MA1S11 (Dotsenko) Solutions to Tutorial/Exercise Sheet 10

Week 12, Michaelmas 2013

1. Evaluate the indefinite integrals

$$a)\int \frac{(2+3\sqrt{x})^{20}}{\sqrt{x}}\,dx \qquad b)\int x^3\sqrt{1+4x}\,dx$$

Solution. a) Let us do the u-substitution $u = 2+3\sqrt{x}$, so that

$$\frac{du}{dx} = \frac{3}{2\sqrt{x}},$$

and

$$du = \frac{du}{dx}\,dx = \frac{3}{2\sqrt{x}}\,dx,$$

making the integral become

$$\int \frac{(2+3\sqrt{x})^{20}}{\sqrt{x}}\,dx = \frac{2}{3}\int u^{20}\,du = \frac{2u^{21}}{63} + C = \frac{2(1+\sqrt{x})^{21}}{63} + C$$

For b), we use $u = 1+4x$ so $du = 4\,dx$, and $dx = \frac{1}{4}\,du$. Plus, we have to substitute $x = \frac{1}{4}(u-1)$ back in to the equation, the equation has x^3 in it and

$$x^3 = \left(\frac{1}{4}(u-1)\right)^3 = \frac{1}{64}(u^3-3u^2+3u-1) \tag{1}$$

giving

$$\int x^3\sqrt{1+4x}\,dx = \frac{1}{256}\int (u^3-3u^2+3u-1)\sqrt{u}\,du =$$
$$= \frac{1}{256}\int (u^{7/2}-3u^{5/2}+3u^{3/2}-u^{1/2})\,du = \frac{u^{9/2}}{1152} - \frac{3u^{7/2}}{896} + \frac{3u^{5/2}}{640} - \frac{u^{3/2}}{384} + C =$$
$$= \frac{(1+4x)^{9/2}}{1152} - \frac{3(1+4x)^{7/2}}{896} + \frac{3(1+4x)^{5/2}}{640} - \frac{(1+4x)^{3/2}}{384} + C$$

2. Evaluate the integrals

$$a)\int_0^{\pi/2} \sin x\,dx \qquad b)\int_1^2 \left(y^2-y^{-3}\right)dy \qquad \text{and} \qquad \int_2^1 \left(y^2-y^{-3}\right)dy$$

Solution. a) is fairly obvious: we know an antiderivative of $\sin x$, that is $-\cos x$ hence

$$\int_0^{\pi/2} \sin x dx = -\cos x]_0^{\pi/2} = -\cos\frac{\pi}{2} + \cos 0 = -0 + 1 = 1.$$

For b), we know an antiderivative of y^2, it is $y^3/3$ and of y^{-3}, it is $-\frac{y^{-2}}{2}$, hence

$$\int_1^2 (y^2 - y^{-3})dy = \left(\frac{y^3}{3} + \frac{1}{2y^2}\right)\Big]_1^2 = \frac{8}{3} + \frac{1}{8} - \frac{1}{3} - \frac{1}{2} = \frac{47}{24}$$

and

$$\int_2^1 (y^2 - y^{-3})dy = \left(\frac{y^3}{3} + \frac{1}{2y^2}\right)\Big]_2^1 = \frac{1}{3} + \frac{1}{2} - \frac{8}{3} - \frac{1}{8} = -\frac{47}{24}$$

3. Evaluate the integrals

$$a) \int_{-1}^5 (3+2w)(3w+w^2)^5\, dw \qquad b) \int_{-\pi}^{\pi/2} \cos x e^{\sin x}\, dx$$

Solution. For a), let us consider the u-substitution $u = 3w + w^2$, so that $du = (3 + 2w)\, dw$. When $w = -1$ by substituting back in we have $u = -2$ and when $w = 5$, $u = 40$, hence

$$\int_{-1}^5 (3+2w)(3w+w^2)^5\, dw = \int_{-2}^{40} u^5\, du = \left[\frac{u^6}{6}\right]_{-2}^{40} = \frac{40^6 - 64}{6}$$

For b) let $u = \sin x$, so that $du = \cos x\, dx$. When $x = -\pi$ we have $u = 0$ and when $x = \pi/2$, we have $u = 1$, giving

$$\int_{-\pi}^{\pi/2} \cos x e^{\sin x}\, dx = \int_0^1 e^u\, du = e^u]_0^1 = e - 1.$$

4. Evaluate the integral

$$\int_{-1}^2 \sqrt{2+|x|}\, dx.$$

Solution. We note that the function $\sqrt{2+|x|}$ is piecewise defined, and evaluate our integral as

$$\int_{-1}^2 \sqrt{2+|x|}\, dx = \int_{-1}^0 \sqrt{2+|x|}\, dx + \int_0^2 \sqrt{2+|x|}\, dx =$$

$$= \int_{-1}^0 \sqrt{2-x}\, dx + \int_0^2 \sqrt{2+x}\, dx = -\frac{2}{3}(2-x)^{3/2}\Big]_{-1}^0 + \frac{2}{3}(2+x)^{3/2}\Big]_0^2 =$$

$$= -\frac{2}{3}2^{3/2} + \frac{2}{3}3^{3/2} + \frac{2}{3}4^{3/2} - \frac{2}{3}2^{3/2} = \frac{16}{3} - \frac{8}{3}\sqrt{2} + 2\sqrt{3}.$$

5. Show that

$$\int_{1/2}^{2} \frac{1}{x} \sin\left(x - \frac{1}{x}\right)\, dx = 0$$

Solution. Let us consider the u-substitution $u = 1/x$, so that $du = -\frac{1}{x^2}\, dx$, that is $dx = -\frac{1}{u^2}\, du$. We note that for $x = 1/2$, we have $u = 2$, and for $x = 2$, we have $u = 1/2$, so

$$\int_{1/2}^{2} \frac{1}{x} \sin\left(x - \frac{1}{x}\right)\, dx = \int_{2}^{1/2} u \sin\left(\frac{1}{u} - u\right)\left(-\frac{1}{u^2}\right)\, du = -\int_{1/2}^{2} \frac{1}{u} \sin\left(u - \frac{1}{u}\right)\, du,$$

so this integral is equal to its negative, and therefore vanishes.