MA1S11 (Dotsenko) Solutions to Tutorial/Exercise Sheet 5

Week 6, Michaelmas 2013

1. Let $f(x) = x^{5/3}$ and $g(x) = \sin x$. Compute $f'(0)$ and $g'(0)$.

   Solution. By definition,

$$f'(0) = \lim_{h\to 0} \frac{h^{5/3} - 0^{5/3}}{h} = \lim_{h\to 0} \frac{h^{5/3}}{h} = \lim_{h\to 0} h^{2/3} = 0.$$

   Also,

$$g'(0) = \lim_{h\to 0} \frac{\sin h - \sin 0}{h} = \lim_{h\to 0} \frac{\sin h}{h} = 1,$$

   as proved last week.

2. Find the equation for the tangent line to the graph $y = \sqrt{x}$ at $x = 4$.

   Solution. Since, as we checked in class,

$$y'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{(x+h) - x}{h\sqrt{x+h} + \sqrt{x}} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}},$$

   we have the equation of the tangent line at $x = x_0$

$$y = \frac{1}{2\sqrt{x_0}}(x - x_0) + \sqrt{x_0} = \frac{x}{2\sqrt{x_0}} + \frac{1}{2}\sqrt{x_0},$$

   so for $x_0 = 4$ we get $y = \frac{x}{4} + 1$.

3. Is the function $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$ differentiable at the point $x = 0$?

   Solution. By definition, we would like to know if the limit

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

   exists. We check that directly:

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin \frac{1}{h} - 0}{h} = \lim_{h\to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h\to 0} h \sin \frac{1}{h} = 0$$

   by Squeezing Theorem (see last question of the previous tutorial).

4. Find the values of x_0 for which the tangent line to the graph $y = x^3 - x$ at $x = x_0$ is parallel to the line $y = x$, and write down equations for the corresponding tangent lines.

   Solution. Since
$$y'(x) = 3x^2 - 1,$$
   and the slope of the tangent line is equal to the derivative, the points where the tangent line is parallel to $y = x$ are given by $3x^2 - 1 = 1$, that is $x = \pm\sqrt{\frac{2}{3}}$. The corresponding equations
$$y = (3x_0^2 - 1)(x - x_0) + (x_0^3 - x_0) = (3x_0^2 - 1)x - 2x_0^3$$
   simplify to
$$y = x \mp \frac{4\sqrt{2}}{3\sqrt{3}}.$$

5. A person drops a coin from the roof of a skyscraper which is 218 metres above the street level. The position of the coin (in metres above water) as a function of time (in seconds) is given by
$$s(t) = 218 - \frac{1}{2}gt^2,$$
   where $g = 9.81m/s^2$ (metres per seconds squared) is the acceleration due to the gravitational force.

   - How long will it take for the coin to reach the street level?
   - What is the instantaneous velocity of the coin as a function of t?
   - What is the instantaneous velocity at the end of the fall?

   Solution. Solving the equation $0 = s(t) = 218 - \frac{1}{2}gt^2$, we obtain the time needed to reach the street level. From this equation,
$$t^2 = \frac{218m}{\frac{1}{2}g\,m/s^2} = \frac{400}{9}s^2,$$
   so $t = \frac{20}{3}s \approx 6.66s$. The instantaneous velocity is the derivative of the position $s(t)$, that is $s'(t) = -gt$. (It is negative because the velocity has its direction towards the earth, opposite the direction of growth of the height.) For the end of the fall, $t = \frac{20}{3}s$, so the (absolute value of) velocity is $9.81m/s^2 \cdot \frac{20}{3}s = 65.4m/s$.