1S11: Calculus for students in Science

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TCD

Lecture 2

Natural domain of a function (reminder slide)

With the expressions like $\sqrt{x}$ or $1/x$ we saw how the domain of a function may be restricted by a formula from which we compute values of a function.

Definition. The *natural domain* of a function f defined by a formula consists of all values of x for which $f(x)$ has a well defined real value.

Example 1. The natural domain of $f(x) = x^3$ consists of all real numbers $(-\infty, \infty)$, since for each real x its cube is a well defined real number.

Natural domain of a function

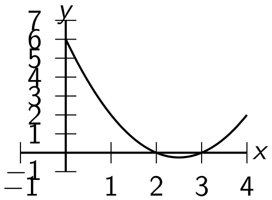
Example 2. The natural domain of $f(x) = \frac{1}{(x-1)(x-3)}$ consists of all real numbers except for $x = 1$ and $x = 3$ since for those numbers division by zero occurs.
One can write the answer as either $\{x\colon x \neq 1 \text{ and } x \neq 3\}$ or, using our previous notation, $(-\infty, 1) \cup (1, 3) \cup (3, +\infty)$. Here a mathematical symbol $\cup$ for union (joining together two sets) is used.

Example 3. The natural domain of $f(x) = \tan x = \frac{\sin x}{\cos x}$ consists of all numbers except for those where $\cos x = 0$. The cosine vanishes precisely at odd multiples of $\pi/2$, so the natural domain of f is

$$\{x\colon x \neq \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \dots\}$$

Natural domain of a function

Example 4. The natural domain of $f(x) = \sqrt{x^2 - 5x + 6}$ consists of all numbers x for which $x^2 - 5x + 6 \geq 0$ (in order for the square root to assume a real value). Since $x^2 - 5x + 6 = (x-2)(x-3)$ (we use the formula for roots of a quadratic equation, $x = \frac{5 \pm \sqrt{5^2 - 4 \cdot 6}}{2}$), we would like to find x for which $(x-2)(x-3) \geq 0$.



Thus, the natural domain of f is $(-\infty, 2] \cup [3, +\infty)$.

Example 5. The natural domain of $f(x) = \frac{x^2-5x+6}{x-2}$ consists of all $x \neq 2$ since the only reason for the value of f to be undefined in this case is where division by zero occurs.

However, since $x^2 - 5x + 6 = (x - 2)(x - 3)$, algebraically this expression can be simplified to $x - 3$. The latter expression is defined for $x = 2$ as well. Therefore, when performing algebraic operations, it is important to keep track of original domains, and in this particular case we may write $f(x) = x - 3, x \neq 2$, thus not losing information on the original domain.

Range of a function

Let us look at the two functions we already examined when talking about domains: $f(x) = 1/x$ and $f(x) = \sqrt{x}$. What values do those functions assume for the values of x ranging over their natural domains?

The first function $f(x) = 1/x$ assumes all nonzero values. Indeed, to attain the value $v \neq 0$, we should just observe that $1/(1/v) = v$.

The second function $f(x) = \sqrt{x}$ assumes all nonnegative values. Indeed, to attain the value $v \geq 0$, we should just observe that for such values we have $\sqrt{v^2} = v$.

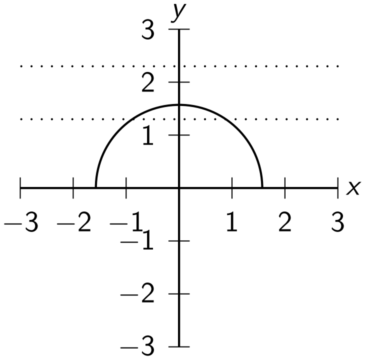
Definition. The *range* of a function f consists of all values $f(x)$ it assumes when x ranges over its domain.

For a function defined by a table, its range consists of numbers in the second row:

x	1	1.5	2	2.5	3	3.5	4
y	0	1.25	3	5.25	8	11.25	15

Range of a function

For a function defined by a graph, its range is the projection of its graph on the y-axis



Range of a function

Example 1. The range of $f(x) = 2 + \sqrt{x-1}$ is $[2, +\infty)$. To see that, we observe that the natural domain of this function is $[1, +\infty)$ since we request that the expression from which we extract the square root is nonnegative. As x varies in $[1, +\infty)$, $x - 1$ varies in $[0, +\infty)$, and $\sqrt{x-1}$ varies in $[0, +\infty)$, so $2 + \sqrt{x-1}$ varies in $[2, +\infty)$.

Example 2. The range of $f(x) = \frac{x+1}{x-1}$ is harder to guess, so we shall approach it directly from the definition. We would like to find the values of a for which the equation $\frac{x+1}{x-1} = a$ has solutions. Solving it, we get

$$x + 1 = ax - a,$$
$$a + 1 = ax - x = x(a - 1),$$
$$x = \frac{a+1}{a-1},$$

so $a = 1$ is not in the natural range. We should be careful when going from $\frac{x+1}{x-1} = a$ to $x + 1 = a(x - 1)$ because we changed the natural domain. However, $x = 1$ is not a solution to $x + 1 = a(x - 1)$ anyway.

New functions from old ones: arithmetic operations

Given two functions f and g, they can be added, subtracted, multiplied and divided in a natural way. For $f+g$, $f-g$, and fg to be defined, both f and g should be defined, and for f/g to be defined, both f and g should be defined, and also the value of g should be non-zero.

Example 1. Let $f(x) = 1+\sqrt{x-2}$, and $g(x) = x-3$. Then

- $(f+g)(x) = 1+\sqrt{x-2}+x-3 = x-2+\sqrt{x-2}$, the domain is $[2,+\infty)$,
- $(f-g)(x) = 1+\sqrt{x-2}-(x-3) = 4-x+\sqrt{x-2}$, the domain is $[2,+\infty)$,
- $(fg)(x) = (1+\sqrt{x-2})(x-3)$, the domain is $[2,+\infty)$,
- $(f/g)(x) = \frac{1+\sqrt{x-2}}{x-3}$, the domain is $[2,3)\cup(3,+\infty)$.

In these examples, the domains of $f+g$, $f-g$, fg, f/g are their natural domains. That is not always the case.

New functions from old ones: arithmetic operations

Example 2. Let $f(x) = \sqrt{x-2}$, and $g(x) = \sqrt{x-3}$. Then

$$(fg)(x) = \sqrt{x-2}\sqrt{x-3} = \sqrt{(x-2)(x-3)} = \sqrt{x^2-5x+6}.$$

The domain of fg is $[3, +\infty)$, which *does not* coincide with the natural domain of $\sqrt{x^2-5x+6}$, that is $(-\infty, 2] \cup [3, +\infty)$.

Example 3. Let $f(x) = x$, and $g(x) = 1/x$. Then

$$(f/g)(x) = \frac{x}{\frac{1}{x}} = x^2.$$

The domain of f/g is $(-\infty, 0) \cup (0, +\infty)$, which *does not* coincide with the natural domain of x^2, that is $(-\infty, +\infty)$.

New functions from old ones: composition

The arithmetic operations on functions were not "genuinely" new operations, since they just used aritmetics of real numbers at different points x independently. Now we shall define a truly new way to construct new functions, not having numeric analogues.

Definition. The *composition* of two functions f and g, denoted by $f \circ g$, is the function whose value at x is $f(g(x))$:

$$(f \circ g)(x) = f(g(x)).$$

Its domain is defined as the set of all x in the domain of g for which the value $g(x)$ is in the domain of f.

Example 1. Recall the usual method for solving quadratic equations:

$$x^2+px+q = x^2+2\frac{p}{2}x+q = x^2+2\frac{p}{2}x+\frac{p^2}{4}-\frac{p^2}{4}+q = \left(x+\frac{p}{2}\right)^2-\frac{p^2}{4}+q,$$

which is the composition of $f(x) = x^2 - \left(\frac{p^2}{4} - q\right)$ and $g(x) = x + \frac{p}{2}$.