

1S11: CALCULUS FOR STUDENTS IN SCIENCE

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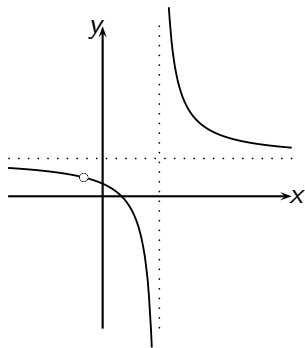
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Lecture 23

ANALYSIS OF RATIONAL FUNCTIONS

Unlike polynomials, rational functions in principle need more effort in handling, since one has variations in what the domain is, what the asymptotes are etc. The recipe on the following slide gives a way to identify most important features of the graph of a given rational function, and to sketch its graph correctly.

We shall assume that we are dealing with a rational function $f(x) = \frac{p(x)}{q(x)}$ written in lowest terms, so that p and q have no common factors. If they do, we can cancel all common factors, make a note that f actually is not defined where those factors vanish, and remove the respective points from the graph, as we did before.



ANALYSIS OF RATIONAL FUNCTIONS

Graphing $f(x) = p(x)/q(x)$ (for p and q without common factors).

- Determine if the graph has any symmetries (about the y -axis or about the origin).
- Find the points where the graph meets the x -axis ($(r, 0)$ for each root r of $p(x)$) and the point where the graph meets the y -axis ($(0, f(0))$).
- Find vertical asymptotes of the graph ($x = s$ for each root s of $q(x)$).
- Points where f can potentially change sign are at the x -intercepts or vertical asymptotes. For each interval between those points, determine the sign of f on that interval.
- Determine the limit of $f(x)$ at $-\infty$ and at $+\infty$. This would determine the horizontal asymptote of the graph, if any.
- Compute $f'(x)$ and $f''(x)$. Analyse the signs of these.
- Using the sign analysis of the derivatives, determine where f is increasing, decreasing, concave up, and concave down. Determine all stationary points, relative extrema, and inflection points. Use the sign analysis of $f(x)$ to determine how the graph behaves near the vertical asymptotes. Based on these conclusions, sketch the graph.

ANALYSIS OF RATIONAL FUNCTIONS: EXAMPLE

Let us analyse the rational function $f(x) = \frac{x^2-1}{x^3}$.

Symmetries: replacing x by $-x$ and y by $-y$ gives an equation that simplifies to the original equation, so the graph is symmetric about the origin.

Intercepts: setting $y = 0$ yields the x -intercepts $x = -1$ and $x = 1$. The point $x = 0$ is not in the domain of f , so there are no y -intercepts.

Vertical asymptotes: $x = 0$ is the only one.

Sign analysis: the points $-1, 0, 1$ (corresponding to the only vertical asymptote and the two x -intercepts) divide the x -axis into the intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, $(1, +\infty)$. Signs of the corresponding factors result in the following signs for f :

interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, +\infty)$
signs of factors x^3 , $(x - 1)$, and $(x + 1)$	$(-)(-)(-)$	$(-)(-)(+)$	$(+)(-)(+)$	$(+)(+)(+)$
sign of f	$-$	$+$	$-$	$+$

ANALYSIS OF RATIONAL FUNCTIONS: EXAMPLE

Let us analyse the rational function $f(x) = \frac{x^2-1}{x^3}$.

Limits at infinity: We have

$$\lim_{x \rightarrow +\infty} \frac{x^2 - 1}{x^3} = \lim_{x \rightarrow +\infty} \left(\frac{1}{x} - \frac{1}{x^2} \right) = 0,$$

$$\lim_{x \rightarrow -\infty} \frac{x^2 - 1}{x^3} = \lim_{x \rightarrow -\infty} \left(\frac{1}{x} - \frac{1}{x^2} \right) = 0,$$

so $y = 0$ is a horizontal asymptote.

Derivatives: We have

$$f'(x) = \frac{2x \cdot x^3 - (x^2 - 1) \cdot 3x^2}{(x^3)^2} = \frac{3 - x^2}{x^4} = \frac{(\sqrt{3} - x)(x + \sqrt{3})}{x^4},$$

$$f''(x) = \frac{(-2x) \cdot x^4 - (3 - x^2) \cdot 4x^3}{(x^4)^2} = \frac{2(x^2 - 6)}{x^5} = \frac{(x - \sqrt{6})(x + \sqrt{6})}{x^5}.$$

The first derivative vanishes for $x = \pm\sqrt{3}$, the second one vanishes for $x = \pm\sqrt{6}$.

ANALYSIS OF RATIONAL FUNCTIONS: EXAMPLE

Let us analyse the rational function $f(x) = \frac{x^2-1}{x^3}$.

Sign analysis of the first derivative: Signs of the corresponding factors result in the following signs for f' :

interval	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, +\infty)$
signs of factors x^4 , $(\sqrt{3} - x)$, and $(x + \sqrt{3})$	$(+)(+)(-)$	$(+)(+)(+)$	$(+)(+)(+)$	$(+)(-)(+)$
sign of f'	-	+	+	-

This suggests that f is decreasing on $(-\infty, -\sqrt{3}]$ and on $[\sqrt{3}, +\infty)$ and is increasing on $[-\sqrt{3}, 0)$ as well as on $(0, \sqrt{3}]$. Therefore, there is a relative minimum at $x = -\sqrt{3}$ and a relative maximum at $x = \sqrt{3}$.

ANALYSIS OF RATIONAL FUNCTIONS: EXAMPLE

Let us analyse the rational function $f(x) = \frac{x^2-1}{x^3}$.

Sign analysis of the second derivative: Signs of the corresponding factors result in the following signs for f'' :

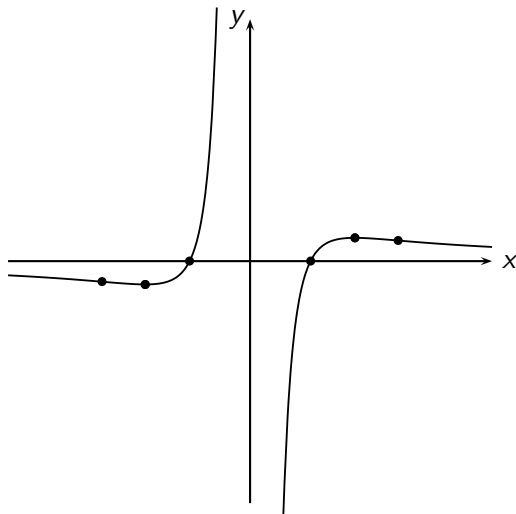
interval	$(-\infty, -\sqrt{6})$	$(-\sqrt{6}, 0)$	$(0, \sqrt{6})$	$(\sqrt{6}, +\infty)$
signs of factors x^5 , $(x - \sqrt{6})$, and $(x + \sqrt{6})$	$(-)(-)(-)$	$(-)(-)(+)$	$(+)(-)(+)$	$(+)(+)(+)$
sign of f''	-	+	-	+

This suggests that f is concave down on $(-\infty, -\sqrt{6}]$ and on $(0, \sqrt{6}]$ and is concave up on $[-\sqrt{6}, 0)$ as well as on $[\sqrt{6}, +\infty)$. Therefore, both $x = -\sqrt{6}$ and $x = \sqrt{6}$ are inflection points.

ANALYSIS OF RATIONAL FUNCTIONS: EXAMPLE

Summarising all the available information, we plot the graph of

$f(x) = \frac{x^2-1}{x^3}$ as follows:



ABSOLUTE EXTREMA

A function f is said to have an *absolute minimum* (“global minimum”) at c , if for all x we have $f(x) \geq f(c)$. Similarly, it is said to have an *absolute maximum* (“global maximum”) at c , if for all x we have $f(x) \leq f(c)$.

Theorem. (*Extreme Value Theorem*) If a function f is continuous on the closed interval $[a, b]$, then it has both an absolute maximum and an absolute minimum on $[a, b]$.

It is an existence theorem (no formula or anything for the numbers which this theorem guarantees to exist), and different proofs give different methods to compute the extremal values. Using derivatives, we can make this very constructive: on the open interval (a, b) , relative extremal values occur at critical points, hence an absolute extremum must occur at either a critical point of f or the endpoints of $[a, b]$.

ABSOLUTE EXTREMA: EXAMPLE

Example. Let us determine the absolute extrema of $f(x) = 2x^3 - 15x^2 + 36x$ on the interval $[1, 5]$. We have

$$f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x - 2)(x - 3).$$

Therefore, the stationary points of $f(x)$ are $x = 2$ and $x = 3$. There are no points of non-differentiability, so we should compare the values of our function at the endpoints and the stationary points. We have

$$f(1) = 2 - 15 + 36 = 23,$$

$$f(2) = 16 - 60 + 72 = 28,$$

$$f(3) = 54 - 135 + 108 = 27,$$

$$f(5) = 250 - 375 + 180 = 55.$$

We conclude that the absolute maximum of f on $[1, 5]$ is 55, attained at $x = 5$, and the absolute minimum is 23, occurring at $x = 1$.

DERIVATIVES AND ROOTS: NEWTON'S METHOD

Theorem. (*Intermediate Value Theorem, particular case*) If a function f is continuous on the closed interval $[a, b]$, and the values $f(a)$ and $f(b)$ have the opposite signs, then the equation $f(x) = 0$ has a solution on $[a, b]$.

Note that now computing roots feels like a more “useful” task, since roots of the derivative f' , that is stationary points, are points where a relative extremum may occur. Once again, the intermediate value theorem is an existence theorem, and depending on how you prove it, there are more and less efficient methods to actually find a solution. One way to do so, applicable when f is differentiable, is called “Newton’s method”.

The idea behind Newton’s method is very simple: if we already have a rough estimate x_0 of our solution, take the tangent line at $(x_0, f(x_0))$, and find the point where it meets the x -axis. That point is our new estimate of the solution, and we continue in the same way.

DERIVATIVES AND ROOTS: NEWTON'S METHOD

To implement this method, we should look at the actual formulas. The tangent line has the equation $y - f(x_0) = f'(x_0)(x - x_0)$. It meets the x -axis at a point of the form $(x_1, 0)$ so $-f(x_0) = f'(x_0)(x_1 - x_0)$, and therefore

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Of course, when $f'(x_0) = 0$, the method would not work, since in that case the tangent line is parallel to the x -axis.

Iterating the process we just described, we obtain $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, and in general

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

for subsequent stages of the method.