A HIGH-ORDER UNCONDITIONALLY STABLE RELAXATION SCHEME

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1 INTRODUCTION

MODELING. We deal with the modeling of **multi-scales physical phenomena** described by a nonlinear hyperbolic system

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = 0 \quad \text{with } \boldsymbol{U} \in \mathbb{R}^N \quad (1)$$

and especially want to design a numerical scheme in order to **capture the slowest scale** and to filter the others.

EXAMPLES. Multi-scales phenomena encompass low Mach number flows, low Froude number hydrodynamic motions, low β-limit in magneto-hydrodynamics, etc.

CLASSICAL NUMERICAL METHODS.

Explicit methods
✗ Restrictive stability conditions

Implicit methods
✗ High CPU and memory costs
✗ Ill-conditioned matrices

AIMS OF THE NEW SCHEME.
✓ Unconditionally stable
| large time steps associated to the slowest scale might be used
✓ High-order in time and space
✓ Without matrices inversion nor matrices storage
✓ With non-cartesian grids

2 KINETIC REPRESENTATION

2.1 The kinetic BGK model

PREREQUISITE. At a mesoscopic scale, the **Boltzmann equation** on the particules distribution function f models gas dynamics

$$\underbrace{\partial_t f(t,x,v) + v\partial_x f(t,x,v)}_{\text{transport part}} = \underbrace{Q(f)}_{\text{collision part}},$$

with $Q(f)$ a collision operator often chosen of BGK type $Q(f) = \frac{1}{\varepsilon}(f^{eq} - f)$.
| The collision is thus a **relaxation towards an equilibrium state** f^{eq}.

KEY POINT [AN99]. As a generalisation, the hyperbolic system (1) is approximated by a **kinetic BGK model** (2)

$$\partial_t f(t,x,v) + v\partial_x f(t,x,v) = \frac{1}{\varepsilon}(f^{eq} - f). \quad (2)$$

Under some consistency conditions on the moments of f^{eq}, kinetic BGK model (2) converges asymptotically to the hyperbolic system (1).

2.2 Starting point : the Lattice-Boltzmann method

IDEA. It is a discrete version of the Boltzmann continuous equation.

DISCRETIZATION : D$_1$Q$_d$.

Discrete framework
✓ Finite set of velocities
$\mathcal{V} = \{\lambda_1, ..., \lambda_d\}$
✓ Distribution vector $\boldsymbol{f}$
| such that $f_j(t,x) = f(t,x,\lambda_j)$
✗ Space lattice $x \in \mathcal{L}$
✗ Time step Δt
| such that $x + \lambda_i \Delta t \in \mathcal{L}$, $\forall x \in \mathcal{L}$
✓ Choice of a discrete $\boldsymbol{f}^{eq}$
| submitted to discrete consistency
| conditions on its moments

Splitting algorithm
✓ collision step :
$f_j^*(t,x) = f_j(t,x) + \omega(f_j^{eq} - f_j(t,x))$
with $\omega \in [0,2]$ the relaxation parameter

✗ exact transport :
$f_j(t+\Delta t, x) = f_j^*(t, x - \lambda_j \Delta t)$

Remark. ✗ Particles are required to move exactly on the lattice
| which imposes a CFL type condition and a cartesian grid.
✓ Hereinafter, we want to design a scheme with only ✓ items
| to **get rid of** the previous restrictions.

VECTORIAL SCHEME. We will focus on a specific kinetic scheme called "vectorial" which consists of repeating a 1D representation for each component of $\boldsymbol{U}$: $[\mathrm{D}_1\mathrm{Q}_d]^N$.

3 NUMERICAL SCHEME

VELOCITIES. Add a **central/zero velocity** to mimic and to better treat the steady or quasi steady scale of the hyperbolic system (1).

$$\mathcal{V} = \{\lambda_-, \lambda_0, \lambda_+\}^N. \quad (3)$$

EQUILIBRIUM [Bou03]. Suppose that we could split $\boldsymbol{F}(\boldsymbol{U}) - \lambda_0 \boldsymbol{U}$ into two parts (abusively called "positive" and "negative" parts)

the flux vector splitting : $\boldsymbol{F}(\boldsymbol{U}) - \lambda_0 \boldsymbol{U} = \boldsymbol{F}_0^+(\boldsymbol{U}) + \boldsymbol{F}_0^-(\boldsymbol{U})$.

The consistency conditions on the moments of $\boldsymbol{f}^{eq}$ together with this flux vector splitting enable to construct the equilibrium $\boldsymbol{f}^{eq}$:

$$\begin{cases} \boldsymbol{f}_-^{eq}(\boldsymbol{U}) = -\dfrac{1}{(\lambda_0 - \lambda_-)}\boldsymbol{F}_0^-(\boldsymbol{U}), \\ \boldsymbol{f}_0^{eq}(\boldsymbol{U}) = \boldsymbol{U} - \left(\dfrac{\boldsymbol{F}_0^+(\boldsymbol{U})}{(\lambda_+ - \lambda_0)} - \dfrac{\boldsymbol{F}_0^-(\boldsymbol{U})}{(\lambda_0 - \lambda_-)}\right), \\ \boldsymbol{f}_+^{eq}(\boldsymbol{U}) = \dfrac{1}{(\lambda_+ - \lambda_0)}\boldsymbol{F}_0^+(\boldsymbol{U}). \end{cases} \quad (4)$$

SPLITTING ALGORITHM $[\mathbf{D}_1\mathbf{Q}_3]^N$. First order in time: $T(\Delta t) \circ R(\Delta t, 1)$ or second order in time: $T(\frac{\Delta t}{2}) \circ R(\Delta t, \frac{1}{2}) \circ T(\frac{\Delta t}{2})$ with

✓ **Semi-Lagrangian transport** step: $T(\Delta t)$

$$f_i^*(t,x) = \mathbb{I}_{\Delta x}\left(f_i^n(t, x - \lambda_i \Delta t)\right)$$

| with $\mathbb{I}_{\Delta x}$ the interpolation operator associated to the semi-lagrangian method

✓ **θ-scheme relaxation** step: $R(\Delta t, \theta)$

$$\frac{\boldsymbol{f}^{n+1} - \boldsymbol{f}^*}{\Delta t} = \theta \frac{\boldsymbol{f}^{eq} - \boldsymbol{f}^{n+1}}{\varepsilon} + (1-\theta)\frac{\boldsymbol{f}^{eq} - \boldsymbol{f}^*}{\varepsilon}$$

| rewritten as $\boldsymbol{f}^{n+1}(t,x) = \boldsymbol{f}^*(t,x) + \omega(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^*(t,x))$, with $\omega = \frac{\Delta t}{\varepsilon + \theta \Delta t}$

4 PROPERTIES

CONSISTENCY LEMMA. The numerical scheme $[\mathrm{D}_1\mathrm{Q}_3]^N$ with velocities (3) and equilibrium (4) admits the following PDE limit

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \left(\frac{1}{\omega} - \frac{1}{2}\right)\Delta t \; \partial_x\left(D(\boldsymbol{U})\partial_x \boldsymbol{U}\right) + O(\Delta t^2)$$

with the following diffusion term

$$D(\boldsymbol{U}) = \lambda_+ \partial \boldsymbol{F}_0^+(\boldsymbol{U}) + \lambda_- \partial \boldsymbol{F}_0^-(\boldsymbol{U}) + \lambda_0 \partial \boldsymbol{F}(\boldsymbol{U}) - |\, \partial \boldsymbol{F}(\boldsymbol{U}) \,|^2 .$$

✓ The diffusion depends on $\boldsymbol{U}$
| less diffusive than with 2 velocities

✓ More parameters to tune.

STABILITY LEMMA. The numerical scheme $[\mathrm{D}_1\mathrm{Q}_3]^N$ with velocities (3) and equilibrium (4) is

✓ **entropy stable** [Dub13] (for the entropy η) if
- the transport is exact
- $\omega \in [0,1]$
- $\partial \boldsymbol{f}_-^{eq}$, $\partial \boldsymbol{f}_0^{eq}$, $\partial \boldsymbol{f}_+^{eq}$ are nonnegative matrices
- the flux vector splitting is entropic
| existence of a entropy flux $\zeta_0^{\pm}$ w.r.t η

✓ **linearly stable** (for a linear flux $\boldsymbol{F}$) if
- $\omega \in [0,1]$
| and even $\omega \in [0,2]$
- $\partial \boldsymbol{f}_-^{eq}$, $\partial \boldsymbol{f}_0^{eq}$, $\partial \boldsymbol{f}_+^{eq}$ are nonnegative matrices

References

[AN99] D.Aregba-Driollet and R.Natalini. *Discrete Kinetic Schemes for Systems of Conservation Laws*. Birkhäuser Basel, 1999.

[Bou03] F.Bouchut. Entropy satisfying flux vector splittings and kinetic BGK models. *Numer. Math.*, 94:623–672, 2003.

[Dub13] F.Dubois. Stable lattice Boltzmann schemes with a dual entropy approach for monodimensional nonlinear waves. *Computers and Mathematics with Applications*, 65(2):142–159, 2013.

[LS93] M.-S.Liou and C.J.Steffen. A New Flux Splitting Scheme. *Journal of Computational Physics*, 107(1):23–39, 1993.

5 NUMERICAL RESULTS

FLUX VECTOR SPLITTINGS. (tested on all examples)
✓ Rusanov splitting ($\lambda_- < \lambda_0 = 0 < \lambda_+$)

$$\boldsymbol{F}_0^-(\boldsymbol{U}) = -\lambda_- \frac{\boldsymbol{F}(\boldsymbol{U}) - \lambda_+ \boldsymbol{U}}{\lambda_+ - \lambda_-}, \quad \boldsymbol{F}_0^+(\boldsymbol{U}) = \lambda_+ \frac{\boldsymbol{F}(\boldsymbol{U}) - \lambda_- \boldsymbol{U}}{\lambda_+ - \lambda_-}$$

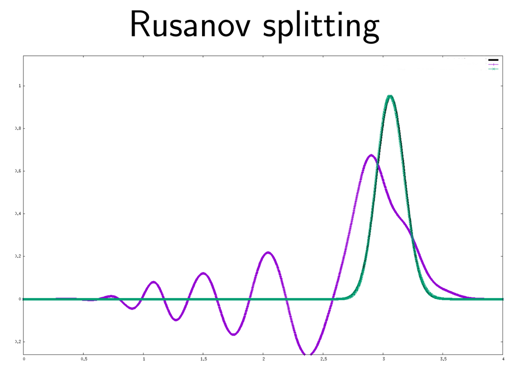
✓ Upwind splitting ($\lambda_- < \lambda_0 < \lambda_+$)

$$F_0^-(u) = \mathbb{1}_{F'(u)<\lambda_0}(F(u) - \lambda_0 u), \quad F_0^+(u) = \mathbb{1}_{F'(u)>\lambda_0}(F(u) - \lambda_0 u)$$

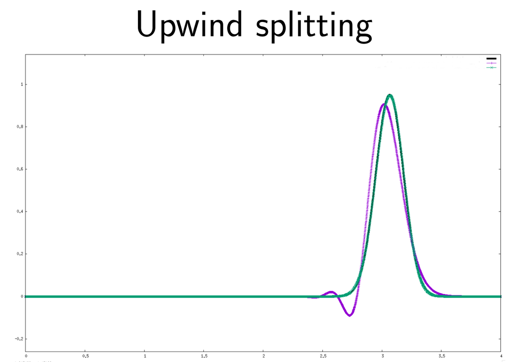
5.1 Advection equation : $\partial_t u + \partial_x(a(x)u) = 0$

Parameters. $a(x) = 1.0 + 0.01x^2$, orders : 2 in time and 17 in space, $\Delta x = 4.0.10^{-4}$ and $\Delta t = 0.1$

Rusanov splitting



Upwind splitting



$\mathcal{V} = \{-1.5, 0, 1.5\}$ and $\mathcal{V} = \{0.5, 1.0, 1.5\}$

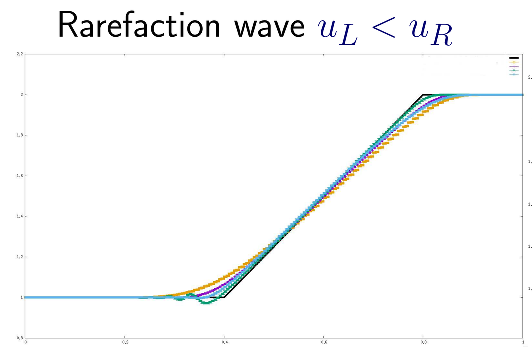
5.2 Burgers equation : $\partial_t u + \partial_x\left(\frac{u^2}{2}\right) = 0$

Other splitting. Lax-Wendroff splitting ($-\lambda_- = \lambda_+ = \lambda$, $\lambda_0 = 0$)

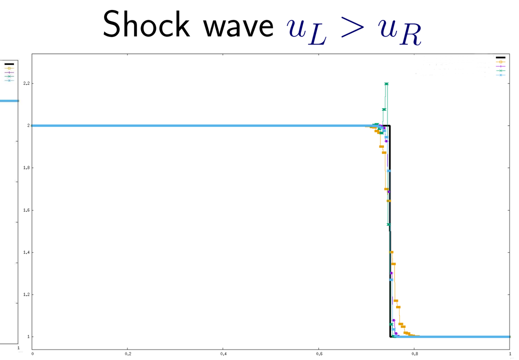
$$F_0^-(u) = \frac{u^2}{4} - \alpha\frac{u^3}{6\lambda}, \quad F_0^+(u) = \frac{u^2}{4} + \alpha\frac{u^3}{6\lambda}$$

Parameters. $\Delta x = 5.0.10^{-4}$ and $\Delta t = 2.0.10^{-3}$ (CFL$_{\text{finite volume}}$ $= 10$), orders : 1 in time and 11 in space

Rarefaction wave $u_L < u_R$



Shock wave $u_L > u_R$



Reference, Rusanov, Upwind, Lax-Wendroff(α=1), Lax-Wendroff(α=1.5)

5.3 Low Mach Euler system

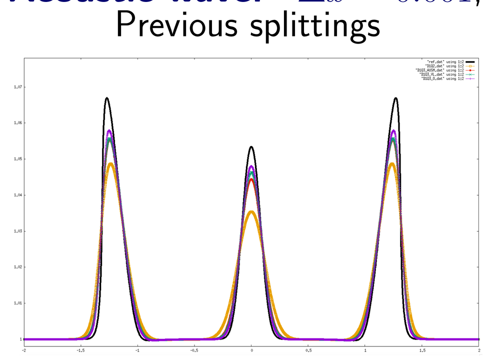
Other splittings. Lax-Wendroff splitting, AUSM splitting [LS93] (Advection Upstream Splitting Method), Van-Leer splitting, etc.

OUR "LOW MACH" SPLITTING.

$$\boldsymbol{F}_{\text{Euler}}(\boldsymbol{U}) = \underbrace{\boldsymbol{F}_{\text{fluid}}(\boldsymbol{U})}_{\text{Lax-Wendroff splitting}} + \underbrace{\boldsymbol{F}_{\text{acoustic}}(\boldsymbol{U})}_{\sim \text{ AUSM splitting}}$$

Acoustic wave. $\Delta x = 0.001$, orders : 1 in time and 11 in space

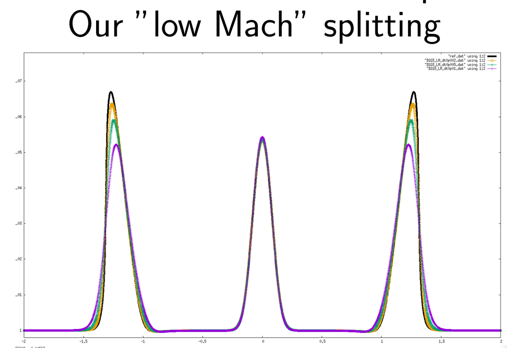
Previous splittings



Reference, Rusanov, Van-Leer, Osher, AUSM

$\Delta t = 0.002$

Our "low Mach" splitting

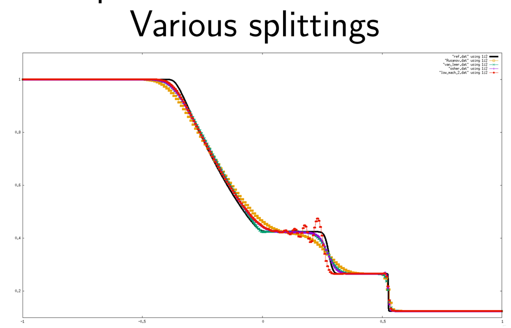


Reference, $\Delta t = 0.002$, $\Delta t = 0.005$, $\Delta t = 0.01$

✓ Material wave: well captured,
✓ Acoustic wave: less dissipated
✓ Same results with Δt five times larger than previously

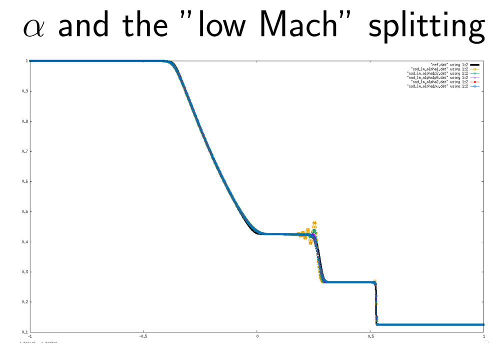
Sod problem. $\Delta x = 5.0.10^{-4}$ and $\Delta t = 0.002$, orders : 2 in time and 1 in space

Various splittings



Reference, Rusanov, Van-Leer, Osher, "low Mach" ($\alpha = 1$)

α and the "low Mach" splitting



Reference, $\alpha = 1$, $\alpha = 1.2$, $\alpha = 1.5$, $\alpha = 2$, $\alpha = 1 + u$

✓ α adds some numerical diffusion

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