

MA1S11 (Dotsenko) Solutions to Tutorial/Exercise Sheet 2

Week 3, Michaelmas 2013

1. Determine the symmetry properties (symmetry about x -axis, about y -axis, about the origin or none at all) of the following curves in the xy plane (you do not need to plot the graphs!):

$$y = x^4 + \frac{3}{x^2} - 2, \quad y^5 = x^5 + x \cos(x), \quad 2x^2 + \frac{y^2}{5} = 1, \quad y^5 = \frac{1}{x} + x^3 + 1.$$

Solution. It suffices to perform the relevant substitutions ($-y$ instead of y , $-x$ instead of x or both), and check if the equation changes. The equation $y = x^4 + \frac{3}{x^2} - 2$ determines a curve symmetric about the y -axis only, since plugging $-y$ instead of y changes the equation, but plugging $-x$ instead of x does not. The equation $y^5 = x^5 + x \cos(x)$ determines a curve symmetric about the origin only, since (since plugging $-y$ instead of y changes the equation, plugging $-x$ instead of x also does (it is useful to recall that \cos is an even function, so $(-x) \cos(-x) = -x \cos(x)$), but changing both signs simultaneously just multiplies both sides of our equation by -1 , so makes it into an equivalent equation. The equation $2x^2 + \frac{y^2}{5} = 1$ has all the symmetries: about the x -axis, about the y -axis, and about the origin, since each substitution does not change the equation. Finally, the equation $y^5 = \frac{1}{x} + x^3 + 1$ has no symmetries at all, each substitution changes it.

2. Show that the function $f(x) = 2\sqrt[3]{x-7} + 1$ has an inverse, and compute its inverse.

Solution. One of the conditions the inverse satisfies is $f(g(x)) = x$, so we would like to have $2\sqrt[3]{g(x)-7} + 1 = x$. This can be rewritten as $2\sqrt[3]{g(x)-7} = x-1$, so $8(g(x)-7) = (x-1)^3$, and $g(x) = \frac{1}{8}(x-1)^3 + 7$, with the natural domain $(-\infty, \infty)$. We can check directly that the other condition is satisfied:

$$g(f(x)) = \frac{1}{8}((2\sqrt[3]{x-7} + 1) - 1)^3 + 7 = x$$

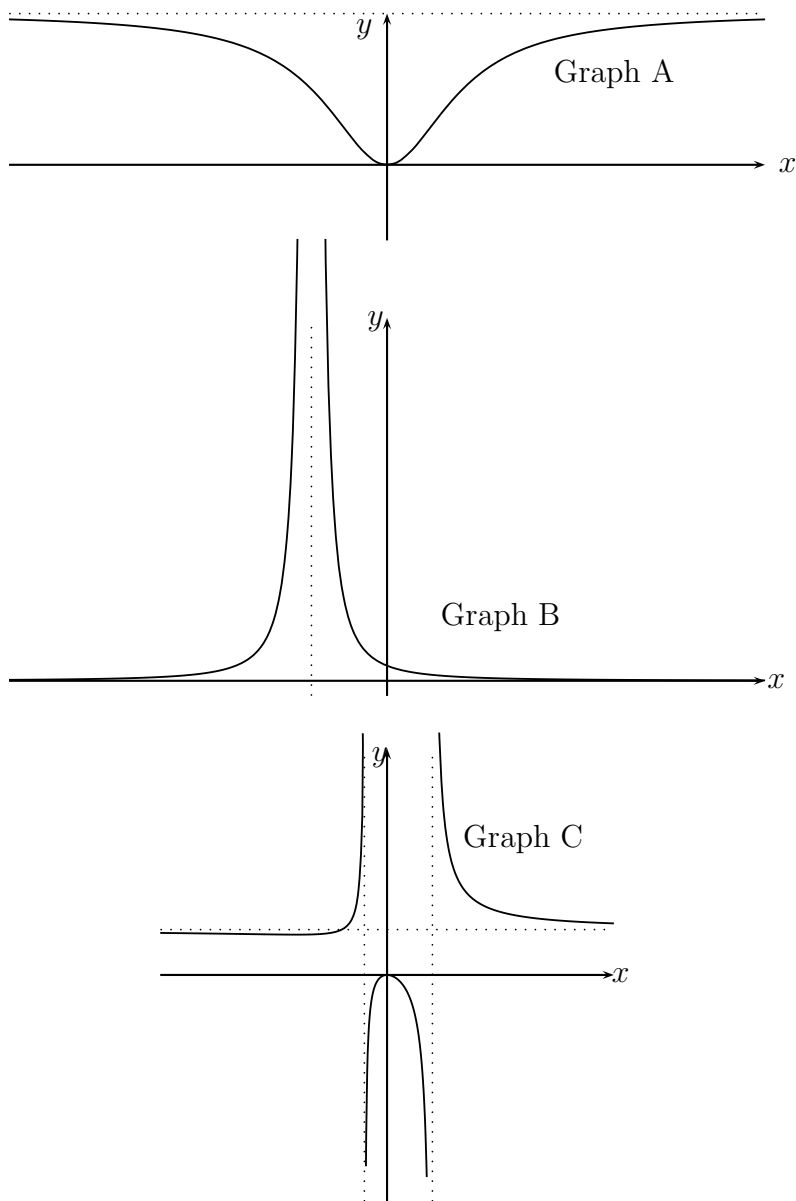
for all x . Therefore, $f(x)$ has an inverse, and its inverse is $f^{-1}(x) = \frac{1}{8}(x-1)^3 + 7$.

3. State the geometric property common to all lines in the family $y = -x + b$. State the geometric property common to all lines in the family $y = mx + 3$.

Solution. The lines in the first family all have the same slope, so they are all parallel to one another.

The lines in the second family all pass through the point $(0, 3)$, since for $x = 0$ we get $y = 3$ regardless of m .

4. The following three graphs A, B, and C correspond (in some order) to the equations (1) $y = \frac{x^2}{x^2-x-2}$, (2) $y = \frac{2x^2}{x^2+1}$, (3) $y = \frac{4}{(x+2)^2}$. Match the graphs with the equations, and explain your reasoning.



Solution. The domain of function (1) consists of all x except for those where $x^2 - x - 2 = 0$, that is $x = -1$ or $x = 2$. The domain of function (2) consists of all real x since $x^2 + 1$ is always positive. The domain of function (3) consists of all $x \neq -2$. This immediately shows that the graph of (1) is C, the graph of (2) is A, and the graph of (3) is B.

5. In the previous question, give equations for horizontal and vertical asymptotes of the given graphs.

Solution.

The horizontal asymptote of graph A is the value which the function (2) approaches when $|x|$ is large. Since $\frac{2x^2}{x^2+1} = 2 - \frac{2}{x^2+1}$, it is clear that the horizontal asymptote is $y = 2$.

The horizontal asymptote of graph B is $y = 0$, since $\frac{4}{(x+2)^2}$ is very small for large $|x|$. The vertical asymptote is $x = -2$, since that is the value where y is not defined (and the values approach $+\infty$ close to -2).

The horizontal asymptote of graph C is $y = 1$, since $\frac{x^2}{x^2-x-2} = 1 + \frac{x+2}{x^2-x-2}$ is very small for large $|x|$. The vertical asymptotes are $x = -1$ and $x = 2$, since those are the values where y is not defined (and the values approach $\pm\infty$ close to -1 and 2).