1S11: Calculus for students in Science

Dr. Vladimir Dotsenko

TCD

Lecture 17

Derivatives of polynomials: reminder

Theorem. Every polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

is differentiable everywhere, and

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1}.$$

Proof. It is enough to prove that the derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$, since then we can use rules for scalar factors and addition. We note that similar to the fact that $a^2 - b^2 = (a-b)(a+b)$, we have

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2),$$

$$\ldots$$

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}),$$

so

$$\lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0} = \lim_{x \to x_0} (x^{n-1} + x^{n-2}x_0 + \cdots + xx_0^{n-2} + x_0^{n-1}) = nx_0^{n-1}.$$

Derivatives of polynomials: another approach

There is another way to prove the same theorem; it relies on the *binomial formula*

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n.$$

Here $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, where in turn $n!$ is the "factorial" number, $n! = 1 \cdot 2 \cdot 3 \cdots n$. In particular, $\binom{n}{1} = \frac{n!}{1!(n-1)!} = n$.
To compute the derivative of x^n, we need to examine the limit

$$\lim_{h\to 0} \frac{(x+h)^n - x^n}{h}.$$

Expanding $(x+h)^n$ by the binomial formula, we get $(x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n$, which allows us to conclude that

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + \binom{n}{n-1}xh^{n-2} + h^{n-1},$$

which manifestly has the limit nx^{n-1} as $h \to 0$.

Computing derivatives: reminder

Theorem. Suppose that two functions f and g are both differentiable at $x = x_0$. Then $f + g$ and $f - g$ are differentiable at $x = x_0$, and

$$(f+g)'(x_0) = f'(x_0) + g'(x_0), \quad (f-g)'(x_0) = f'(x_0) - g'(x_0).$$

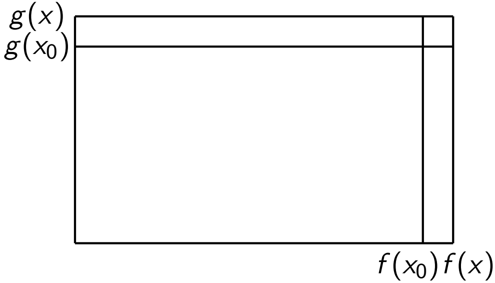
Since the derivative function of x^2 is $2x$, and the derivative function of x is 1, the derivative of the product $x \cdot x$ is not equal to the product of derivatives. Instead, the following more complex product rule for derivatives holds.

Product rule

Theorem. Suppose that two functions f and g are both differentiable at $x = x_0$. Then fg is differentiable at $x = x_0$, and

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0).$$

Informally, if f and g describe how two sides of a rectangle change with time, fg describes the change of the area, and the change of area can be read from the following figure:



Proof of the product rule

Proof. We have

$$(fg)'(x_0) = \lim_{x\to x_0} \frac{(fg)(x)-(fg)(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} =$$

$$= \lim_{x\to x_0} \frac{f(x)g(x)-f(x_0)g(x)+f(x_0)g(x)-f(x_0)g(x_0)}{x-x_0} =$$

$$= \lim_{x\to x_0} \left(\frac{f(x)-f(x_0)}{x-x_0}g(x)+f(x_0)\frac{g(x)-g(x_0)}{x-x_0}\right) =$$

$$= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \lim_{x\to x_0} g(x) + f(x_0) \lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} =$$

$$= f'(x_0)g(x_0)+f(x_0)g'(x_0),$$

where we used the fact that a function that is differentiable at x_0 is also continuous at x_0.

Product rule: examples

Example 1. Let $f(x) = g(x) = x$, so that $f(x)g(x) = x^2$. We have $f'(x) = g'(x) = 1$, so

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x) = 1 \cdot x + x \cdot 1 = 2x,$$

as expected. Moreover, one can use product rule to prove $(x^n)' = nx^{n-1}$ in a yet another way.

Example 2. Let $f(x) = (1+x)\sqrt{x}$. Then

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}[1+x]\sqrt{x} + (1+x)\frac{d}{dx}[\sqrt{x}] = \\ = \sqrt{x} + (1+x)\frac{1}{2\sqrt{x}} = \frac{1+3x}{2\sqrt{x}}.$$

Derivative of the quotient

Theorem. Suppose that the function g is differentiable at $x = x_0$, and that $g(x_0) \neq 0$. Then $\frac{1}{g}$ is differentiable at $x = x_0$, and

$$\left(\frac{1}{g}\right)'(x_0) = -\frac{g'(x_0)}{g(x_0)^2}.$$

Proof. We have

$$\left(\frac{1}{g}\right)'(x_0) = \lim_{x\to x_0} \frac{\frac{1}{g(x)} - \frac{1}{g(x_0)}}{x - x_0} = \lim_{x\to x_0} \frac{\frac{g(x_0)-g(x)}{g(x_0)g(x)}}{x - x_0} =$$

$$= -\frac{1}{g(x_0)} \lim_{x\to x_0} \frac{1}{g(x)} \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} = -\frac{g'(x_0)}{g(x_0)^2},$$

as required.

Derivative of the quotient

Theorem. Suppose that two functions f and g are both differentiable at $x = x_0$, and that $g(x_0) \neq 0$. Then $\frac{f}{g}$ is differentiable at $x = x_0$, and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.$$

Proof. Let us apply the previous result and the product rule to the product $f \cdot \frac{1}{g} = \frac{f}{g}$:

$$\left(\frac{f}{g}\right)'(x_0) = f'(x_0)\frac{1}{g(x_0)} + f(x_0)\left(\frac{1}{g}\right)'(x_0) = \frac{f'(x_0)}{g(x_0)} - \frac{f(x_0)g'(x_0)}{g(x_0)^2},$$

which is equal to $\frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$, as required.

Derivatives of trigonometric functions

Theorem. We have $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$.

Proof. We shall use the addition formulas for sines and cosines:

$$\sin(a+b) = \sin a \cos b + \sin b \cos a, \quad \cos(a+b) = \cos a \cos b - \sin a \sin b.$$

Using these formulas, we can evaluate the respective limits:

$$\lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} =$$
$$= \cos x \lim_{h\to 0} \frac{\sin h}{h} - \sin x \lim_{h\to 0} \frac{1-\cos h}{h} = \cos x,$$

$$\lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} =$$
$$= \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h} = -\sin x.$$

Derivatives of trigonometric functions

Theorem. We have

$$(\tan x)' = \sec^2 x, \quad (\cot x)' = -\csc^2 x,$$
$$(\sec x)' = \sec x \tan x, \quad (\csc x)' = -\csc x \cot x.$$

Proof. This follows from the rule for computing derivatives of quotients. For example,

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} =$$
$$= \frac{\cos x \cos x - \sin x (-\sin x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

Exercise. Prove the three remaining formulas yourself; this would be an excellent way to start getting used to computing derivatives!