1S11: Calculus for students in Science

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TCD

Lecture 19

Derivatives of implicitly defined functions

Implicit differentiation refers to the situation when the dependent variable y is not given explicitly, but rather implicitly, that is a relationship between x and y is expressed by an equation (which may be possible to solve directly or not); in this case the derivative $y'(x)$ can be found by differentiating the equation using the chain rule. Using implicit differentiation, we can compute many different derivatives.

Example 1. To compute the derivative of $1/x$ in an alternative way, one can apply implicit differentiation to $xy = 1$:

$$\frac{d}{dx}[xy] = \frac{d}{dx}[1],$$

$$y + xy' = 0,$$

$$xy' = -y,$$

$$y' = -\frac{y}{x} = -\frac{\frac{1}{x}}{x} = -\frac{1}{x^2}.$$

Derivatives of implicitly defined functions

Example 2. To compute the derivative of $y = \sqrt{1-x^2}$, we can of course use the chain rule:

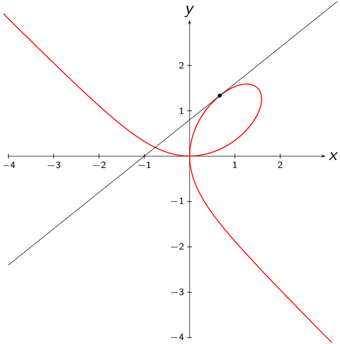
$$\left(\sqrt{1-x^2}\right)' = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = -\frac{x}{\sqrt{1-x^2}}.$$

However, we can approach it differently, applying implicit differentiation to $x^2 + y^2 = 1$:

$$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[1],$$
$$2x + 2yy' = 0,$$
$$yy' = -x,$$
$$y' = -\frac{x}{y} = -\frac{x}{\sqrt{1-x^2}}.$$

Derivatives of implicitly defined functions

Let us consider the curve $x^3 + y^3 = 3xy$:



What is the equation of the tangent line at the point $(2/3, 4/3)$? Note that this point is on the curve since

$$(2/3)^3 + (4/3)^3 = 8/27 + 64/27 = 72/27 = 8/3 = 3 \cdot 2/3 \cdot 4/3.$$

Derivatives of implicitly defined functions

This curve is not a graph of a function, but we still can use derivatives! Differentiating implicitly amounts to the following steps:

$$\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[3xy],$$
$$3x^2 + 3y^2 y'(x) = 3y + 3xy'(x),$$
$$x^2 + y^2 y'(x) = y + xy'(x),$$
$$(y^2 - x)y'(x) = y - x^2,$$
$$y'(x) = \frac{y - x^2}{y^2 - x}.$$

Now to compute the slope of the tangent line at a point, we just substitute the x- and y-coordinates, e.g. for $(2/3, 4/3)$ we obtain

$$y'(x) = \frac{4/3 - 4/9}{16/9 - 2/3} = \frac{8/9}{10/9} = 0.8,$$

and using the point-slope formula, we get $y - 4/3 = 0.8(x - 2/3)$, that is

$$y = 0.8x + 0.8.$$

Higher derivatives

Given a differentiable function f, its derivative f' is another function, which is often again differentiable. This new function $(f')'$, if exists, is denoted by f'', and is called the *second derivative* of the function f. Similarly, the derivative of the second derivative is denoted by f''' and is called the *third derivative* of the function f, etc. Starting from the order 4, a more compact notation is used: the fourth derivative is denoted by $f^{(4)}$, the fifth derivative by $f^{(5)}$, etc.

Other common notations for higher derivatives are

$$y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{d}{dx}[f(x)]\right] = \frac{d^2}{dx^2}[f(x)],$$

$$y''' = f'''(x) = \frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{d^2}{dx^2}[f(x)]\right] = \frac{d^3}{dx^3}[f(x)],$$

$$\cdots$$

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n} = \frac{d^n}{dx^n}[f(x)].$$

Higher derivatives: meaning

By definition, the second derivative of a function is "rate of change of the rate of change". If the function f describes the linear motion of a particle, then, as we discussed before, f' describes the instantaneous velocity at each point of the trajectory, and f'' describes the instantaneous acceleration.

On a less serious note,

> *In the fall of 1972 President Nixon announced that the rate of increase of inflation was decreasing. This was the first time a sitting president used the third derivative to advance his case for reelection.*

(Hugo Rossi, in Notices of American Mathematical Society, vol. 43, no. 10, 1996.)

Higher derivatives: product rule

If we denote $f(x) = f^{(0)}(x)$, $f'(x) = f^{(1)}(x)$, $f''(x) = f^{(2)}(x)$, there is a very nice and compact product rule for higher derivatives:

$$(fg)^{(n)}(x) = f^{(0)}(x)g^{(n)}(x) + \binom{n}{1} f^{(1)}(x)g^{(n-1)}(x) + \\ + \binom{n}{2} f^{(2)}(x)g^{(n-2)}(x) + \cdots + \binom{n}{n-1} f^{(n-1)}(x)g^{(1)}(x) + \\ + f^{(n)}(x)g^{(0)}(x),$$

where the coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ are the same coefficients that are featured in the binomial formula for $(a+b)^n$.

Derivatives and analysis of functions

The following facts will be useful for us. We shall use them without proof.

- If f is a constant function, then its derivative is zero.
- If f is differentiable on (a, b), and is an increasing function ($f(x_1) < f(x_2)$ whenever $x_1 < x_2$), then the derivative of a function f is non-negative on (a, b): $f'(x) \geq 0$.
- If f is differentiable on (a, b), and is a decreasing function ($f(x_1) > f(x_2)$ whenever $x_1 < x_2$), then the derivative of a function f is non-positive on (a, b): $f'(x) \leq 0$.

Note that since we pass to limits, inequalities may become non-strict, e.g. $f(x) = x^3$ is increasing, but $f'(0) = 0$.

Derivatives and analysis of functions

- If the derivative of a function f is zero on (a, b), then f is a constant function.
- If the derivative of a function f is positive on (a, b), then f is an increasing function: $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.
- If the derivative of a function f is negative on (a, b), then f is a decreasing function: $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.
- If f is differentiable on (a, b), and attains a (locally) extremal value at the point c inside (a, b) (this means that either for all points x sufficiently close to c we have $f(x) \leq f(c)$ or for all points x sufficiently close to c we have $f(x) \geq f(c)$), then $f'(c) = 0$.

Note that the converse of the last statement is false: not every point where the first derivative is equal to zero gives a locally extremal value, e.g. for the same function $f(x) = x^3$ that we just discussed, we have $f'(0) = 0$, but $f(x) > f(0)$ for positive x, and $f(x) < f(0)$ for negative x.