) = \rho f(\rho)$, which is linked to the virial identity, and

$$F(u) = \int \left(u_r^2 + \frac{d(u)}{r^2} \right) r dr.$$

The following variational Lemma is at the heart of the rigidity theorem. Here is the only point where we use assumption (A3), which ensures that $g'(\rho) \geq 0$ for $\rho \in [-D^*, D^*]$.

Lemma 7. *There exist $c > 0$ and $\delta \in (0, E(Q))$ such that for all u such that $(u, 0) \in \mathcal{V}(\delta)$, we have*

$$cE(u) \leq F(u) \leq \frac{1}{c} E(u).$$

Proof. Fix $\delta < E(Q)$. $g^2(u) \geq \omega(\delta) u^2$ for some function $\omega : [0, E(Q)) \to \mathbb{R}_*^+$, and $|d(x)| \leq \|g'\|_{L^\infty(-C^*,C^*)}^2 x^2$ for $|x| < C^*$, so that

$$F(u) \leq \left(1 + \frac{\|g'\|_{L^\infty(-C^*,C^*)}^2}{\omega(\delta)} \right) E(u),$$

which is the upper bound.

For the lower bound, we need assumption (A3) on g. Hence on $[-D^*, D^*]$, $d(x) \geq 0$, and on $[0, D^*]$, $d(-x) \geq d(x)$. Denote $A = \int_0^{D^*} \sqrt{d(x)} dx > 0$. One easily sees that for a function $v : [a,b] \to [-D^*, D^*]$ such that $v(a) = 0$, $|v(b)| = D^*$ then

$$\int_a^b \left(v_r^2 + \frac{d(v)}{r^2} \right) r dr \geq 2 \int_a^b |v_r \sqrt{d(v(r))}| dr \geq 2 \int_0^{D^*} \sqrt{d(x)} dx = 2A.$$

In the same way,

$$\int_a^b \left(v_r^2 + \frac{g^2(v)}{r^2} \right) r dr \geq 2G(D^*) = G(C^*).$$

Let $\delta > 0$ to be determined later and u be such that $(u, 0) \in \mathcal{V}(\delta)$. Recall that $\|u\|_{L^\infty} \leq K(E(Q) + \delta) < C^*$ (Lemma 2), and hence $g(u) \geq \omega(E(Q) + \delta)|u|$.

Assume first $\|u\|_{L^\infty} > D^*$. Then let A_1, A_2 such that $u \in [-D^*, D^*]$ on both intervals $[0, A_1]$ and $[A_2, \infty)$ and $|u(A_1)| = |u(A_2)| = D^*$. Then

$$\int \left(u_r^2 + \frac{d(u)}{r^2} \right) r dr = \int_0^{A_1} + \int_{A_1}^{A_2} + \int_{A_2}^{\infty} \geq 4A + \int_{A_1}^{A_2} \left(u_r^2 + \frac{d(u)}{r^2} \right) r dr$$

Doing the same with the energy density, one gets

$$\int_0^{A_1} \left(u_r^2 + \frac{g^2(u)}{r^2}\right) rdr + \int_{A_2}^{\infty} \left(u_r^2 + \frac{g^2(u)}{r^2}\right) rdr \geq 4G(D^*) = 2G(C^*) = E(Q).$$

Hence $E_{A_1}^{A_2}(u) < \delta$. Now, we have

$$|d(u)| = |u||g'(u)||g(u)| \leq \|g'\|_{L^\infty}|u|g(u) \leq \frac{\|g'\|_{L^\infty}}{\omega(E(Q)+\delta)} g^2(u),$$

so that

$$\int_{A_1}^{A_2} \left(u_r^2 + \frac{d(u)}{r^2}\right) rdr \geq \int_{A_1}^{A_2} \left(u_r^2 - \frac{\|g'\|_{L^\infty}}{\omega(E(Q)+\delta)} \frac{g^2(u)}{r^2}\right) rdr \geq -\frac{\|g'\|_{L^\infty}}{\omega(E(Q)+\delta)}\delta.$$

Finally, choosing $\delta > 0$ small enough so that $\frac{\|g'\|_{L^\infty}}{\omega(E(Q)+\delta)}\delta \leq 2A$, we get

$$\int \left(u_r^2 + \frac{d(u)}{r^2}\right) rdr \geq 4A - \frac{\|g'\|_{L^\infty}}{\omega(E(Q)+\delta)}\delta \geq 2A \geq \frac{A}{E(Q)}E(u).$$

This gives the lower bound with constant $\frac{A}{E(Q)}$.

Assume now that $\|u\|_{L^\infty} \leq D^*$. Then $d(u) \geq 0$. As $f(x) \sim k^2 x$ as $x \to 0$, let $D > 0$ be such that $|f| \geq k^2/2x$ on the interval $[-D, D]$. If $\|u\|_{L^\infty} \leq D$, then of course

$$F(u) \geq \int u_r^2 rdr + \frac{k^2}{2\|g'\|_{L^\infty}^2} \int \frac{g^2(u)}{r^2} rdr \geq \min\left(1, \frac{k^2}{2\|g'\|_{L^\infty}^2}\right) E(u).$$

Otherwise, arguing as before, $\|u\|_{L^\infty} \in [D, D^*]$ and we see that $F(u) \geq 4\int_0^D \sqrt{d}$ so that (as $E(u) < E(Q) + \delta \leq 2E(Q)$)

$$F(u) \geq \frac{2\int_0^D \sqrt{d}}{E(Q)} E(u).$$

Choosing $\delta > 0$ small enough and $c = \min(2(\int_0^D \sqrt{d})/E(Q), A/E(Q)k^2/(2\|g'\|_{L^\infty}^2), 1)$ ends the proof. □

Let φ be such that $\varphi(r) = 1$ if $r \leq 1$, $\varphi(r) = 0$ if $r \geq 2$, and $\varphi(r) \in [0, 1]$. Denote $\varphi_R(x) = \varphi(r/R)$.

In the notation $\mathcal{O}$, constants are absolute (do not depend on R or t or u).

Lemma 8. *Let $(u, u_t) \in \mathcal{V}(\delta)$ be a solution to (1). One has*

$$\frac{d}{dt}\int u_t u_r r^2 \varphi_R(r) dr = -\int u_t^2 rdr + \mathcal{O}(E_R^\infty(u, u_t)),$$
$$\frac{d}{dt}\int u u_t r \varphi_R(r) dr = \int \left(u_t^2 - u_r^2 - \frac{uf(u)}{r^2}\right) rdr + \mathcal{O}(E_R^\infty(u, u_t)).$$

***Remark* 4.** *For the $\mathcal{O}$, we can consider the rest of the energy E_R^∞ or equivalently the tail in $H \times L^2$*

$$\tau(R, u, u_t) = \int_R^\infty \left(u_t^2 + u_r^2 + \frac{u^2}{r^2}\right) rdr.$$

Proof. One computes

$$
\begin{aligned}
&\frac{d}{dt}\int u_t u_r r^2 \varphi_R(r) dr \\
&= \int u_{tt} u_r r^2 \varphi_R(r) dr + \int u_t u_{rt} r^2 \varphi_R(r) dr \\
&= \int \left(u_{rr} + \frac{1}{r} u_r - \frac{f(u)}{r^2} \right) u_r r^2 \varphi_R(r) dr - \frac{1}{2} \int u_t^2 (2r\varphi_R(r) + r^2 \varphi_R'(r)) dr \\
&= -\frac{1}{2} \int u_t^2 (2r\varphi_R(r) + r^2 \varphi_R'(r)) dr + \int u_r^2 (r\varphi_R(r) - \frac{1}{2}(r^2 \varphi_R(r))')) dr + \frac{1}{2} \int g^2(u) \varphi_R'(r) dr \\
&= -\int u_t^2 r \varphi_R(r) dr + \frac{1}{2} \int \left(u_t^2 - u_r^2 + \frac{g^2(u)}{r^2} \right) r^2 \varphi_R'(r) dr
\end{aligned}
$$

Now, notice that

$$
\begin{aligned}
&\left| -\int u_t^2 r(1-\varphi_R(r)) dr - \frac{1}{2} \int \left(u_t^2 - u_r^2 + \frac{g^2(u)}{r^2} \right) r^2 \varphi_R'(r) dr \right| \\
&\qquad \le \int e(u,u_t)(1-\varphi_R(r)) r dr + \int e(u,u_t) r^2 |\varphi_R'(r)| dr \\
&\qquad \le E_R^\infty(u,u_t) + \frac{1}{R} \int e(u) r^2 |\varphi'(r/R)| dr \\
&\qquad \le (1 + 2\|\varphi'\|_{L^\infty}) E_R^\infty(u,u_t).
\end{aligned}
$$

From this, we immediately deduce

$$
\frac{d}{dt}\int u_t u_r r^2 \varphi_R(r) dr = -\int u_t^2 r dr + \mathcal{O}(E_R^\infty(u,u_t)).
$$

In the same way,

$$
\begin{aligned}
\frac{d}{dt}\int u u_t r \varphi_R(r) dr &= \int u_t^2 r \varphi_R(r) dr + \int u u_{tt} r \varphi_R(r) dr \\
&= \int u_t^2 r \varphi_R(r) dr + \int u \left(u_{rr} + \frac{1}{r} u_r - \frac{f(u)}{r^2} \right) r \varphi_R(r) dr \\
&= \int \left(u_t^2 - u_r^2 - \frac{u f(u)}{r^2} \right) r \varphi_R(r) dr + \frac{1}{2} \int u^2 (r\varphi_R(r))'' dr \\
&\quad - \frac{1}{2} \int u^2 \varphi_R'(r) dr.
\end{aligned}
$$

Then similarly

$$
\begin{aligned}
&\left| \int \left(u_t^2 - u_r^2 - \frac{u f(u)}{r^2} \right) r(1-\varphi_R(r)) dr + \frac{1}{2} \int u^2 (r\varphi_R(r))'' dr - \frac{1}{2} \int u^2 \varphi_R'(r) dr \right| \\
&\quad \le \int \left| u_t^2 - u_r^2 - \frac{u f(u)}{r^2} \right| (1-\varphi_R(r)) r dr + \frac{1}{2} \int \frac{u^2}{r^2} |r^2 \varphi_R''(r) + r\varphi_R'(r)| r dr \\
&\quad \le C \int e(u,u_t)(1-\varphi_R(r)) r dr + C \int \frac{g^2(u)}{r^2} \left| \frac{r^2}{R^2} \varphi''(r/R) - \frac{r}{R} \varphi'(r/R) \right| r dr \\
&\quad \le C E_R^\infty(u,u_t) + C(4\|\varphi''\|_{L^\infty} + 2\|\varphi'\|_{L^\infty}) E_R^\infty(u,u_t).
\end{aligned}
$$

(The bounds on the third line come respectively from the pointwise bounds $|uf(u)| \le Cg^2(u)$ and $u^2 \le Cg^2(u)$, which hold according to the proof of Lemma 7). □

Theorem 3 (Rigidity property). *Let $(u_0, u_1) \in \mathcal{V}(\delta)$, and denote by $u(t)$ the associated solution. Suppose that for all $t \geq 0$, there exist $\lambda(t) \geq A_0 > 0$ such that*

$$K = \left\{ \left(u\left(t, \frac{r}{\lambda(t)}\right), \frac{1}{\lambda(t)} u_t\left(t, \frac{r}{\lambda(t)}\right) \right) \middle| (t, r) \in \mathbb{R}_+ \right\} \text{ is precompact in } H \times L^2.$$

Then $u \equiv 0$.

Proof. Recall that u is global due to Proposition 1.

As K is precompact and $\lambda(t) \geq A_0 > 0$, for all $\varepsilon > 0$, there exists $R(\varepsilon)$ such that

$$\forall t \geq 0, \quad E^{\infty}_{R(\varepsilon)}(u, u_t) < \varepsilon.$$

This means that

$$\lim_{R \to \infty} \sup_{t \geq 0} E^{\infty}_{R}(u, u_t) = 0.$$

Due to Lemma 8 and 7, we have

$$\begin{aligned} \frac{d}{dt}\left(\int u_t u_r r^2 \varphi_R(r) dr + \frac{1}{2} \int u u_t \varphi_R(r) dr \right) &= -\frac{1}{2} \int \left(u_t^2 + u_r^2 + \frac{u f(u)}{r^2} \right) r dr + \mathcal{O}(E^{\infty}_R(u, u_t)) \\ &\leq -\frac{c}{2} E(u, u_t) + \mathcal{O}(E^{\infty}_R(u, u_t)). \end{aligned}$$

Fix R large enough so that $\sup_{t \geq 0} \mathcal{O}(E^{\infty}_R(u, u_t)) \leq \frac{cE(u,u_t)}{4}$. Then by integration between $\tau = 0$ and $\tau = t$ and conservation of energy :

$$\int u_t u_r r^2 \varphi_R(r) dr + \frac{1}{2} \int u u_t r \varphi_R(r) dr \leq -\frac{c}{4} E(u, u_t) t + C_0.$$

However, from finiteness of energy and $u^2 \leq C g^2(u)$, we have for all t,

$$\begin{aligned} &\left| \int u_t u_r r^2 \varphi_R(r) dr + \frac{1}{2} \int u u_t r \varphi_R(r) dr \right| \\ &\leq \frac{1}{2} \int (u_t^2 + u_r^2) r^2 \varphi_R(r) dr + \frac{1}{4} \int \left(u_t^2 + C \frac{g^2(u)}{r^2} \right) r^2 \varphi_R(r) \\ &\leq R E(u, u_t) + \frac{1}{2C} R E(u, u_t), \end{aligned}$$

so that this quantity is bounded, hence $t \leq 4(R + R/(2C) + C_0)/c$. This is a contradiction with the fact that u is global in time. $\square$

5 Proofs of Theorem 1 and Corollary 1

Proof of Theorem 1. The proof follows the general framework of Kenig and Merle [6, 5]. For a detailed exposition of the various steps and lemmas, we refer to [6, Section 4].

Let $\delta \in (0, E(Q)]$ as in Lemma 7. All the wave maps considered below in the proof will have initial data in $\mathcal{V}(\delta)$; from Proposition 1, they are all defined globally in time. Hence we are left to show that all wave maps u with initial data $(u_0, u_1) \in \mathcal{V}(\delta)$ scatter at $t \to \pm\infty$. From Theorem 2, we only need to show that $\|u\|_{S(\mathbb{R})} < \infty$.

We consider the critical energy

$$E_c = \sup\{E \in [0, E(Q) + \delta] | \ \forall (u_0, u_1) \in \mathcal{V}(\delta), \ E(u_0, u_1) < E \Longrightarrow \|u(t)\|_{S(\mathbb{R})} < \infty\}.$$

Theorem 1 is the assertion

$$E_c = E(Q) + \delta.$$

Assume this is not the case, namely $E_c < E(Q) + \delta$, and we will reach a contradiction ; this will complete the proof of Theorem 1. Due to Theorem 2, notice that

$$E_c \geq \delta_0 \stackrel{\text{def}}{=} \delta_0(E(Q) + \delta) > 0. \tag{17}$$

The compensated compactness procedure of Kenig and Merle in [5] provides us with a critical element u^c (in the case $E_c < E(Q) + \delta$) :

Proposition 2. *There exists $(u_0^c, u_1^c) \in H \times L^2$, satisfying $(u_0^c, u_1^c) \in \mathcal{V}(\delta)$, $E(u_0^c, u_1^c) = E_c$ and if we denote $u^c(t)$ the associated solution to Problem (1), $u^c(t)$ is global and $\|u^c\|_{S(\mathbb{R})} = \|u^c\|_{S(\mathbb{R}^+)} = \infty$.*

Moreover, a critical element enjoys the following properties :

Proposition 3. *Let u^c be as in Proposition 2. Then there exist a continuous function $\lambda : \mathbb{R}^+ \to \mathbb{R}^+_*$ such that the set*

$$K = \left\{ v(t) \in H \times L^2 \middle| v(t,r) = \left(u^c\left(t, \frac{r}{\lambda(t)}\right), \frac{1}{\lambda(t)} u^c_t\left(t, \frac{r}{\lambda(t)}\right)\right)\right\}$$

has compact closure in $H \times L^2$.

Up to considering a different critical element, we can furhermore assume that $\lambda(t) \geq A_0$ for all $t \geq 0$, for some $A_0 > 0$.

We thus consider u^c given by Propositions 2 and 3. From Theorem 3, we deduce that $(u^c, u^c_t) = (0,0)$, which is a contradiction with $E(u^c, u^c_t) = E_c > 0$ (in view of (17)). Hence $E_c = E(Q) + \delta$. This completes the proof of Theorem 1.

For the convenience of the reader, we sketch the proof of Proposition 2 and 3 ; a complete proof can be derived (with minor modifications) from Proposition 4.1, 4.2 and Lemma 4.9 of [6, Section 4].

We consider a sequence of (global in time) wave maps u_n and their initial data $(u_{0n}, u_{1n}) \in \mathcal{V}(\delta)$ such that $\|W(t)(u_{0n}, u_{1n})\|_{S(\mathbb{R})} \geq \delta_0$, $\|u_n\|_{S(I_n)} = \infty$ and $E(u_n, u_{nt}) \to E_c$ (hence $E_c \leq E(u_n, u_{nt}) < E(Q) + \delta$).

Using the result by Bahouri and Gerard [1] on the operator $W(t)$, we apply the (linear) profile decomposition to the sequence $(u_{0n}, u_{1n})_{n\geq 1}$:

Lemma 9 (Profile decomposition)**.** *Let $(u_{0n}, u_{1n})_n$ be a bounded sequence of $H \times L^2$. Then there exist sequences $(V_{0,j}, V_{1,j})_{j\geq 1} \in H \times L^2$ and $(\lambda_{j,n}, t_{j,n}) \in \mathbb{R}^+_* \times \mathbb{R}$ with*

$$\frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{|t_{j,n} - t_{j',n}|}{\lambda_{j,n}} \to \infty \quad \textit{as} \quad n \to \infty,$$

for $j \neq j'$ (orthogonal couple), such that the following holds. Denote $V_j(t) = W(t)(V_{0,j}, V_{1,j})$ the linear profiles, then for all $J \geq 1$, there exist $(w_{0n}^J, w_{1n}^J) \in H \times L^2$ such that (up to a subsequence of (u_n), which we still denote (u_n))

$$u_{0n}(r) = \sum_{j=1}^{J} V_j\left(-\frac{t_{j,n}}{\lambda_{j,n}}, \frac{r}{\lambda_{j,n}}\right) + w_{0n}(r)$$

$$u_{1n}(t) = \sum_{j=1}^{J} \frac{1}{\lambda_{j,n}} V_{jt}\left(-\frac{t_{j,n}}{\lambda_{j,n}}, \frac{r}{\lambda_{j,n}}\right) + w_{1n}(r)$$

with

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|W(t)(w_{0n}^J, w_{1n}^J)\|_{S(\mathbb{R})} = 0,$$

$$\forall J \geq 1, \quad E(u_{0n}, u_{1n}) = \sum_{j=1}^{J} E\left(V_j\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right), V_{jt}\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right) + E(w_{0n}^J, w_{1n}^J) + o_{n\to\infty}(1).$$

(Notice there is no shift in the space variable r as we are in a radial setting).

Then one can prove the following technical lemma. First recall the notion of non-linear profile : given data $(V_0, V_1) \in H \times L^2$ and a sequence (s_n), with $s_n \to \bar{s} \in \overline{\mathbb{R}}$, it is the unique wave map U defined on a neighbourhood of $\bar{s}$ such that $\|(U(s_n), U_t(s_n)) - W(s_n)(V_0, V_1)\|_{H\times L^2} \to 0$ as $n \to \infty$. U exists in virtue of Theorem 2, possibly doing a fixed point at infinity. If $(V_0, V_1) \in \mathcal{V}(\delta)$, U is global in time due to Proposition 1.

Lemma 10. *Let $(u_{0n}, u_{1n}) \in H\times L^2$ be such that $E(u_{0n}, u_{1n}) \to E_c$ and $\|W(t)(u_{0n}, u_{1n})\|_{S(\mathbb{R})} \geq \delta_0$. Let $(V_j)_{j\geq 1}$ and $\lambda_{j,n}, t_{j,n}$ be as in Lemma 9. Assume that one the following conditions holds (denote $s_n = -t_{1,n}/\lambda_{1,n}$) :*

- $\liminf_{n\to\infty} E(V_1(s_n), \lambda_{1,n}^{-1} V_{1t}(s_n)) < E_c$, *or*
- $\liminf_{n\to\infty} E(V_1(s_n), \lambda_{1,n}^{-1} V_{1t}(s_n)) = E_c$ *and that after passing to a subsequence so that $s_n \to \bar{s} \in \overline{\mathbb{R}}$ and $E(V_1(s_n), \lambda_{1,n}^{-1} V_{1t}(s_n)) \to E_c$, if U_1 is the non linear profile associated to $(V_{0,1}, V_{1,1})$ and (s_n), then U_1 is global and $\|U_1\|_{S(\mathbb{R})} < \infty$.*

Denote u_n the wave map with initial data (u_{0n}, u_{1n}). Then after passing to a subsequence, u_n is global and $\|u_n\|_{S(\mathbb{R})} < \infty$.

The proof of this lemma relies on the profile decomposition and a perturbation result which is a by-product of Theorem 2. We refer to [6, Lemma 4.9] for further details.

From this, one can prove that the all profiles $(V_{0,j}, V_{1,j})$ associated to $(u_{0n}, u_{1n})_n$ are zero, except for (exactly) one, say $(V_{0,1}, V_{1,1})$ and that $E(V_{0,1}, V_{1,1}) = E_c$. Then denote $s_n = -\frac{t_{1,n}}{\lambda_{1,n}}$ and consider the non-linear profile associated to $(V_{0,1}, V_{1,1})$ and (s_n) (up to a subsequence such that s_n has a limit in $\overline{\mathbb{R}}$). Then one can prove that U is global and $\|U\|_{S(\mathbb{R})} = +\infty$: $U(t)$ or $U(-t)$ satisfies the conclusion of Proposition 2.

We now turn to Proposition 3 ; for the compactness result, we argue by contradiction. Assume that there exists $\eta_0 > 0$ and a sequence $(t_n)_{n\geq 1}$ such that for all $\lambda > 0$, $n \neq n'$,

$$\left\| \left(U\left(t_n, \frac{r}{\lambda}\right), \frac{1}{\lambda} U_t\left(t_n, \frac{r}{\lambda}\right)\right) - (U(t_{n'}, r), U_t(t_{n'}, r)) \right\|_{H\times L^2} \geq \eta_0. \tag{18}$$

Up to considering a subsequence, we can assume that t_n has a limit in $[0, \infty]$; by continuity of the flow, $t_n \to \infty$. Now consider the profile decomposition of the sequence $(U(t_n), U_t(t_n))$. Again using Lemma 10, one can prove that all profiles are zero, except for one. Then one can obtain for this profile a statement similar to (18), and from there, reach a contradiction. Hence there exists $\lambda : \mathbb{R}^+ \to \mathbb{R}^+_*$ such that the set

$$\widetilde{K} = \left\{ v(t) \in H \times L^2 \middle| v(t,r) = \left(U\left(t, \frac{r}{\lambda(t)}\right), \frac{1}{\lambda(t)} U_t\left(t, \frac{r}{\lambda(t)}\right)\right) \right\}$$

has compact closure in $H \times L^2$. It remains to prove that, up to changing the critical element U, one can further assume $\lambda(t) \geq A_0 > 0$. Indeed, if it is not the case for U, by compactness, there exist $\lambda_n \to 0$ and $t_n \to \infty$ such that $U(t_n, \frac{r}{\lambda_n}), U_t(t_n, \frac{r}{\lambda_n}) \to (W_0, W_1)$ in $H \times L^2$. Then one can prove that the wave map u^c with initial data (W_0, W_1) satisfies all the properties of Propositions 2 and 3. □

Proof of Corollary 1. Notice that if (u_0, u_1) is such that $E(u_0, u_1) \leq E(Q)$ and $(u_0, u_1) \notin \mathcal{V}(\delta)$, then (as $u_0(0) = 0$), $|u_0(\infty)| \geq C^*$, and from the pointwise inequality (5), $|u_0(\infty)| = C^*$, $u_0(r) = \varepsilon Q(\lambda r)$ for some $\lambda > 0$ and $\varepsilon \in \{-1, 1\}$, and $u_1 = 0$.

Hence in our case, $(u_0, u_1) \in \mathcal{V}(\delta)$, and the result follows from Theorem 1. □

References

[1] Hajer Bahouri and Patrick Gérard, *High frequency approximation of solutions to critical nonlinear wave equations*, Amer. J. Math. **121** (1999), 131–175.

[2] Thierry Cazenave, Jalal Shatah, and A. Shadi Tahvildar-Zadeh, *Harmonic maps of the hyperbolic space and development of singularities in wave maps and Yang-Mills fields*, Ann. Inst. H. Poincaré Phys. Théor. **68** (1998), no. 3, 315–349. MR MR1622539 (2000g:58042)

[3] Raphaël Côte, *Instability of harmonic maps for the (1+2)-dimensional equivariant Wave map system*, Int. Math. Res. Not. **57** (2005), 3525–3549.

[4] Patrick Gérard, *Description du défaut de compacité de l'injection de Sobolev*, ESAIM Control Optim. Calc. Var. **3** (1998), 213–233 (electronic).

[5] Carlos E. Kenig and Frank Merle, *Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear wave equation*, To appear, Acta. Math.

[6] ______, *Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case*, Invent. Math. **166** (2006), no. 3, 645–675. MR MR2257393 (2007g:35232)

[7] Carlos E. Kenig, Gustavo Ponce, and Luis Vega, *Well-posedness and scattering result for the generalized Korteweg-De Vries equation via contraction principle*, Comm. Pure Appl. Math. **46** (1993), 527–620.

[8] Joachim Krieger, Wilhelm Schlag, and Daniel Tataru, *Renormalization and blow up for charge one equivariant critical wave maps*, Preprint, arXiv:math.AP/0610248 (2006).

[9] Igor Rodnianski and Jacob Sterbenz, *On the formation of singularities in the critical O(3) sigma-model*, Preprint, arXiv:math.AP/0605023 (2006).

[10] Jalal Shatah and Michael Struwe, *Geometric wave equations*, Courant Lecture Notes in Mathematics, vol. 2, New York University Courant Institute of Mathematical Sciences, New York, 1998.

[11] Jalal Shatah and A. Shadi Tahvildar-Zadeh, *On the Cauchy problem for equivariant wave maps*, Comm. Pure Appl. Math. **47** (1994), no. 5, 719–754.

[12] Elias M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970. MR MR0290095 (44 #7280)

[13] Michael Struwe, *Equivariant wave maps in two space dimensions*, Comm. Pure Appl. Math. **56** (2003), no. 7, 815–823.

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