ERROR ESTIMATES OF A FINITE DIFFERENCE SCHEME FOR THE KORTEWEG-DE VRIES EQUATION

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1 INTRODUCTION

EQUATION. We focus on the numerical analysis of the initial value problem on **Korteweg-de Vries equation**

$$\begin{cases} \partial_t u(t,x) + u(t,x)\partial_x u(t,x) + \partial_x^3 u(t,x) = 0, \text{ in } [0,T]\times\mathbb{R}, \\ u_{|t=0} = u_0, \text{ in } \mathbb{R}. \end{cases} \quad (1)$$

AIM. We want to **quantify the rate of convergence** by a **unified method** which takes into account the **two antagonist effects** : the formation of a shock wave due to the Burgers non-linear term $u\partial_x u$ and the dispersive oscillation wave due to the linear Airy term $\partial_x^3 u$.

NUMERICAL SCHEME. We study the general class of **Rusanov θ-finite difference scheme**

$$\frac{v_j^{n+1}-v_j^n}{\Delta t} + D\left(\frac{v^2}{2}\right)_j^n + \theta D_+D_+D_-(v)_j^{n+1} + (1-\theta)D_+D_+D_-(v)_j^n = \frac{c^n\Delta x}{2}D_+D_-(v)_j^n,$$

with $c^n > 0$ and $\theta\in[0,1]$. Let $D_+(v)_j^n = \frac{v_{j+1}^n - v_j^n}{\Delta x}$, $D_-(v)_j^n = \frac{v_j^n - v_{j-1}^n}{\Delta x}$ and $D(v)_j^n = \frac{D_+(v)_j^n + D_-(v)_j^n}{2}$.

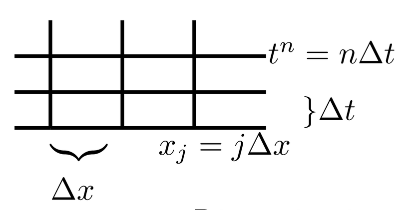

FIGURE : Discretisation

PARTICULAR CASE. $c^n\Delta t = \Delta x$ and $\theta = 1$ in [HKR15].

CONVERGENCE ERROR. We define

$$e_j^n = v_j^n - u_\Delta(t^n, x_j)$$

with $u_\Delta(t,x) = \frac{1}{\Delta x\Delta t}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}} u(s,y)dsdy.$

2 MAIN RESULTS

Theorem 1. [CLR16] : Assume $u_0\in\mathbb{H}^6(\mathbb{R})$ for the Sobolev regularity of the initial data, the Rusanov constant c^n be such as $||(u_\Delta)^n||_{\ell^\infty} < c^n$ and the Courant-Friedrichs-Lewy condition :

for $\theta\in[0,1]\setminus\left\{\frac{1}{2}\right\}$	for $\theta = \frac{1}{2}$,
$\Delta t(1-2\theta) < \frac{\Delta x^3}{4}$ and $c^n\Delta t \le \Delta x$,	$k^n\Delta t = \Delta x$, with $\frac{1}{k^n} < \frac{1}{c^n}$,

then there exists a constant Γ independent of Δt and Δx such as

$$\sup_{n\in[\![0,N]\!]} ||e^n||_{\ell^2_\Delta} \le \Gamma\Delta x. \quad (2)$$

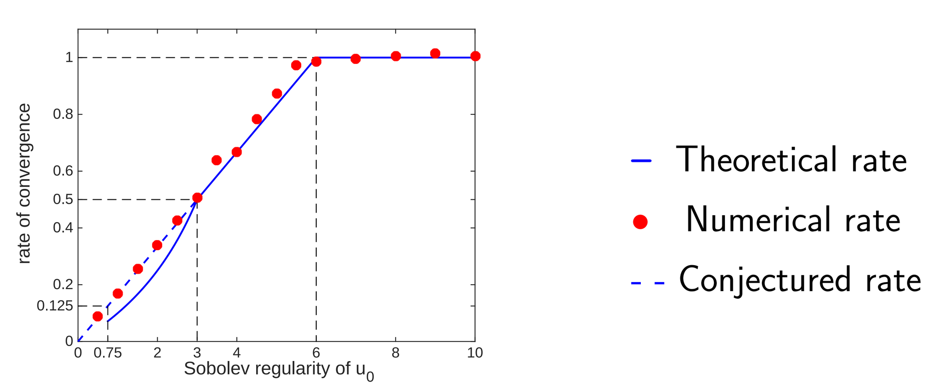

FIGURE : Rate of convergence

Theorem 2. [CLR16] : Assume $u_0\in\mathbb{H}^m(\mathbb{R})$ with $m > \frac{3}{4}$, the Rusanov constant c^n be such as $||(u_\Delta)^n||_{\ell^\infty} < c^n$ and the Courant-Friedrichs-Lewy condition

for $\theta\in[0,1]\setminus\left\{\frac{1}{2}\right\}$	for $\theta = \frac{1}{2}$,
$\Delta t(1-2\theta) < \frac{\Delta x^3}{4}$ and $c^n\Delta t \le \Delta x$,	$k^n\Delta t = \Delta x$, with $\frac{1}{k^n} < \frac{1}{c^n}$,

then there exist constants Γ_i independent of Δt and Δx such as

- if $3 \le m$
$$\sup_{n\in[\![0,N]\!]} ||e^n||_{\ell^2_\Delta} \le \Gamma_1\Delta x^{\frac{\min(m,6)}{6}}, \quad (3)$$
- if $\frac{3}{4} < m < 3$
$$\sup_{n\in[\![0,N]\!]} ||e^n||_{\ell^2_\Delta} \le \Gamma_2\Delta x^{\frac{m}{12-2m}}. \quad (4)$$

3 PROOFS

3.1 Consistency error

DEFINITION. The consistency error is defined as

$$\epsilon_j^n = \frac{(u_\Delta)_j^{n+1} - (u_\Delta)_j^n}{\Delta t} + D\left(\frac{u_\Delta^2}{2}\right)_j^n + \theta D_+D_+D_-(u_\Delta)_j^{n+1} + (1-\theta)D_+D_+D_-(u_\Delta)_j^n - \frac{c^n\Delta x}{2}D_+D_-(u_\Delta)_j^n.$$

Lemma 1. If $\Delta t \lesssim \Delta x$, $\eta > 0$ and $u_0\in\mathbb{H}^6(\mathbb{R})$, there exists a function h (depending on u_0 and T) which controls the consistency error

$$\sup_{n\in[\![0,N]\!]} ||\epsilon^n||_{\ell^2_\Delta} \le h(T, ||u_0||_{\mathbb{H}^{\frac{3}{4}+\eta}(\mathbb{R})}, ||u_0||_{\mathbb{H}^6(\mathbb{R})})\Delta x.$$

3.2 Stability

Property 1. The convergence error satisfies the following ℓ^2_Δ-stability inequality

$$||\mathcal{A}e^{n+1}|| \le ||\mathcal{A}e^n||\left[1 + C_1\Delta t + C_2\int_{t^n}^{t^{n+1}}||\partial_x u(s,.)||_{\mathbb{L}^\infty}ds\right] + ||e^n||\left|C_3\Delta t + C_4||D_+(e)^n||\right| + C_5||D_+D_+D_-(e)^n|| + C_6||D(e)^n|| + C_7||D_+D_-(e)^n|| + C_8||D_+D(e)^n||, \quad (5)$$

with $||.||$ the $||.||_{\ell^2_\Delta}$-norm, *i.e* $||e^n||^2_{\ell^2_\Delta} = \sum_{j\in\mathbb{Z}}\Delta x\left(e_j^n\right)^2$ and $\mathcal{A}$ the operator defined by $\mathcal{A} = I + \theta\Delta t D_+D_+D_-$ with I the identity operator.

REMARK. The constants C_i depend only on Δt, Δx and u_0. However, C_2 depends also on $||e^n||_{\ell^\infty}$.

Property 2. The CFL condition implies $C_{\{4,5,6,7,8\}} \le 0$.

3.3 Convergence

STEP 1. We suppose by induction the existence of $\gamma\in[0,\frac{1}{2}[$ such as $||e^n||_{\ell^\infty} \le \Delta x^{\frac{1}{2}-\gamma}$ in order to control C_2 in (5).

STEP 2. We **need to control** $\int_{t^n}^{t^{n+1}}||\partial_x u(s,.)||_{\mathbb{L}^\infty}ds$ in order to apply Grönwall lemma. In [KPV91], this term is upper bounded as soon as $u_0\in\mathbb{H}^{\frac{3}{4}+\eta}(\mathbb{R})$ with $\eta > 0$.

STEP 3. Grönwall lemma and the consistency error imply $||\mathcal{A}e^n||_{\ell^2_\Delta} \le \Gamma\Delta x$.

STEP 4. We need **return to** $||e^n||_{\ell^2_\Delta}$ thanks to $||e^n||_{\ell^2_\Delta} \le ||\mathcal{A}e^n||_{\ell^2_\Delta}$ and verify the induction hypothesis at rank $n+1$.

3.4 For a less smooth initial data

METHOD. We **regularize** the initial data thanks to a **convolution product with mollifiers** $\left(\varphi^\delta\right)_{\delta>0}$. Let us denote by

- u the exact solution from u_0,
- u^δ the exact one from $u_0^\delta = u_0\star\varphi^\delta$,
- $\left(v_j^n\right)_{(j,n)}$ the numerical solution from u_0^δ.

REMARK. Therefore, we use the triangle inequality to upper bound $||e^n||_{\ell^2_\Delta} \le ||u_\Delta - u_\Delta^\delta||_{\ell^2_\Delta} + ||u_\Delta^\delta - v^n||_{\ell^2_\Delta} := [\alpha] + [\beta]$.

Lemma 2. If $u_0\in\mathbb{H}^m$, with $m > \frac{3}{4}$, then there exists a function G such as $[\alpha] \le G\left(T, ||u_0||_{\mathbb{H}^{\frac{3}{4}+\eta}(\mathbb{R})}\right)\delta^m||u_0||_{\mathbb{H}^m(\mathbb{R})}$ with $\eta > 0$ such as $m \ge \frac{3}{4}+\eta$.

Lemma 3. If $m > \frac{3}{4}$ (cf. STEP 2) and $\frac{1}{\delta^{6-m}} \le \frac{1}{\Delta x^\gamma}$ (cf. STEP 4), then Theorem 1 implies $[\beta] \le \Gamma\frac{\Delta x}{\delta^{6-m}}$.

KEY POINT. We have to find the optimal δ such as

$$\begin{cases} \delta^m = \frac{\Delta x}{\delta^{6-m}}, \\ \text{under the constraint } \frac{1}{\delta^{6-m}} \le \frac{1}{\Delta x^\gamma}. \end{cases}$$

- If $\frac{3}{4} < m < 3$, the constraint is binding ($\delta = \Delta x^{\frac{\gamma}{6-m}}$),
- if $3 < m \le 6$, the optimal δ is $\delta = \Delta x^{\frac{1}{6}}$.

4 NUMERICAL RESULTS

DEFINITION. The numerical rate of convergence is computed with the relation $r = \frac{\log(E_J) - \log(E_{2J})}{\log(2)}$ with E_J the difference between a numerical solution with J spatial meshes and a numerical one with $2J$ spatial meshes.

Cnoidal-wave			
J	Δx	error in $\ell^\infty(0,T,\ell^2_\Delta(\mathbb{Z}))$	numerical order
1600	$6.2500.10^{-4}$	$8.9875.10^{-4}$	
3200	$3.1250.10^{-4}$	$4.5253.10^{-4}$	0.9899
6400	$1.5625.10^{-4}$	$2.2636.10^{-4}$	0.9994
12800	$7.8125.10^{-5}$	$1.1292.10^{-4}$	1.0034
25600	$3.9062.10^{-5}$	$5.7102.10^{-5}$	0.9837

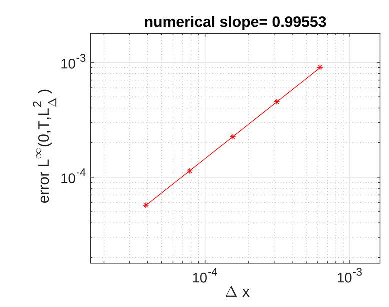

FIGURE : Numerical results for a **theoretical rate of 1**

$u_0\in\mathbb{H}^{\frac{9}{2}}([0,L])$			
J	Δx	error in $\ell^\infty(0,T,\ell^2_\Delta(\mathbb{Z}))$	numerical order
1600	$3.1250.10^{-2}$	$6.5105.10^{-3}$	
3200	$1.5625.10^{-2}$	$3.9541.10^{-3}$	0.71941
6400	$7.8125.10^{-3}$	$2.2620.10^{-3}$	0.80574
12800	$3.9063.10^{-3}$	$1.3091.10^{-3}$	0.78909
25600	$1.9531.10^{-3}$	$7.4923.10^{-4}$	0.80504

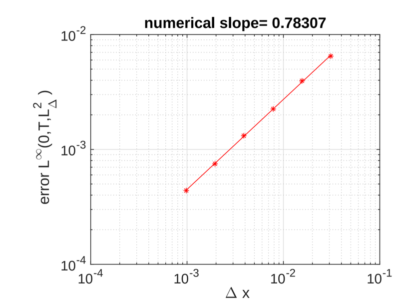

FIGURE : Numerical results for a **theoretical rate of 0.75**

$u_0\in\mathbb{H}^4([0,L])$			
J	Δx	error in $\ell^\infty(0,T,\ell^2_\Delta(\mathbb{Z}))$	numerical order
1600	$3.1250.10^{-2}$	$4.6454.10^{-3}$	
3200	$1.5625.10^{-2}$	$2.8109.10^{-3}$	0.72476
6400	$7.8125.10^{-3}$	$1.7147.10^{-3}$	0.71307
12800	$3.9063.10^{-3}$	$1.0892.10^{-3}$	0.65474
25600	$1.9531.10^{-3}$	$6.8793.10^{-4}$	0.66290

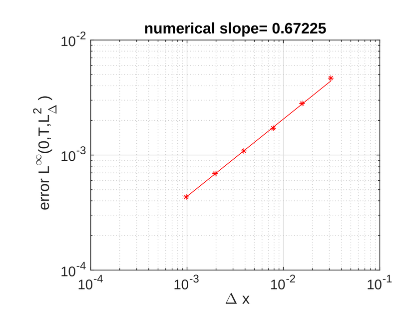

FIGURE : Numerical results for a **theoretical rate of 0.66667**

5 IMPROVEMENTS

CONJECTURED RATE 1. We suppose that the restriction $3 < m \le 6$ comes only from a computational difficulty (linked to our induction hypothesis) so that the rate $\frac{m}{6}$ **should be valid even for** $\frac{3}{4} < m \le 3$.

$u_0\in\mathbb{H}^2([0,L])$			
J	Δx	error in $\ell^\infty(0,T,\ell^2_\Delta(\mathbb{Z}))$	numerical order
1600	$3.125.10^{-2}$	$6.6322.10^{-3}$	
3200	$1.5625.10^{-2}$	$5.2115.10^{-3}$	0.34779
6400	$7.8125.10^{-3}$	$4.0950.10^{-3}$	0.34783
12800	$3.9063.10^{-3}$	$3.2699.10^{-3}$	0.32461
25600	$1.9531.10^{-3}$	$2.5937.10^{-3}$	0.33426

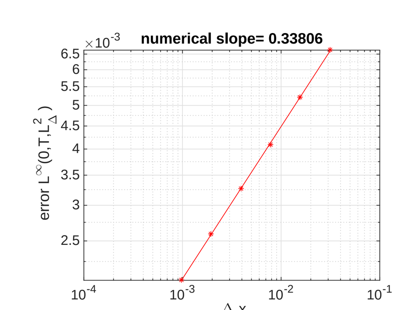

FIGURE : Numerical result for a **conjectured rate of 0.33333**

CONJECTURED RATE 2. In addition, we suppose that the **lower bound** $\frac{3}{4} < m$ **could be stamped out** by the use of [CKS+03] instead of [KPV91].

$u_0\in\mathbb{H}^{\frac{1}{2}}([0,L])$			
J	Δx	error in $\ell^\infty(0,T,\ell^2_\Delta(\mathbb{Z}))$	numerical order
3200	$1.5625.10^{-2}$	$1.0567.10^{-2}$	
6400	$7.8125.10^{-3}$	$9.8843.10^{-3}$	0.0964
12800	$3.9063.10^{-3}$	$9.2992.10^{-3}$	0.0880
25600	$1.9531.10^{-3}$	$8.7490.10^{-3}$	0.0879

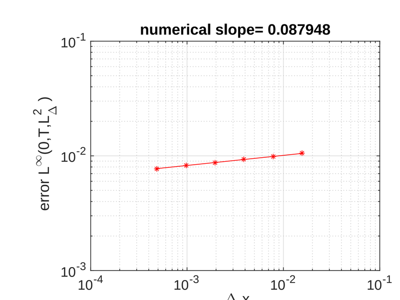

FIGURE : Numerical results for a **conjectured rate of 0.08333**

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