1S11: Calculus for students in Science

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TCD

Lecture 25

Antiderivatives: reminder

It is often important for applications to solve the reverse problem: assume that we know the derivative of a function, what can we say about the original function? For example, we see the instantaneous velocity on the speedometer all the time, and want to know how far we progressed over the given period of time.

Definition. A function F is called an *antiderivative* of the given function f if

$$\frac{dF(x)}{dx} = f(x).$$

Suppose that F_1 and F_2 are two antiderivatives of the same function f. Then

$$\frac{d(F_1(x) - F_2(x))}{dx} = \frac{dF_1(x)}{dx} - \frac{dF_2(x)}{dx} = f(x) - f(x) = 0,$$

which implies that $F_1(x) - F_2(x)$ is a constant c. Vice versa, if F is an antiderivative of f, then $G(x) = F(x) + c$ is also an antiderivative of f.

Integral calculus: antiderivatives

There is an alternative "integral" notation for antiderivatives:

$$\int f(x)\,dx = F(x) + C$$

denotes the fact that F is an antiderivative of f. The expression $\int f(x)\,dx$ on the left side of that equation is called the *indefinite integral*.

Examples. Some examples of indefinite integrals are shown below:

derivative formula	equivalent integration formula
$(x^2)' = 2x$	$\int 2x\,dx = x^2 + C$
$(\sin x)' = \cos x$	$\int \cos x\,dx = \sin x + C$
$(\tan x) = \frac{1}{\cos^2 x}$	$\int \frac{dx}{\cos^2 x} = \tan x + C$
$(\ln x)' = \frac{1}{x}$	$\int \frac{dx}{x} = \ln x + C$
$\left(\frac{x^{n+1}}{n+1}\right)' = x^n$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

Rules for antiderivatives

Theorem. Suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$ respectively, and that c is a constant. Then

- $\int cf(x)\,dx = cF(x) + C$,
- $\int[f(x) + g(x)]\,dx = F(x) + G(x) + C$,
- $\int[f(x) - g(x)]\,dx = F(x) - G(x) + C$.

Proof. Just use rules for differentiation. For example,
$(F(x) - G(x))' = F'(x) - G'(x) = f(x) - g(x)$.

Another way to write the same is

- $\int cf(x)\,dx = c\int f(x)\,dx$,
- $\int[f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$,
- $\int[f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$.

Rules for antiderivatives

These rules should be handled with care, leaving the arbitrary constant till the very end of the computation, so that we do not write anything like that

$$x^2 + C = \int 2x\,dx = 2\int x\,dx = 2\left(\frac{x^2}{2} + C\right) = x^2 + 2C,$$

concluding then that $C = 2C$, so $C = 0$.

Of course, there is a more general rule which combines addition and multiplication by constants:

$$\int [c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)]\,dx =$$

$$= c_1 \int f_1(x)\,dx + c_2 \int f_2(x)\,dx + \cdots + c_n \int f_n(x)\,dx.$$

Rules for antiderivatives

Theorem. (u-substitution) Suppose that $F(x)$ is an antiderivative of $f(x)$. Then

$$\int [f(g(x))g'(x)]\,dx = F(g(x)) + C.$$

Proof. Applying the chain rule, we obtain

$$(F(g(x)))' = F'(g(x))g'(x) = f(g(x))g'(x),$$

as required.

Example. Let us evaluate the integral

$$\int (x^2+1)^{50} \cdot 2x\,dx.$$

Let $g(x) = x^2 + 1$. Noticing that $(x^2+1)' = 2x$, we can write the integral as

$$\int (x^2+1)^{50} \cdot 2x\,dx = \int g(x)^{50} g'(x)\,dx = \frac{g(x)^{51}}{51} + C = \frac{(x^2+1)^{51}}{51} + C.$$

u-SUBSTITUTION: MORE EXAMPLES

Example. Let us evaluate the integral

$$\int \frac{\cos x}{\sin^2 x}\,dx.$$

Denoting $g(x) = \sin x$, we get

$$\int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{1}{g(x)^2} g'(x)\,dx = -\frac{1}{g(x)} + C = -\frac{1}{\sin x} + C.$$

Example. Let us evaluate the integral

$$\int \frac{\cos \sqrt{x}}{\sqrt{x}}\,dx.$$

Denoting $g(x) = \sqrt{x}$, we get

$$\int \frac{\cos \sqrt{x}}{\sqrt{x}}\,dx = \int \cos g(x) \cdot 2g'(x)\,dx = 2\sin g(x) + C = 2\sin \sqrt{x} + C.$$

Rules for antiderivatives

We already know that differentiating products is not as easy as differentiating sums. Same is true for antiderivatives.

Theorem. (Integration by parts) Suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$ respectively. Then

$$\int [f(x)G(x)]\,dx + \int [F(x)g(x)]\,dx = F(x)G(x) + C.$$

Proof. Applying the product rule, we obtain

$$(F(x)G(x))' = F'(x)G(x) + F(x)G'(x) = f(x)G(x) + F(x)g(x),$$

as required.

This rule is usually applied in the following way:

$$\int [f(x)G(x)]\,dx = F(x)G(x) - \int [F(x)g(x)]\,dx,$$

meaning that computing the antiderivative of $f(x)G(x)$ can be reduced to computing the antiderivative of $F(x)g(x)$, and the latter one can be substantially simpler to compute.

Integration by parts: examples

Example. Let us evaluate $\int xe^x\,dx$. Using integration by parts with $G(x) = x$, $f(x) = e^x$ (in which case $g(x) = 1$, and we can take $F(x) = e^x$), we get

$$\int xe^x\,dx = xe^x - \int e^x \cdot 1\,dx = xe^x - e^x + C.$$

Note that if we instead apply integration by parts with $f(x) = x$, $G(x) = e^x$ (so that $F(x) = \frac{x^2}{2}$, $g(x) = e^x$), we get

$$\int xe^x\,dx = \frac{x^2}{2}e^x - \int \frac{x^2}{2}e^x\,dx,$$

and our computation expressed the given integral using what is a more complicated indefinite integral!

Integration by parts: examples

Example. Let us evaluate $\int x^2 e^x \, dx$. Using integration by parts with $G(x) = x^2$, $f(x) = e^x$ (in which case $g(x) = 2x$, and we can take $F(x) = e^x$), we get

$$\int x^2 e^x \, dx = x^2 e^x - \int 2xe^x \, dx = x^2 e^x - 2xe^x + 2e^x + C,$$

since we already know $\int xe^x \, dx$.

Example. Let us evaluate $\int \ln x \, dx$. Using integration by parts with $G(x) = \ln x$, $f(x) = 1$ (in which case $g(x) = \frac{1}{x}$, and we can take $F(x) = x$), we get

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - x + C.$$

Integration by parts: examples

Example. Let us evaluate $\int e^x \sin x\,dx$. Using integration by parts with $G(x) = \sin x$, $f(x) = e^x$ (in which case $g(x) = \cos x$, and we can take $F(x) = e^x$), we get

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx.$$

Now we use integration by parts with $G(x) = \cos x$, $f(x) = e^x$ (in which case $g(x) = -\sin x$, and we can take $F(x) = e^x$), so that we get

$$\int e^x \cos x\,dx = e^x \cos x + \int e^x \sin x\,dx,$$

so that $\int e^x \sin x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$, and

$$\int e^x \sin x\,dx = \frac{1}{2}(e^x \sin x - e^x \cos x) + C.$$