MA1S11 (Dotsenko) Solutions to Tutorial/Exercise Sheet 4

Week 5, Michaelmas 2013

1. Which of the following limits exist (as finite or infinite limits)? Explain your answer and compute them.

   - $\lim\limits_{x\to+\infty} \frac{3-5x^3}{1+4x+x^3}$.
   - $\lim\limits_{x\to-\infty} \frac{3-5x^3}{1+4x+x^3}$.

   Solution. Clearly, $\frac{3-5x^3}{1+4x+x^3} = \frac{-5x^3}{x^3} \cdot \frac{1-\frac{3}{5x^3}}{1+\frac{4}{x^2}+\frac{1}{x^3}}$, so $\lim\limits_{x\to+\infty} \frac{3-5x^3}{1+4x+x^3} = \lim\limits_{x\to+\infty} \frac{-5x^3}{x^3} = -5$ and also $\lim\limits_{x\to-\infty} \frac{3-5x^3}{1+4x+x^3} = \lim\limits_{x\to-\infty} \frac{-5x^3}{x^3} = -5$.

2. Which of the following limits exist (as finite or infinite limits)? Explain your answer and compute them.

   - $\lim\limits_{x\to-\infty} \frac{\sqrt{x^2+x}}{3x-1}$.
   - $\lim\limits_{x\to+\infty} \left(\sqrt{x+1}-\sqrt{x}\right)$.

   Solution. For the first limit, we note that $\frac{\sqrt{x^2+x}}{3x-1} = \frac{\sqrt{x^2}\sqrt{1+\frac{1}{x}}}{3x(1-\frac{1}{3x}}$, so

   $$\lim_{x\to-\infty} \frac{\sqrt{x^2+x}}{3x-1} = \lim_{x\to-\infty} \frac{|x|}{3x} = -\frac{1}{3}.$$

   For the second limit, we note that $\sqrt{x+1}-\sqrt{x} = \frac{(\sqrt{x+1})^2-(\sqrt{x})^2}{\sqrt{x+1}+\sqrt{x}} = \frac{1}{\sqrt{x+1}+\sqrt{x}}$, so

   $$\lim_{x\to+\infty} \left(\sqrt{x+1}-\sqrt{x}\right) = 0.$$

3. Which of the following limits exist (as finite or infinite limits)? Explain your answer and compute them.

   - $\lim\limits_{x\to0} \frac{1-\cos x}{\sin x}$.
   - $\lim\limits_{x\to0} \frac{\tan x-\sin x}{x^3}$.

   Solution. For the first limit, we have

$$\lim_{x\to0} \frac{1-\cos x}{\sin x} = \lim_{x\to0} \frac{1-\cos x}{x^2} \cdot \frac{x^2}{\sin x} = \lim_{x\to0} \frac{1-\cos x}{x^2} \cdot \lim_{x\to0} \frac{x}{\sin x} \cdot \lim_{x\to0} x = \frac{1}{2} \cdot 1 \cdot 0 = 0.$$

For the second one, we have

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}-\sin x}{x^3}=\lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^3\cos x}=$$
$$=\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{1-\cos x}{x^2}\lim_{x\to 0}\frac{1}{\cos x}=1\cdot\frac{1}{2}\cdot 1=\frac{1}{2}.$$

4. Show that
$$-|x|\le x\sin\frac{1}{x}\le |x|$$
for all $x\neq 0$, and explain why $\lim\limits_{x\to 0} x\sin\frac{1}{x}=0$.

Solution. Since $|\sin t|\le 1$ for all t, we have $|x\sin\frac{1}{x}|=|x||\sin\frac{1}{x}|\le |x|$, therefore $-|x|\le x\sin\frac{1}{x}\le |x|$. Since $\lim\limits_{x\to 0}|x|=\lim\limits_{x\to 0}(-|x|)=0$, the Squeezing Theorem guarantees that $\lim\limits_{x\to 0} x\sin\frac{1}{x}=0$.

5. Let us consider function $f(x)=\begin{cases}\sin\frac{1}{x}, x\neq 0,\\ a, \quad x=0.\end{cases}$ Does there exist a choice of a for which this function is continuous at $x=0$? Explain your answer.

Solution. No. If this were the case, this would mean that $\lim\limits_{x\to 0}\sin\frac{1}{x}=a$. However, this limit does not exist, since as x approaches zero, since $\sin\frac{1}{x}$ oscillates between -1 and 1 in every interval containing zero.