1S11: Calculus for students in Science

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TCD

Lecture 24

Newton's method: reminder

The tangent line to the graph of f at the point $(x_0, f(x_0))$ has the equation $y - f(x_0) = f'(x_0)(x - x_0)$. It meets the x-axis at a point of the form $(x_1, 0)$ so $-f(x_0) = f'(x_0)(x_1 - x_0)$, and therefore

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Of course, when $f'(x_0) = 0$, the method would not work, since in that case the tangent line is parallel to the x-axis.

Iterating the process we just described, we obtain $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, and in general

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

for subsequent stages of the method.

Newton's method: example

Example. Let us look at the equation $x^3 - x - 1 = 0$. We have $f'(x) = 3x^2 - 1$, so our method suggests

$$x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}.$$

Starting from the initial approximation $x_0 = 1.5$, we get

$$x_1 = 1.5 - \frac{1.5^3 - 1.5 - 1}{3(1.5)^2 - 1} \approx 1.34782609,$$

$$x_2 = x_1 - \frac{x_1^3 - x_1 - 1}{3x_1^2 - 1} \approx 1.32520040,$$

$$x_3 \approx 1.32471817,$$

$$x_4 \approx 1.32471796,$$

$$x_5 \approx 1.32471796,$$

after which within our precision nothing changes. Therefore $x \approx 1.32471796$ is the approximation of our solution to eight decimal places.

Newton's method: applicability

It is important to have in mind that Newton's method may sometimes fail because of arriving at a point with a horisontal tangent line, or converge to a different root, overlooking the root you are aiming for, or simply not converge at all. Details on its applicability can be found in a book on numerical analysis, and we just conclude with one more example.

Let us take $f(x) = \sqrt[3]{x}$. Newton's method formulas in this case become

$$x_{n+1} = x_n - \frac{\sqrt[3]{x_n}}{\frac{1}{3\sqrt[3]{x_n^2}}} = x_n - 3x_n = -2x_n.$$

So, whichever nonzero value x_0 we start with, we get $x_n = (-2)^n x_0$ which of course gets larger and larger in magnitude, and does not converge.

Exponential and logarithmic functions

One type of functions which we have not discussed yet is the *exponential function* $f(x) = a^x$. If a is negative, this function can't be defined for (most) non-integer values. For positive a, however, it is possible to define it for all x, not just fractions (which we already did by putting $a^{\frac{m}{n}} = \sqrt[n]{a^m}$). This function satisfies the properties

$$a^{x_1+x_2} = a^{x_1}a^{x_2}, \quad a^{xy} = (a^x)^y.$$

Theorem. The function $f(x) = a^x$ is continuous everywhere. It is increasing for $a > 1$, and is decreasing for $0 < a < 1$.

We conclude that for positive $a \neq 1$ the function $f(x) = a^x$ admits an inverse function, which is continuous. It is denoted by $\log_a x$ and is called the logarithm with respect to base a; therefore $\log_a(a^x) = a^{\log_a x} = x$. The domain of $\log_a$ consists of all positive real numbers. This function satisfies the properties

$$\log_a(x_1x_2) = \log_a(x_1) + \log_a(x_2), \quad \log_a(x^s) = s\log_a x, \quad \log_a b = \frac{\log_c b}{\log_c a}.$$

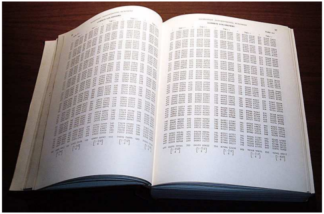
Exponential and logarithmic functions

Historically, logarithms used to be absolutely crucial for computations. Since

$$M \cdot N = a^{\log_a M} \cdot a^{\log_a N} = a^{\log_a M + \log_a N},$$

multiplying two numbers can be done by adding their logarithms, and then finding in the table of logarithms the number whose logarithm is equal to the result. Since addition is much faster than multiplication, it actually is a big improvement.Moreover, since $\log_a(x^{\frac{1}{n}}) = \frac{1}{n}\log_a x$, logarithms are equally useful for extracting roots.

Altogether, logarithms proved to be really indispensable for fast computations, and tables of logarithms used to be one of the main books that scientists and engineers would use in everyday life before computers and advanced calculators arrived.



Exponential and logarithmic functions

Definition. The number e is defined as the limit

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}.$$

(One can show that this limit exists and is approximately equal to 2.71828.) The logarithm $\log_e x$ is usually denoted by $\ln x$, and is called the *natural logarithm*.

Theorem. We have $\lim\limits_{x\to 0}\frac{e^x-1}{x}=1$.

Proof. Denoting $e^x-1=t$, we see that $x=\ln(1+t)$, so

$$\frac{e^x-1}{x}=\frac{t}{\ln(1+t)}=\frac{1}{\frac{1}{t}\ln(1+t)}=\frac{1}{\ln(1+t)^{\frac{1}{t}}},$$

so we conclude that the limit is equal to 1 because the exponential function and the logarithm function are continuous.

Exponential and logarithmic functions

Using the previous limit and properties of the exponential function, one can prove the following general result:

Theorem. The function $f(x) = a^x$ is differentiable everywhere. We have

$$(a^x)' = a^x \ln a,$$
$$(\log_a x)' = \frac{1}{x \ln a}.$$

In particular,

$$(e^x)' = e^x,$$
$$(\ln x)' = \frac{1}{x}.$$

For more complicated functions, it is often beneficial to write them as $f(x) = e^{\ln f(x)}$, and apply the chain rule. For example, let us compute the derivative of $f(x) = x^x$. This is not the kind of a function we know how to differentiate. Let us try the outlined approach.
We have

$$\ln f(x) = \ln(x^x) = x \ln x.$$

Therefore,

$$(x^x)' = (e^{x \ln x})' = e^{x \ln x}(x \ln x)' =$$

$$= e^{x \ln x}\left(\ln x + x \cdot \frac{1}{x}\right) = x^x(\ln x + 1).$$

Differential calculus: summary

- Derivative of f at x_0 is the limit $f'(x_0) := \lim\limits_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$. Alternatively, f has the derivative m at x_0, if
$$f(x) = f(x_0) + m(x - x_0) + \alpha(x),$$
where $\lim\limits_{x\to x_0} \frac{\alpha(x)}{x-x_0} = 0$. Informally, the line $y = f(x_0) + m(x - x_0)$ approximates $f(x)$ with error much less than $|x - x_0|$.
- Derivative can be used to locate intervals where f is increasing or decreasing, and relative maxima/minima.
- The second derivative can be used to locate intervals where f is concave up or down, and the inflection points.
- Derivatives are used in Newton's method for finding roots.
- There are standard ways to compute derivatives which you are expected to master: rules for differentiating arithmetic operations (sums, products, fractions), chain rule, implicit differentiation.

Integral calculus: antiderivatives

It is often important for applications to solve the reverse problem: assume that we know the derivative of a function, what can we say about the original function? For example, we see the instantaneous velocity on the speedometer all the time, and want to know how far we progressed over the given period of time.

Definition. A function F is called an *antiderivative* of the given function f if

$$\frac{dF(x)}{dx} = f(x).$$

Suppose that F_1 and F_2 are two antiderivatives of the same function f. Then

$$\frac{d(F_1(x) - F_2(x))}{dx} = \frac{dF_1(x)}{dx} - \frac{dF_2(x)}{dx} = f(x) - f(x) = 0,$$

which implies that $F_1(x) - F_2(x)$ is a constant c. Vice versa, if F is an antiderivative of f, then $G(x) = F(x) + c$ is also an antiderivative of f.