MA1S11 (Dotsenko) Solutions to Tutorial/Exercise Sheet 7

Week 9, Michaelmas 2013

1. Using implicit differentiation, find the equation of the tangent line to the curve $y^2 - y = x^3 - x$ at the point $(2, 3)$.

   Solution. Differentiating the equation, we get

$$2yy' - y' = 3x^2 - 1,$$
$$y' = \frac{3x^2 - 1}{2y - 1}.$$

   Therefore, at $(2, 3)$ we have $y'(x) = \frac{12-1}{6-1} = \frac{11}{5}$, and the point-slope equation gives the tangent line $y - 3 = \frac{11}{5}(x - 2)$, or $y = \frac{11}{5}x - \frac{7}{5}$.

2. Determine the roots, the relative extrema, the regions of concavity up/down and the limit behaviour for $x \to \pm\infty$ for

$$f(x) = (x + 1)^2(x - 1) = x^3 + x^2 - x - 1$$

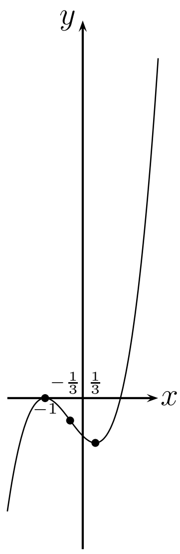
   Then sketch the graph of f using all the gathered information.

   Solution. Since $f(x) = (x + 1)^2(x - 1)$, the roots of f are $x = -1$ and $x = 1$. Let us compute f' and f'':

$$f'(x) = 3x^2 + 2x - 1,$$
$$f''(x) = 6x + 2.$$

   The critical points of f are stationary, since f is differentiable everywhere. Computing roots of $f'(x)$ using the formula for roots of a quadratic equation, we get $x = -1$ and $x = 1/3$. We get $f''(-1) = -4 < 0$ and $f''(1/3) = 4 > 0$. Therefore, $x = -1$ gives a relative maximum, and $x = 1/3$ relative minimum. Also, the only root of $f''(x)$ is $x = -1/3$, and for $x < -1/3$ we have $f''(x) < 0$, while for $x > -1/3$ we have $f''(x) > 0$. Therefore, on $(-\infty, -1/3)$ the function f is concave down, and on $(-1/3, +\infty)$ the function f is concave up, while

$x = -1/3$ is an inflection point. Using all this information, we obtain the following graph:



3. Determine the relative extrema, the inflection points and the regions of concavity up or down of the function

$$f(x) = \frac{x^4}{4} + \frac{2}{9}x^3 - \frac{5}{6}x^2 + \frac{10}{9}.$$

Solution. Let us compute f' and f'':

$$f'(x) = x^3 + \frac{2}{3}x^2 - \frac{5}{3}x,$$
$$f''(x) = 3x^2 + \frac{4}{3}x - \frac{5}{3}.$$

The critical points of f are stationary, since f is differentiable everywhere. We have $f'(x) = x(x^2 + \frac{2}{3}x - \frac{5}{3})$, so the roots of f' are $x = 0$, $x = 1$ and $x = -\frac{5}{3}$. Computing f'' at the stationary points, we get $f''(0) = -\frac{5}{3}$, $f''(1) = \frac{8}{3}$, $f''(-\frac{5}{3}) = \frac{40}{9}$. Therefore, $x = -\frac{5}{3}$ and $x = 1$ give relative minima, and $x = 0$ a relative maximum. Roots of $f''(x)$ are $x = -1$ and $x = \frac{5}{9}$, and for $-1 < x < \frac{5}{9}$ we have $f''(x) < 0$, while for $x < -1$ or $x > \frac{5}{9}$ we have $f''(x) > 0$. Therefore, on $(-1, \frac{5}{9})$ the function f is concave down, and both on $(-\infty, -1)$ and on $(\frac{5}{9}, +\infty)$ the function f is concave up, while $x = -1$ and $x = \frac{5}{9}$ are inflection points.