1S11: Calculus for students in Science

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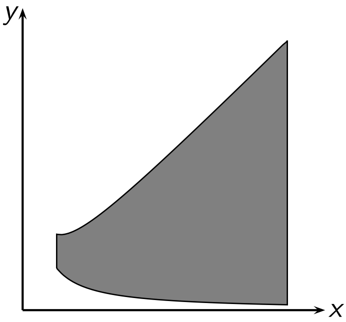
TCD

Lecture 31

Outline of the remaining lectures

- today: applications of definite integrals in geometry (areas, volumes, lengths);
- tomorrow: applications of definite integrals in physics and engineering (work, energy, center of gravity);
- Thursday: practise at home with sample exam problems (will be on the course webpage from Wednesday; answers will be made available online next week).

Area between two curves

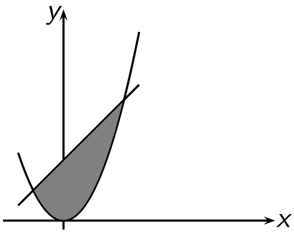


Suppose that f and g are continuous functions on $[a, b]$, and $f(x) > g(x)$ for all x in $[a, b]$. Then the area of the region bounded by these graphs and the lines $x = a$, $x = b$, is equal to

$$\int_a^b (f(x) - g(x))\,dx.$$

Area between two curves

If we have $f(a) = g(a)$ and / or $f(b) = g(b)$, the formula for the area is also applicable. For example, let us find the area of the region enclosed between the parabola $y = x^2$ and the line $y = x + 2$.

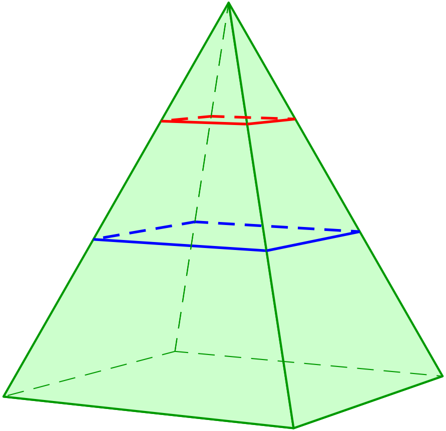


Solving the equation $x^2 = x + 2$, we get $x = 2$ and $x = -1$, so the points $(2, 4)$ and $(-1, 1)$ are the points where these curves meet. The area in question is equal to the integral

$$\int_{-1}^{2} (x + 2 - x^2)\, dx = \frac{x^2}{2} + 2x - \frac{x^3}{3}\Big]_{-1}^{2} = 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3} = 4.5.$$

Volumes by slicing

To compute volumes of solids, we can slice the solids, and integrate the area of the slice with respect to the position of the slice.



In this case, it is beneficial to make horisontal slices; the area of each slice is proportional to the square of the distance to the top, and we recover the formula $V = \frac{1}{3}hS$ for the volume of a pyramid.

Solids of revolution

This method is especially efficient in the cases of solids of revolution: if f is positive on $[a, b]$, then the volume of the solid bounded by the surface of revolution of the graph of f about the x-axis is equal to

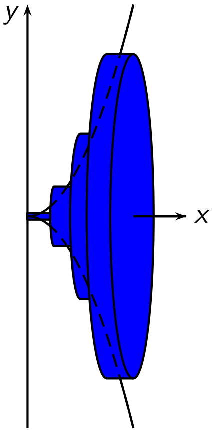
$$\pi \int_a^b [f(x)]^2 \, dx,$$

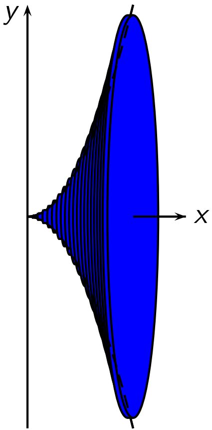
since the slice at the point x is a circle of radius $[f(x)]^2$.

If a solid of revolution has hollow spaces, that is obtained by rotating the region between the graphs of f and g about the x-axis, then the formula for the respective volume becomes

$$\pi \int_a^b ([f(x)]^2 - [g(x)]^2) \, dx.$$

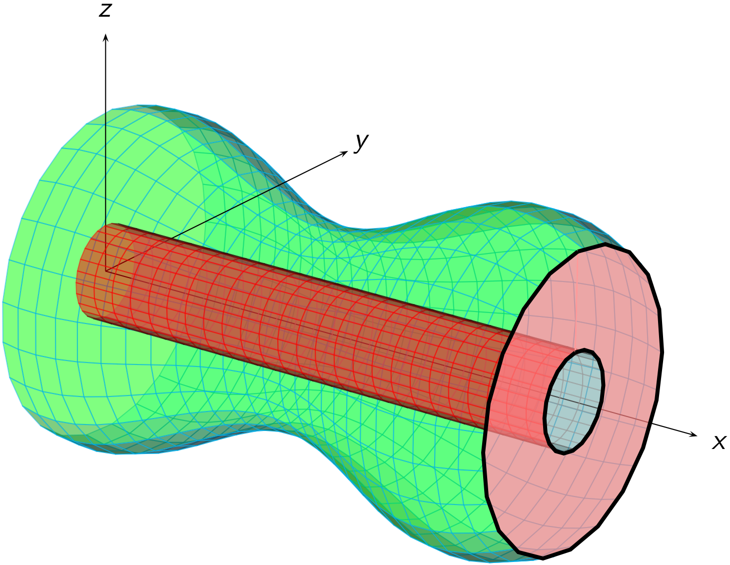
Solids of revolution





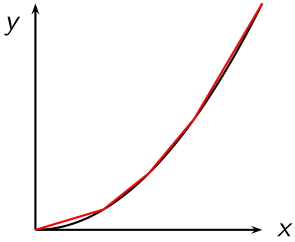
For the graph $y = x^2$ on $[0,1]$ we compute the volume of the solid of revolution as $\pi \int_0^1 (x^2)^2\, dx = \pi \int_0^1 x^4\, dx = \frac{\pi}{5}$.

Solids of revolution



Arc lengths

To compute the arc length of the plane curve $y = f(x)$, one approximates that curve by polygons:



The total length of this polygon approximation is equal to the sum of lengths of segments:

$$\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \Delta x_k \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \rightsquigarrow \sqrt{1 + (f'(x))^2}\, dx.$$

Arc lengths

Theorem. The arc length of the curve $y = f(x)$ on $[a, b]$ is equal to

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx,$$

provided $f'(x)$ is continuous on $[a, b]$.
Example. Let $f(x) = x^{3/2}$ on $[1, 2]$. Since $f'(x) = \frac{3}{2}\sqrt{x}$, the arc length of the curve $y = x^{3/2}$ on $[1, 2]$ is equal to

$$\int_1^2 \sqrt{1 + \frac{9}{4}x}\, dx \overset{u=1+\frac{9}{4}x,\, du=\frac{9}{4}\, dx,\, u(1)=\frac{13}{4},\, u(2)=\frac{22}{4}}{=}$$

$$= \frac{4}{9}\int_{13/4}^{22/4} u^{1/2} du = \frac{8}{27}u^{3/2}\Big]_{13/4}^{22/4} = \frac{22\sqrt{22} - 13\sqrt{13}}{27}.$$

Arc lengths

Example. Let $f(x) = \sqrt{1-x^2}$, so that the curve in question is the half-circle $x^2 + y^2 = 1$, $y \geq 0$. Then $f'(x) = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = -\frac{x}{\sqrt{1-x^2}}$. This means that our formula

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

for the arc length would make sense when $-1 < a < b < 1$, since at $x = \pm 1$ the derivative is not continuous. For such a and b, we get

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_a^b \sqrt{1 + \frac{x^2}{1-x^2}}\, dx = \int_a^b \frac{1}{\sqrt{1-x^2}}\, dx.$$

Arc lengths

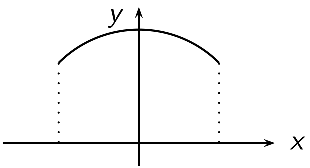
Therefore, the arc length of the half-circle between $x = a$ and $x = b$ is

$$\int_a^b \frac{1}{\sqrt{1-x^2}}\,dx.$$

Recall (Q5 in tutorial 6) that $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$, so we can use the Fundamental Theorem of Calculus to conclude that the arc length in question is $\sin^{-1} b - \sin^{-1} a$.

For example, for the quarter circle, we may take $a = -\frac{1}{\sqrt{2}}$, $b = \frac{1}{\sqrt{2}}$, and the answer is

$$\frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.$$



Solids of revolution revisited

Our approach for computing arc lengths can be easily adapted to compute the area of a surface of revolution. Namely, if f is positive on $[a, b]$ and $f'(x)$ is continuous on $[a, b]$, then the area of the surface of revolution of the graph of f about the x-axis is equal to

$$2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx,$$

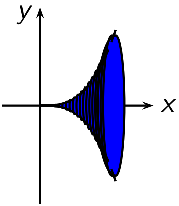
since we rotate the element of length $\sqrt{(\Delta x_k)^2+(\Delta y_k)^2}$ around the circle of radius approximately equal to y_k, thus creating the element of area

$$2\pi y_k\sqrt{(\Delta x_k)^2+(\Delta y_k)^2} =$$

$$= 2\pi y_k \Delta x_k\sqrt{1+\left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \rightsquigarrow 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx.$$

Solids of revolution revisited

Let us consider the example of $y = x^3$ on $[0, 1]$.



In this case, we have to compute the quantity

$$2\pi \int_0^1 x^3 \sqrt{1+(3x^2)^2}\,dx =$$

$$= 2\pi \int_0^1 x^3 \sqrt{1+9x^4}\,dx \overset{u=1+9x^4,\, du=36x^3\,dx,\, u(0)=1,\, u(1)=10}{=}$$

$$= \frac{2\pi}{36} \int_1^{10} \sqrt{u}\,du = \frac{2\pi}{36} \cdot \frac{2}{3} u^{3/2} \Big]_1^{10} = \frac{\pi}{27}(10\sqrt{10} - 1).$$