1S11: Calculus for students in Science

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TCD

Lecture 5

Discussion of the first tutorial

Using the factorisation $x^2 - x - 6 = (x+2)(x-3)$, find the natural domain of $\sqrt{x^2 - x - 6}$.

Solution. For the square root to make sense in real numbers, the value $x^2 - x - 6$ must be nonnegative, which happens when either both factors are nonnegative or both factors are nonpositive:

$$x + 2 \geq 0, x - 3 \geq 0,$$
$$x + 2 \leq 0, x - 3 \leq 0,$$

in other words $x \leq -2$ or $x \geq 3$. Answer: $(-\infty, -2] \cup [3, +\infty)$.

Explain why the domain of $\sqrt{x+2}\sqrt{x-3}$ is different from that of $\sqrt{x^2-x-6}$.

Solution. For $\sqrt{x+2}\sqrt{x-3}$ to be defined, both factors have to be defined, so in this case the domain is $[3,+\infty)$.

Discussion of the first tutorial

Plot the graph of the function

$$\mathrm{sign}(x) := \frac{x}{|x|},$$

and determine the natural domain and the range of this function.

A couple of remarks before we start solving this one. First, the symbol $:=$ above means "by definition", that is we define the function sign by that formula. Second, similarly to how the name of the independent variable is something you can choose, it applies to some extent to names of functions. You already know functions sin (sine), cos (cosine), tan (tangent) etc., and now we define this new function sign (sign), but we could also call it f, or α, or $\diamond$, or even $\mathrm{BlackPudding}$, so that we would write

$$\mathrm{BlackPudding}(x) := \frac{x}{|x|}$$

which is not only too silly but also too long.

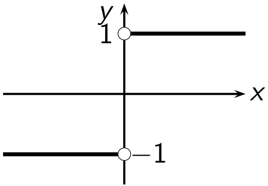
Discussion of the first tutorial

Plot the graph of the function

$$\mathrm{sign}(x) := \frac{x}{|x|},$$

and determine the natural domain and the range of this function.

Solution. Since $|x|$ is equal to x for positive x and to $-x$ for negative x, we get the graph

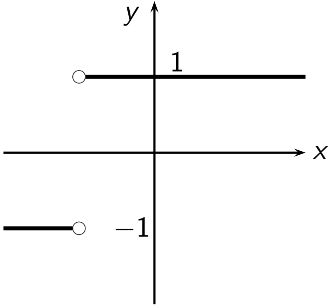


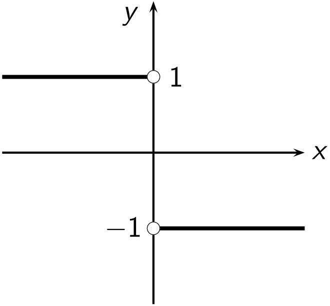
The domain of this function consists of all nonzero x (since it is only undefined when the denominator is equal to zero). The range consists of two numbers, 1 and -1.

Discussion of the first tutorial

Plot the graphs of $\operatorname{sign}(x+1)$ and of $\operatorname{sign}(-x)$.

Solution. Since adding 1 to the independent variable shifts graphs by 1 to the left, and multiplying by -1 reflects about the vertical axis, we get the following graphs:





Discussion of the first tutorial

What is the domain of $f \circ g \circ h$, if $f(x) = 1 - x$, $g(x) = \frac{1}{x}$, and $h(x) = x^2 + 1$?

Solution. The function $h(x)$ is defined for all x, and assumes positive values, since x^2 is nonnegative for all x. Thus, $g \circ h$ is defined for all x. Finally, since f is defined everywhere, the composition $f \circ g \circ h$ is defined everywhere. Answer: $(-\infty, +\infty)$.

Classical types of functions

Definition. A *polynomial* is a function which is a sum of finitely many power terms cx^n, where c is a real number, and n is a nonnegative integer. For example,

$$2x-1, \quad 5x^3+x-\sqrt{3}, \quad x^2, \quad 3, \quad x^7-x^3-2.$$

The function $(x-1)(x-2)^2$ is also a polynomial since we can perform the multiplication and collect similar terms:

$$(x-1)(x-2)^2 = (x-1)(x^2-4x+4) =$$
$$= x^3-4x^2+4x-x^2+4x-4 = x^3-5x^2+8x-4.$$

Classical types of functions

A general polynomial has the form

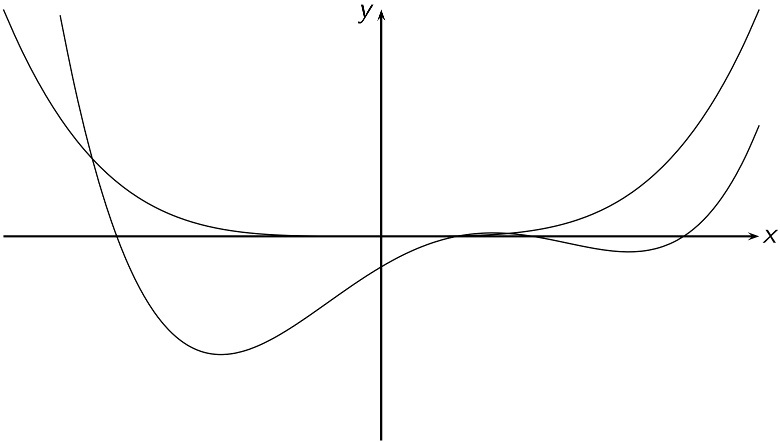
$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0,$$

which is the same as

$$c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} + c_n x^n.$$

The numbers c_0, c_1, ..., c_n are called the *coefficients* of the polynomial. The highest power n occuring with a nonzero coefficient is called the *degree* of the polynomial. A polynomial of degree 0 is a constant $c = cx^0$. Polynomials of degrees $1, 2, 3, 4, 5$ are referred to as *linear*, *quadratic*, *cubic*, *quartic*, *quintic* polynomials respectively.

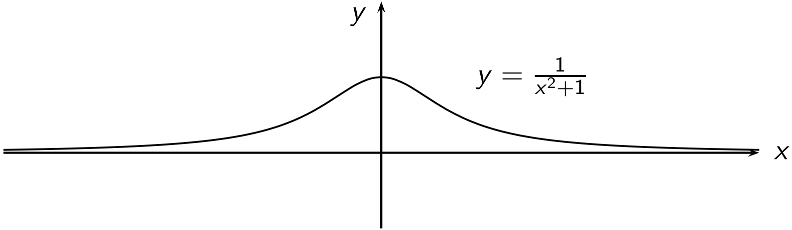
Classical types of functions



For large $|x|$ there is little difference between the graphs of functions $c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$ and just $c_n x^n$: lower terms get negligible.

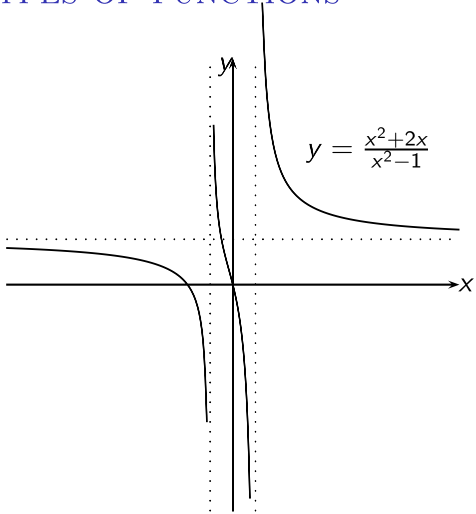
Classical types of functions

Definition. A *rational function* is a ratio of two polynomials $\frac{p(x)}{q(x)}$, for example $\frac{1}{x^2+1}$, $\frac{x^2+2x}{x^2-1}$, $\frac{x^2+1}{x}$.
Rational functions are undefined where the denominator vanishes. In such cases, the corresponding graphs have *vertical asymptotes* (vertical lines that they closely approximate). Also, rational functions often have *horisontal asymptotes*, although that's not always the case.



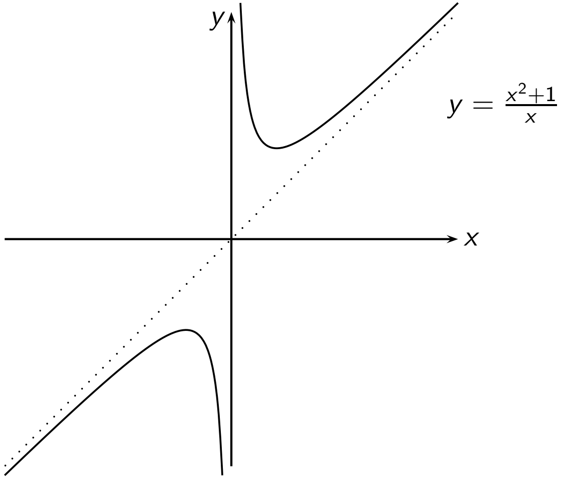
The graph of $\frac{1}{x^2+1}$ has a horisontal asymptote, and no vertical asymptotes.

Classical types of functions



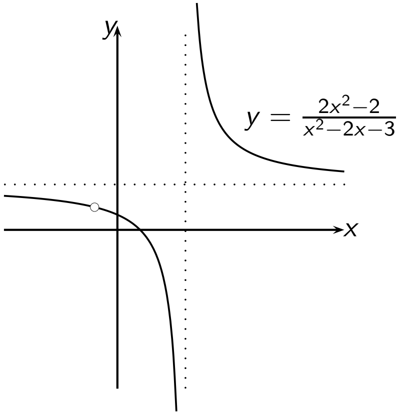
The graph of $\frac{x^2+2x}{x^2-1}$ has vertical asymptotes $x = 1$ and $x = -1$, and a horisontal asymptote $y = 1$. To see the latter, write $\frac{x^2+2x}{x^2-1} = 1 + \frac{2x+1}{x^2-1}$.

Classical types of functions



The graph of $\frac{x^2+1}{x}$ has a vertical asymptotes $x = 0$, and no horisontal asymptotes. It however has another straight line asymptote $y = x$, because $\frac{x^2+1}{x} = x + \frac{1}{x}$.

Classical types of functions



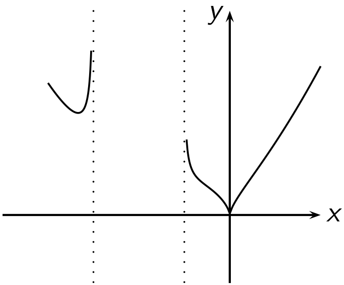
The graph of $\frac{2x^2-2}{x^2-2x-3}$ has a vertical asymptote $x = 3$, a horisontal asymptote $y = 2$, and also the point $(-1, 1)$ which it approaches both on the left and on the right but does not touch. Indeed, for $x \neq -1$ we have

$$\frac{2x^2 - 2}{x^2 - 2x - 3} = \frac{2(x-1)(x+1)}{(x+1)(x-3)} = \frac{2x-2}{x-3} = 2 + \frac{4}{x-3}.$$

Classical types of functions

Algebraic functions are built from the four arithmetic operations, and extracting roots.

Algebraic functions are too varied in behaviour to make any general statements. They are only defined for some values, their graphs may have corners etc.



$$y = \frac{\sqrt[3]{x^2}(x+2)^2}{\sqrt{x^2+4x+3}}$$