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The Entringer-Poupard Matrix Sequence

Dominique Foata and Guo-Niu Han

Abstract. The so-called Entringer-Poupard matrices naturally occur when the distribution of the statistical pair ("last letter", "greater neighbor of maximum") is under study on the set of alternating permutations. Closed formulas for three-variate generating functions are derived. An extension to the ℓ-tangent numbers provides similar results.

1. Introduction

The first aim of this paper is to construct a well-defined sequence of matrices $(A_n = (a_n(m,k))_{(1\le m,k\le n)})$ $(n\ge 1)$ with integral entries, called the *Entringer-Poupard matrix sequence*, which provides a *matrix-analog refinement* $\sum_{m,k} a_n(m,k) = E_n$ of the tangent/secant numbers, in such a way that the row and column sums $\sum_k a_n(m,k)$ and $\sum_m a_n(m,k)$ are themselves *Poupard* and *Entringer numbers*, respectively. The sequence (A_n) is defined by a system of *partial finite difference equations* and, moreover, the three-variate generating function for the entries $a_n(m,k)$ of the matrices A_n can be explicitly evaluated. See §§1.3–1.4 for definitions and results.

This characterization of the Entringer-Poupard matrix sequence completes the program initiated in our previous papers, where matrix-analog refinements of the tangent and secant numbers have been found having the property that *both* row and column sums were equal to Poupard numbers as in [FH14] and [FH14a], and to Entringer numbers as done in [FH14b] and [FH15]. There remains to say something relevant when both Entringer and Poupard numbers are involved.

The second aim is to work out two linear refinements $E_{2n+1}^{(\ell)} = \sum_k f_{n+1}^{\ell,0}(k) = \sum_k f_{n+1}^{\ell,1}(k)$ of the so-called *ℓ-tangent numbers*. The latter numbers occur naturally when the initial conditions in the definitions of the Entringer and Poupard numbers are slightly modified. See §1.1–1.2 for all details. It will be convenient to discuss the latter refinements of those ℓ-tangent numbers, before introducing the combinatorial and analytical aspects of this Entringer-Poupard Matrix Sequence.

Key words and phrases. Entringer numbers, Poupard numbers, tangent numbers, secant numbers, linear refinement, Entringer-Poupard Matrix Sequence, matrix-analog refinement, alternating permutations, three-variate generating function calculus, ℓ-tangent numbers, Seidel Triangle Sequence, Dual Seidel Triangle sequence, formal Laplace transform.

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1.1. *The ℓ-tangent numbers.* For each positive integer ℓ the coefficients $E_{2n+1}^{(\ell)}$ $(n\ge 0)$, further called *ℓ-tangent numbers*, appearing in the Taylor expansion of the function

$$\sec^{\ell-1}(x)\,\tan x=\sum_{n\ge 0}E_{2n+1}^{(\ell)}\frac{x^{2n+1}}{(2n+1)!}\tag{1.1}$$

are all *positive integers*, since it is the case for the *tangent numbers* E_{2n+1} and *secant numbers* E_{2n} $(n\ge 0)$, which occur in the Taylor expansions of

$$\tan u=\sum_{n\ge 0}\frac{u^{2n+1}}{(2n+1)!}E_{2n+1}=\frac{u}{1!}1+\frac{u^3}{3!}2+\frac{u^5}{5!}16+\frac{u^7}{7!}272+\frac{u^9}{9!}7936+\cdots$$

$$\sec u=\sum_{n\ge 0}\frac{u^{2n}}{(2n)!}E_{2n}=1+\frac{u^2}{2!}1+\frac{u^4}{4!}5+\frac{u^6}{6!}61+\frac{u^8}{8!}1385+\frac{u^{10}}{10!}50521+\cdots$$

See, e.g., [Ni23, p. 177-178], [Co74, p. 258-259] for the last two expansions.

Note that $E_{2n+1}^{(1)}=E_{2n+1}$ (tangent number). Also, $E_{2n+1}^{(2)}=E_{2n+2}$ (secant number) $(n\ge 0)$, since $(d/dx)\sec x=\sec x\tan x$. Those two sequences of numbers are registered as A000182 and A000364, respectively, in Sloane's Encyclopedia [OEIS]. The numbers $E_{2n+1}^{(3)}$ also occur therein under no. A024283, as coefficients of the Taylor expansion of $\frac12\tan^2 x$, whose $\sec^2(x)\,\tan x$ is the derivative.

Before stating the first result of this paper let us introduce the following notation. Consider a triangle of numbers

$$f=\begin{matrix} & & & & f_1(1) & & & & \\ & & & f_2(1) & f_2(2) & f_2(3) & & & \\ & & f_3(1) & f_3(2) & f_3(3) & f_3(4) & f_3(5) & & \\ & f_4(1) & f_4(2) & f_4(3) & f_4(4) & f_4(5) & f_4(6) & f_4(7) & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\end{matrix}\tag{1.2}$$

and define its *left* and *right sides* to be the sequences

$$(f_1(1),f_2(1),\ldots,f_n(1),\ldots)\quad\text{and}\quad(f_1(1),f_2(3),\ldots,f_n(2n-1),\ldots),$$

respectively, and the generating function for f to be the *even* series in x and y

$$\Gamma^f(x,y):=\sum_{n\ge 0}\ \sum_{1\le m\le 2n+1}f_{n+1}(m)\frac{x^{2n+1-m}}{(2n+1-m)!}\frac{y^{m-1}}{(m-1)!}.\tag{1.3}$$

This implies that the generating functions for the left and right sides of f are simply the *even* series:

$$\Gamma^f(x,0):=\sum_{n\ge 0}f_{n+1}(1)\frac{x^{2n}}{(2n)!};\quad \Gamma^f(0,x):=\sum_{n\ge 0}f_{n+1}(2n+1)\frac{x^{2n}}{(2n)!}.\tag{1.4}$$

Definition 1.1. For each $\ell \geq 1$ let $f^{\ell,0} = (f_n^{\ell,0}(m))$ and $f^{\ell,1} = (f_n^{\ell,1}(m))$ $(n \geq 1,\ 1 \leq m \leq 2n-1)$ be the two triangles of numbers defined by the following axioms:

(1) for $a = 0,1$ the entries $f_n^{\ell,a}(m)$ satisfy the second-order finite difference equation system

$$\Delta^2 f_n^{\ell,a}(m) + (a+1)^2 f_{n-1}^{\ell,a}(m) = 0 \quad (n \geq 2,\, 1 \leq m \leq 2n-3), \tag{1.5}$$

where Δ stands for the classical finite difference operator (see, e.g. [Jo39]): $\Delta f_n^{\ell,a}(m) := f_n^{\ell,a}(m+1) - f_n^{\ell,a}(m)$;

(2) for $a = 0,1$ the generating functions for the left and right sides of $f^{\ell,a}$ are equal to $\Gamma f^{\ell,a}(x,0) = \sec^\ell(x)\cos(ax)$ and $\Gamma f^{\ell,a}(0,x) = \sec^\ell(x)\cos x$.

It is straightforward to verify that the two triangles $f^{\ell,0}$ and $f^{\ell,1}$ are uniquely defined, their entries $f_n^{\ell,a}(m)$ $(a = 0,1)$ being inductively derived by means of those two axioms. Furthermore, those entries are all *non-negative integers.*

Theorem 1.1. *For each $\ell \geq 1$ let $f^{\ell,0}$, $f^{\ell,1}$ be the two triangles introduced in Definition* 1.1*. Then, the generating function for $f^{\ell,a}$, as defined in* (1.3)*, is equal to:*

$$\Gamma f^{\ell,a}(x,y) = \sec^\ell(x+y)\cos(ax-y) \quad (a = 0,1). \tag{1.6}$$

Moreover, let $(E_{2n+1}^{(\ell)})$ $(n \geq 0)$ be the sequence of all ℓ-tangent numbers defined by (1.1)*. Then, each $E_{2n+1}^{(\ell)}$ $(n \geq 0)$ can be expressed as the two sum refinements*

$$E_{2n+1}^{(\ell)} = \sum_{1 \leq m \leq 2n+1} f_{n+1}^{\ell,0}(m) = \sum_{1 \leq m \leq 2n+1} f_{n+1}^{\ell,1}(m) \quad (n \geq 0), \tag{1.7}$$

The displays of the two triangles $f^{\ell,0}$ and $f^{\ell,1}$ for $\ell = 1,2,3,4$ are reproduced in Appendix I. The proof of Theorem 1.1 will be given in Section 2.

1.2. *Entringer and Poupard numbers.* The triangles $f^{1,0}$ and $f^{2,0}$ will be further denoted by $E^{(\tan)} = (E_{2n-1}(k))$ and $E^{(\sec)} = (E_{2n}(k))$, instead of $f_n^{1,0} = (f_n^{1,0}(k))$ and $f_n^{2,0} = (f_n^{2,0}(k))$ $(1 \leq k \leq 2n-1)$, respectively. Their entries are called *Entringer numbers.*

Likewise, the triangles $f^{1,1}$, $f^{2,1}$ are further denoted by $P^{(\tan)} = (P_{2n-1}(k))$ and $P^{(\sec)} = (P_{2n}(k))$, instead of $f_n^{1,1} = (f_n^{1,1}(k))$ and $f_n^{2,1} = (f_n^{2,1}(k))$ $(1 \leq k \leq 2n-1)$ $(1 \leq k \leq 2n-1)$, respectively. Their entries are called *Poupard numbers.* The first values of both Entringer and Poupard numbers are reproduced in Table 1.1.

$E^{(\tan)} =$	E_{2n-1}	$= P^{(\tan)}$
1	1	1
1 1 0	2	0 2 0
5 5 4 2 0	16	0 4 8 4 0
61 61 56 46 32 16 0	272	0 32 64 80 64 32 0
⋯	⋯	⋯

$E^{(\sec)} =$	E_{2n}	$= P^{(\sec)}$
1	1	1
2 2 1	5	1 3 1
16 16 14 10 5	61	5 15 21 15 5
272 272 256 224 178 122 61	1385	61 183 285 327 285 183 61
⋯	⋯	⋯

Table 1.1. The Entringer and Poupard Numbers

The *Entringer numbers* are traditionally defined by a *first*-order difference equation system. See, e.g., Sloane's Encyclopedia of integers [OEIS], where they are registered as the A008282 sequence. The *Poupard numbers* are registered as the A236934 and A125053 sequences, respectively, in that Encyclopedia. A full study of those latter two sequences was made in our previous paper [FH13].

Following Definition 1.1 the Entringer numbers $E_n(k)$ are defined by

$$(1.8)\qquad \Delta^2 E_n(k) + E_{n-2}(k) = 0 \quad \begin{cases} (1 \le k \le n-1) & \text{for } n \text{ odd;} \\ (1 \le k \le n-2) & \text{for } n \text{ even;} \end{cases}$$

with the initial conditions:

$$(1.9)\qquad E_n(1) = E_{n-1}\ (n \ge 1);$$
$$E_n(n) = 0\ (n \text{ odd} \ge 3);\quad E_n(n-1) = E_{n-2}\ (n \text{ even} \ge 2);$$

and the Poupard numbers $P_n(k)$ by:

$$(1.10)\quad \Delta^2 P_n(k) + 4\,P_{n-2}(k) = 0 \quad \begin{cases} (1 \le k \le n-1) & \text{for } n \text{ odd;} \\ (1 \le k \le n-2) & \text{for } n \text{ even;} \end{cases}$$

with the initial conditions:

$$(1.11)\qquad P_1(1) = 1;\quad P_n(1) = P_n(n) = 0\ (n \text{ odd} \ge 3);$$
$$P_n(1) = P_n(n-1) = E_{n-2}\ (n \text{ even} \ge 2).$$

Theorem 1.1 implies that those numbers provide linear refinements of tangent and secant numbers: $E_{2n-1} = \sum_k E_{2n-1}(k) = \sum_k P_{2n-1}(k)$ and $E_{2n} = \sum_k E_{2n}(k) = \sum_k P_{2n}(k)$ $(1 \le k \le 2n-1)$, as can be seen in Table 1.1 where the row sums have been written between triangles. Those linear refinements have been proved combinatorially by Entringer himself [En66], for the Entringer numbers, and by Christiane Poupard [Po82] for the Poupard numbers, only for E_{2n-1}, the even case being completed in our previous paper [FH13].

Theorem 1.1 also provides a global approach for the calculations of the generating functions for those four triangles by specializing (1.6) for $(\ell, a) = (1,0), (2,0), (1,1)$ and (2,1):

$$\Gamma^{E^{(\tan)}}(x,y) = \sec(x+y)\cos(y); \tag{1.12}$$

$$\Gamma^{E^{(\sec)}}(x,y) = \sec^2(x+y)\cos(y); \tag{1.13}$$

$$\Gamma^{P^{(\tan)}}(x,y) = \sec(x+y)\cos(x-y); \tag{1.14}$$

$$\Gamma^{P^{(\sec)}}(x,y) = \sec^2(x+y)\cos(x-y); \tag{1.15}$$

where each left-hand side is to be normalized as shown in (1.3). Thus, (1.13) is to be rewritten as:

$$\Gamma^{E^{(\sec)}}(x,y) = \sum_{1\le m\le 2n-1} E_{2n}(m)\frac{x^{2n-1-m}}{(2n-1-m)!}\frac{y^{m-1}}{(m-1)!} \tag{1.16}$$
$$= \sec^2(x+y)\cos(y),$$

which is the derivative with respect of x of the formula

$$\sum_{1\le m\le 2n} E_{2n}(m)\frac{x^{2n-m}}{(2n-m)!}\frac{y^{m-1}}{(m-1)!} = \sec(x+y)\sin x, \tag{1.17}$$

already derived in our previous papers [FH14, FH15].

Note that the calculation of (1.14) (resp of (1.12), (1.13) and (1.15)) has already been made, using different methods, by Christiane Poupard [Po82] (resp. in our previous papers [FH13], [FH14].)

1.3. *The Entringer-Poupard Matrix Sequence.* The matrix-analog of the tangent or secant number we look for is now constructed by getting back to Désiré André's old model [An1879, An1881] of *alternating permutations*, a model still thoroughly studied (see, e.g., [GHZ11], [KPP94], [MSY96], [St10]), and by calculating the joint distribution of a pair of statistics $(\mathbf{grn}, \mathbf{L})$ attached to each permutation.

Recall that each permutation $w = x_1x_2\cdots x_n$ of $12\cdots n$ is said to be *alternating* if $x_1 < x_2$, $x_2 > x_3$), $x_3 < x_4$, etc. in an alternating way. For each $n \ge 1$ let Alt_n denote the set of all alternating permutations of $12\cdots n$. Désiré André proved that $\#\mathrm{Alt}_n = E_n$. As for the statistics "**L**" and "**grn**" in question, $\mathbf{L}\,w$ is the *last letter* x_n of the permutation w and $\mathbf{grn}\,w$ the **gr**eater **n**eighbor of the maximum in w, defined as follows.

Definition 1.2. Let $w = x_1x_2\cdots x_n$ be a permutation of $12\cdots n$, so that $x_i = n$ for a certain i $(1 \le i \le n)$. By convention, let $x_0 = x_{n+1} := 0$. Define the **gr***eater* **n***eighbor of* n *in* w to be

$$\mathbf{grn}\,w := \max\{x_{i-1}, x_{i+1}\}. \tag{1.18}$$

Let

$$\mathrm{Alt}_n(m,k) := \{w \in \mathrm{Alt}_n : \mathbf{grn}\, w = m,\ \mathbf{L}\, w = k\}; \tag{1.19}$$

$$a_n(m,k) := \#\mathrm{Alt}_n(m,k); \tag{1.20}$$

and for each $n \geq 2$ let $A_n = (a_n(m,k))_{(1\leq m,k\leq n)}$ be the $n \times n$-matrix, whose entries are the non-negative integers $a_n(m,k)$. Call *Entringer-Poupard Matrix Sequence* the sequence (A_n) $(n \geq 2)$. The table of the first matrices A_n of that sequence is reproduced in Appendix II.

1.4. *The main result.* As will be further discussed in the paper, the Entringer-Poupard Matrix Sequence (A_n) $(n \geq 2)$ can be calculated by means of a partial difference equation system (see Proposition 3.3) that leads to the following closed forms for their generating functions, the bottom rows of the A_n's having zero entries.

Theorem 1.2. *Let $(A_n = (a_n(m,k))$ $(n \geq 2)$ denote the Entringer-Poupard Matrix Sequence. Then, the generating function is evaluated as follows.*

For the upper triangles:

$$\sum_{2\leq m+1\leq k\leq n-1} a_n(m,k)\frac{x^{n-k-1}}{(n-k-1)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^{m-1}}{(m-1)!} = \frac{(\sin x+\cos x)\,\sin(2z)}{\cos^2(x+y+z)}; \tag{1.21}$$

for the lower triangles:

$$\sum_{2\leq k+1\leq m\leq n-1} a_n(m,k)\frac{x^{k-1}}{(k-1)!}\frac{y^{m-k-1}}{(m-k-1)!}\frac{z^{n-m-1}}{(n-m-1)!} = \frac{(\sin x+\cos x)\,\cos(x+y-z)}{\cos^2(x+y+z)}; \tag{1.22}$$

for the rightmost columns and the diagonals:

$$\sum_{1\leq k\leq n-1} (a_n(k,n)+a_n(k,k))\frac{x^{k-1}}{(k-1)!}\frac{y^{n-k-1}}{(n-k-1)!} = \frac{\sin x+\cos x}{\cos(x+y)}. \tag{1.23}$$

The properties of the Entringer-Poupard Matrix Sequence (A_n) $(n \geq 2)$, in particular the fact that the A_n's satisfy a partial difference equation system, are proved in Section 3. The following Section contains the proof of Theorem 1.2. The last two Sections are devoted to the *analytical* derivations of the marginal distributions of the pair $(\mathbf{grn}, \mathbf{L})$.

2. Proof of Theorem 1.1; ℓ-tangent numbers

Start with a lemma connecting finite difference equation systems with the calculation of bivariate generating functions. Keep the notations used in (1.2), (1.3) and (1.4) and let a be an arbitrary real number.

Lemma 2.1. *Let $f = (f_n(m))$ be a triangle of numbers, as displayed in (1.2). Then, the following two statements are equivalent.*

(1) *The finite difference equation system*

$$\Delta^2 f_n(m) + (a+1)^2 f_{n-1}(m) = 0 \quad (n \geq 2,\, 1 \leq m \leq 2n-3). \tag{2.1}$$

holds and the generating functions for the left and right sides of f are equal to $\Gamma^f(x,0) = g(x)\cos(ax)$, $\Gamma^f(0,x) = g(x)\cos x$, respectively, for some even formal power series $g(x) = 1 + \sum_{n\geq 1} g_{2n} x^{2n}/(2n)!$

(2) *The generating function for f, as defined in (1.3), is equal to:*

$$\Gamma^f(x,y) = g(x+y)\,\cos(ax-y). \tag{2.2}$$

Proof. Let $\Gamma^f(x,y) := \sum_{i\geq 0,\, j\geq 0} \gamma_{i,j} \frac{x^i}{i!}\frac{y^j}{j!}$. In terms of the entries γ_{ij} relation (2.1) may be written in the form

$$2\,\gamma_{i,j} = (a+1)^2\gamma_{i-1,j-1} + \gamma_{i-1,j+1} + \gamma_{i+1,j-1} \quad (i \geq 1,\, j \geq 1); \tag{2.3}$$

$$\gamma_{ij} = 0, \qquad \text{if } i+j \text{ odd}. \tag{2.4}$$

Then, it is easily verified that the series $\Gamma^f(x,y)$ satisfies the partial differential equation

$$2\,\frac{\partial^2\Gamma^f(x,y)}{\partial x\,\partial y} = (a+1)^2\Gamma^f(x,y) + \frac{\partial^2\Gamma^f(x,y)}{\partial x^2} + \frac{\partial^2\Gamma^f x,y)}{\partial y^2}, \tag{2.5}$$

if and only if the coefficients $\gamma_{i,j}$ satisfy relations (2.3)–(2.4). But for any given formal power series in one variable $g(x) = 1 + \sum_{n\geq 1} g_{2n}\frac{x^{2n}}{(2n)!}$ it can be also verified that the bivariate formal power series

$$\Gamma^f(x,y) = \sum_{i\geq 0,\, j\geq 0} \gamma_{i,j}\frac{x^i}{i!}\frac{y^j}{j!} = g(x+y)\cos(ax-y) \tag{2.6}$$

satisfies (2.5). ∎

We shall work out the particular cases $g(x) = \sec^\ell x$ ($\ell \geq 1$). When $g(x) = \sec^\ell x$, the corresponding triangle will still be denoted by $f^{\ell,a}$ (with a not necessarily equal to 0 or 1), its entries by $f_n^{\ell,a}(m)$ and the sum $f_n^{\ell,a}(1)+f_n^{\ell,a}(2)+\cdots+f_n^{\ell,a}(2n-1)$ of the n-th row by $f_n^{\ell,a}(\bullet)$. It follows from the previous lemma that $\sec^\ell(x+y)\,\cos(ax-y)$ is the generating function of the triangle $f^{\ell,a}$. In particular, identity (1.6) of Theorem 1.1 is proved.

For the proof of identity (1.7) we have recourse to the formal Laplace transformation. Recall that the formal Laplace transform maps a function $f(x)$ onto a function $\mathcal{L}(f(x), x, s)$ defined by

$$\mathcal{L}(f(x), x, s) := \int_0^\infty f(x) e^{-xs}\, dx.$$

In particular, $\mathcal{L}(\bullet, x, s)$ maps $\dfrac{x^\ell}{\ell!}$ onto $\dfrac{1}{s^{\ell+1}}$: $\mathcal{L}\Big(\dfrac{x^\ell}{\ell\,!}, x, s\Big) = \dfrac{1}{s^{\ell+1}}$. When $g(x,y) = \sec^\ell(x+y)$, identity (1.6) reads:

$$\sum_{n\ge 0} \sum_{1\le m\le 2n+1} f_{n+1}^{\ell,a}(m) \frac{x^{2n+1-m}}{(2n+1-m)!} \frac{y^{m-1}}{(m-1)!} = \sec^\ell(x+y)\, \cos(ax-y). \tag{2.7}$$

Apply the Laplace transform to the latter identity twice with respect to (x,s), (y,t) respectively. We get

$$\sum_{\substack{n\ge 0,\\ 1\le m\le 2n+1}} \frac{1}{s^m} \frac{1}{t^{2n-m+2}} f_{n+1}^{\ell,a}(m) = \int_0^\infty \int_0^\infty \sec^\ell(x+y) \cos(ax-y) e^{-xs-ys} dx\, dy$$

which becomes with $s \leftarrow t$ and $r = x+y$:

$$\sum_{\substack{n\ge 0,\\ 1\le m\le 2n+1}} \frac{1}{s^{2n+2}} f_{n+1}^{\ell,a}(m) = \int_0^\infty \int_0^\infty \sec^\ell(x+y) \cos(ax-y) e^{-xs-ys} dx\, dy$$

or still,

$$\mathcal{L}\Big(\sum_{n\ge 0} f_{n+1}^{\ell,a}(\bullet) \frac{r^{2n+1}}{(2n+1)!}, r, s\Big) = \int_0^\infty \int_0^r \sec^\ell(r) \cos(ax-y) e^{-rs} dx\, dr.$$

Let $a \ne -1$. The right-hand side becomes:

$$\begin{aligned}
&\int_0^\infty e^{-rs} dr \int_0^r \sec^\ell r \Big(\cos(ar)\cos((a+1)y) + \sin(ar)\sin((a+1)y)\Big) dy\\
&\quad = \int_0^\infty e^{-rs} \sec^\ell r \cos(ar)\, dr \int_0^r \cos((a+1)y)\, dy\\
&\qquad + \int_0^\infty e^{-rs} \sec r \sin(ar)\, dr \int_0^r \sin((a+1)y)\, dy.\\
&\quad = \int_0^\infty e^{-rs} \sec^\ell r \cos(ar)\, \frac{\sin((a+1)r)}{a+1}\, dr
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_0^\infty e^{-rs} \sec^\ell r \frac{\sin(ar)}{a+1}(1-\cos((a+1)r))\,dr \\
&= \int_0^\infty e^{-rs} \frac{\sec^\ell r}{a+1}\Big(\cos(ar)\sin((a+1)r) - \sin(ar)\cos(a+1)r)\Big)dr \\
&\quad + \int_0^\infty e^{-rs} \sec^\ell r \frac{\sin(ar)}{a+1}\,dr \\
&= \int_0^\infty e^{-rs} \sec^\ell r \frac{\sin r + \sin(ar)}{a+1}\,dr = \mathcal{L}\Big(\sec^\ell r \frac{\sin r + \sin(ar)}{a+1}, r, s\Big).
\end{aligned}
$$

Thus,

$$
\sum_{n\ge 0} f_{n+1}^{\ell,a}(\bullet)\frac{r^{2n+1}}{(2n+1)!} = \sec^\ell r \frac{\sin r + \sin(ar)}{a+1} \quad (a \neq -1). \tag{2.8}
$$

For an arbitrary a the right-hand side of (2.8) appears to be a Taylor series in r with coefficients in $\mathbb{Z}[a]$. However, when $a=0$ *and* 1, the right-hand side is equal to: $\sec^{\ell-1}(r)\,\tan r$, the series introduced in (1.1), whose coefficients are the *positive integers* $E_{2n+1}^{(\ell)}$. We then have the two sum refinements expressed in (1.7). This achieves the proof of Theorem 1.1.

3. Properties of the Entringer-Poupard Matrix Sequence

Recall that the Entringer-Poupard Matrix Sequence consists of $n \times n$ matrices $A_n = (a_n(m,k)_{(1\le m,k\le n)})$ $(n \ge 2)$, whose entries $a_n(m,k)$ are defined by $a_n(m,k) := \#\{w \in \mathrm{Alt}_n : \mathbf{grn}\, w = m,\ \mathbf{L}\, w = k\}$. As shown in Fig. 3.1, it will be convenient to split each $n\times n$-matrix A_n $(n\ge 2)$ into five regions:

(1) the upper triangle $\mathrm{Up}_n := (a_n(m,k))\quad (1 \le m < k \le n-2)$;

(2) the lower triangle $\mathrm{Low}_n := (a_n(m,k))\quad (1 \le k < m \le n-2)$;

(3) the diagonal $\mathrm{Diag}_n := (a_n(m,m))\quad (1 \le m \le n-1)$, which has $(n-1)$ entries, by convention;

(4) the rightmost column $\mathrm{Col}_n := (a_n(m,n))\quad (1 \le m \le n-1)$, which also has $(n-1)$ entries, by convention;

(5) the bottom row $\mathrm{Row}_n := (a_n(n,k))\quad (1 \le k \le n)$, which has n entries.

By convention, the *partial difference operators* $\underset{m}{\Delta}$, $\underset{k}{\Delta}$, act on the entries of the matrices A_n as follows:

$$
\underset{m}{\Delta}\, a_n(m,k) := a_n(m+1,k) - a_n(m,k); \tag{3.1}
$$

$$
\underset{k}{\Delta}\, a_n(m,k) := a_n(m,k+1) - a_n(m,k). \tag{3.2}
$$

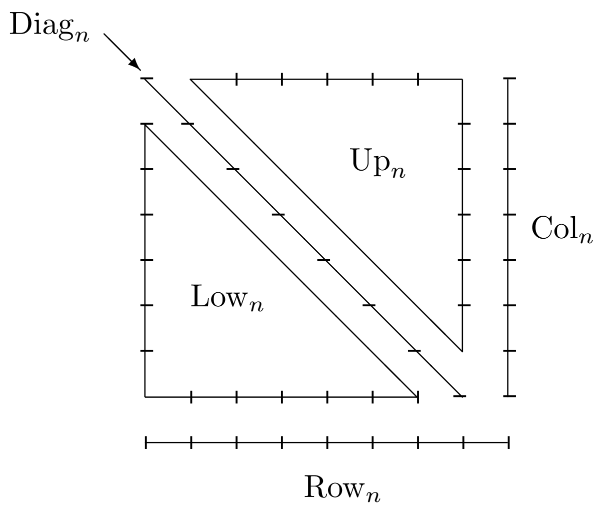

Fig. 3.1. Splitting a matrix in five regions

Also, denote the *row* and *column sums* of $A_n = (a_n(m,k))$ by

$$a_n(m,\bullet) := \sum_{1\le k\le n} a_n(m,k) \quad (1\le m\le n); \tag{3.3}$$

$$a_n(\bullet,k) := \sum_{1\le m\le n} a_n(m,k) \quad (1\le k\le n). \tag{3.4}$$

Proposition 3.1. *We have the identities:*

$$a_n(m,\bullet) = P_n(m); \tag{3.4}$$

$$a_n(\bullet,k) = \begin{cases} E_n(k), & \text{if } n \text{ is odd;}\\ E_n(n+1-k), & \text{if } n \text{ is even;}\end{cases} \tag{3.5}$$

$$\sum_m a_n(m,\bullet) = \sum_k a_n(\bullet,k) = \sum_{m,k} a_n(m,k) = E_n. \tag{3.6}$$

Proof. The first one is is proved in [FH13], the second one is due to Entringer himself [En66], who classified the alternating permutations according to their *first* letters. When using the *last* letters, as we do, the change $E_n(k) \leftarrow E_n(n+1-k)$ is to be made when n is even. The third identity in (3.6) is due to Désiré André! ∎

Clearly, $A_2 := \begin{pmatrix}0&1\\0&0\end{pmatrix}$. Also, the bottom row Row$_n$ has zero entries for all $n\ge 2$. For $n\ge 3$ each matrix A_n will be characterized by four recurrence relations: (1) for Diag$_n$; (2) for Col$_n$; (3) for the rightmost column of Up$_n$; (4) for the leftmost column of Low$_n$, and two partial difference equation systems for Up$_n$ and Low$_n$.

Proposition 3.2 (Special values)**.**

(1) *Each diagonal* Diag$_n$ *has zero entries when n is even, and equal*

to $(E_{n-1}(n-1), E_{n-1}(n-2), \ldots, E_{n-1}(1)) = (a_{n-1}(\bullet,1), a_{n-1}(\bullet,2), \ldots, a_{n-1}(\bullet, n-1))$, *when* n *is odd.*

(2) *The rightmost column* Col_n *has zero entries when* n *is odd, and equal to* $(E_{n-1}(1), E_{n-1}(2), \ldots, E_{n-1}(n-1)) = (a_{n-1}(\bullet,1), a_{n-1}(\bullet,2), \ldots, a_{n-1}(\bullet, n-1))$ *from top to bottom, when* n *is even.*

(3) *The rightmost column of* Up_n *has zero entries when* n *is odd, and equal to* $(P_{n-1}(1), P_{n-1}(2), \ldots, P_{n-1}(n-2)) = (a_{n-1}(1,\bullet), a_{n-1}(2,\bullet), \ldots, a_{n-1}(n-2,\bullet))$ *from top to bottom when* n *is even.*

(4) *The leftmost column of* Low_n *has zero entries when* n *is even, and equal to* $(P_{n-1}(1), P_{n-1}(2), \ldots, P_{n-1}(n-2)) = (a_{n-1}(1,\bullet), a_{n-1}(2,\bullet), \ldots, a_{n-1}(n-2,\bullet))$ *from top to bottom when* n *is odd.*

Proof. (1) When n is even, each permutation w from Alt_n ends with a factor $l\,m$ with $l < m$. Either $m = n$ and then $\mathbf{grn}\, w = l \neq m$, or $m < n$ and the neighbors of n in w can never be m, so that $a_n(m,m) = 0$ for all m. When n is odd, each permutation w from Alt_n such that $\mathbf{grn}\, w = \mathbf{L}\, w = m$ ends with the factor $l\,n\,m$ with $l \le m-1$. Delete n from w. What it is left is a permutation w' from Alt_{n-1} ending with m. Hence, $a_n(m,m) = E_{n-1}(n-m)$, also equal to $a_{n-1}(\bullet, m)$ by (3.5).

(2) When n is odd, the rightmost letter of each w from Alt_n is less than n, so that $a_n(m,n) = 0$ for all m. When n is even, each permutation from Alt_n ending with n has a penultimate letter equal to $m < n$. Delete n from w; we get a permutation w' from Alt_{n-1} such that $\mathbf{L}\, w' = m$. Hence, $a_n(m,n) = E_{n-1}(m) = a_{n-1}(\bullet, m)$ by (3.5).

(3) When n is odd, each permutation w from Alt_n such that $\mathbf{grn}\, w \le n-2$ and $\mathbf{L}\, w = n-1$ necessarily ends with $n\,(n-1)$, so that $\mathbf{grn}\, w = n-1$. Hence, if $m \le n-2$, then $a_n(m, n-1) = 0$. When n is even, delete the last letter $(n-1)$ and replace the letter n by $(n-1)$. We get a permutation w' from Alt_{n-1} such that $\mathbf{grn}\, w' = \mathbf{grn}\, w \le n-2$. Thus, $a_n(m, n-1) = P_{n-1}(m) = a_{n-1}(m, \bullet)$ with $m \le n-2$ by (3.4).

(4) When n is even, each permutation from Alt_n cannot end with 1, so that $a_n(m,1) = 0$ for all m. When n is odd, each alternating permutation w ending with 1 and such that $\mathbf{grn}\, w = m \ge 2$ can be mapped, in a bijective manner, on a permutation w' from Alt_{n-1} such that $\mathbf{grn}\, w' = m-1$ by deleting 1 and subtracting 1 from all the other letters. Thus, $a_n(m,1) = P_{n-1}(m-1) = a_{n-1}(m-1, \bullet)$ with $m \ge 2$ by (3.4). ∎

Proposition 3.3 (Partial difference equation systems).

(5) *For the upper triangles, that is, for* $1 \le m < k \le n-2$,

$$(R\,1) \qquad \Delta_k\, a_n(m,k) + (-1)^{n-1}\, a_{n-1}(m,k) = 0.$$

(6) *For the lower triangles, that is, for* $3 \le k+2 \le m \le n-2$,

$$(R\,2) \qquad \Delta_k\, a_n(m,k) + (-1)^{n-1}\, a_{n-1}(m-1,k) = 0;$$

Proof. (5) When n is even (resp. odd), the transposition $\varphi=(k,k+1)$ transforms each permutation from $\mathrm{Alt}_n(m,k)$ (resp. from $\mathrm{Alt}_n(m,k)$ not ending with $(k+1)\,k$) onto a permutation from $\mathrm{Alt}_n(m,k+1)$ not ending with $k\,(k+1)$ (resp. from $\mathrm{Alt}_n(m,k+1)$) in a bijective manner. Hence, the set $\mathrm{Alt}_n(m,k+1)\setminus\varphi\bigl(\mathrm{Alt}_n(m,k)\bigr)$ (resp. $\mathrm{Alt}_n(m,k)\setminus\varphi\bigl(\mathrm{Alt}_n(m,k+1)\bigr)$) consists of all the permutations from Alt_n ending with $l\,k\,(k+1)$ for some $l\ge k+2$ (resp. from Alt_n ending with $l\,(k+1)k$ for some $l\le k-1$), and this set is of cardinality $a_{n-1}(m,k)$. Thus, when n is even (resp. odd), we have: $a_n(m,k+1)-a_n(m,k)=a_{n-1}(m,k)$ (resp. $a_n(m,k)-a_n(m,k+1)=a_{n-1}(m,k)$) and $(R\,1)$ holds.

(6) The proof is quite analogous to the previous one. This time, $k<m$, so that because of the deletion of the letter $(k+1)$ the greater neighbor of maximum in the permutation becomes $(m-1)$. $\square$

Remark. The entries of the matrix A_n are derived from A_{n-1} by first applying recurrence relations (1)–(4) of Proposition 3.2 that fully determine the diagonal, rightmost column, the rightmost column of Up_n and the leftmost column of Low_n of A_n. Starting with $m=1$ up to $m=n-3$, for each $k=n-2$ down to $m+1$ we can evaluate $a_n(m,k)$ by means of equation $(R\,1)$, the entries $a_n(m,k+1)$, $a_{n-1}(m,k)$ being already calculated. The same procedure can be applied for the entries from Low_n by applying rule $(R\,2)$. Accordingly, Propositions 3.2 and 3.3 uniquely determine the Entringer-Poupard Matrix Sequence. See Appendix II. In the spirit of Brualdi-Ryser's Combinatorial Matrix Theory [BR91] the combinatorial setting then serves to calculate the Entringer-Poupard Matrix Sequence explicitly.

4. Generating function for the Entringer-Poupard Matrix Sequence

For calculating the generating functions displayed in (1.21) and (1.22) we shall recourse to the techniques developed in our previous paper [FH14] dealing with the so-called *Seidel triangle sequences.* Only definitions will be stated, as well as the main result. With the triangle f as shown in (1.2) associate the infinite matrix

$$\Gamma^f=(\gamma_{ij})_{(i\ge0,j\ge0)}:=\begin{pmatrix} f_1(1) & \cdot & f_2(3) & \cdot & f_3(5) & \cdot & f_4(7)\cdots\\ \cdot & f_2(2) & \cdot & f_3(4) & \cdot & f_4(6) & \\ f_2(1) & \cdot & f_3(3) & \cdot & f_4(5) & & \\ \cdot & f_3(2) & \cdot & f_4(4) & & & \\ f_3(1) & \cdot & f_4(3) & & & & \\ \cdot & f_4(2) & & & & & \\ f_4(1) & & & & & & \\ \vdots & & & & & & \end{pmatrix}, \tag{4.1}$$

where the zero entries are written as simple dots "$\cdot$". As was already defined in Section 2, let $\gamma_{ij}:=0$ when $i+j$ is odd, and $\gamma_{ij}:=f_n(m)$

with $m := j + 1$, $2n = 2 + i + j$ when $i + j$ is even. For $i + j$ even the mapping $(i, j) \mapsto (n, m)$ is one-to-one, the reverse mapping being for $n \geq 1$, $1 \leq m \leq 2n - 1$ given by $i = 2n - 1 - m$, $j = m - 1$.

We attach the following exponential generating function

$$\Gamma^f(x, y) := \sum_{i,j \geq 0} \gamma_{ij} \frac{x^i}{i!} \frac{y^j}{j!} \tag{4.2}$$

to Γ^f itself. Clearly, the generating function for f, as defined in (1.3), is identical to $\Gamma^f(x, y)$, as defined in (4.2). In particular,

$$\Gamma^{P^{(\tan)}} = \begin{pmatrix} P_1(1) & \cdot & P_3(3) & \cdot & P_5(5) & \cdot & P_7(7) \cdots \\ \cdot & P_3(2) & \cdot & P_5(4) & \cdot & P_7(6) & \\ P_3(1) & \cdot & P_5(3) & \cdot & P_7(5) & & \\ \cdot & P_5(2) & \cdot & P_7(4) & & & \\ P_5(1) & \cdot & P_7(3) & & & & \\ \cdot & P_7(2) & & & & & \\ P_7(1) & & & & & & \\ \vdots & & & & & & \end{pmatrix} = \begin{pmatrix} 1 & \cdot & 0 & \cdot & 0 & \cdot & 0 \cdots \\ \cdot & 2 & \cdot & 4 & \cdot & 32 & \\ 0 & \cdot & 8 & \cdot & 64 & & \\ \cdot & 4 & \cdot & 80 & & & \\ 0 & \cdot & 64 & & & & \\ \cdot & 32 & & & & & \\ 0 & & & & & & \\ \vdots & & & & & & \end{pmatrix};$$

(4.3)

$$\Gamma^{P^{(\sec)}} = \begin{pmatrix} P_2(1) & \cdot & P_4(3) & \cdot & P_6(5) & \cdot & P_8(7) \cdots \\ \cdot & P_4(2) & \cdot & P_6(4) & \cdot & P_8(6) & \\ P_4(1) & \cdot & P_6(3) & \cdot & P_8(5) & & \\ \cdot & P_6(2) & \cdot & P_8(4) & & & \\ P_6(1) & \cdot & P_8(3) & & & & \\ \cdot & P_8(2) & & & & & \\ P_8(1) & & & & & & \\ \vdots & & & & & & \end{pmatrix} = \begin{pmatrix} 1 & \cdot & 1 & \cdot & 5 & \cdot & 61 \cdots \\ \cdot & 3 & \cdot & 15 & \cdot & 163 & \\ 1 & \cdot & 21 & \cdot & 285 & & \\ \cdot & 15 & \cdot & 327 & & & \\ 5 & \cdot & 285 & & & & \\ \cdot & 183 & & & & & \\ 61 & & & & & & \\ \vdots & & & & & & \end{pmatrix};$$

and recall that $\Gamma^{P^{(\tan)}}(x, y) = \sec(x + y)\cos(x - y)$ and $\Gamma^{P^{(\sec)}}(x, y) = \sec^2(x + y)\cos(x - y)$ by (1.14) and (1.15).

4.1. *Seidel Triangle Sequences.*

Definition 4.1. A sequence of square matrices (C_n) $(n \geq 1)$ is called a *Seidel triangle sequence* if the following three conditions are fulfilled:

(STS1) each matrix C_n is of dimension n;

(STS2) each matrix C_n has null entries along and below its diagonal; let $(c_n(m, k))$ $(0 \leq m < k \leq n - 1)$ denote its entries strictly above its diagonal, so that

$$C_1 = (\,\cdot\,); \quad C_2 = \begin{pmatrix} \cdot & c_2(0,1) \\ \cdot & \cdot \end{pmatrix}; \qquad C_3 = \begin{pmatrix} \cdot & c_3(0,1) & c_3(0,2) \\ \cdot & \cdot & c_3(1,2) \\ \cdot & \cdot & \cdot \end{pmatrix}; \ \ldots ;$$

$$C_n = \begin{pmatrix} \cdot & c_n(0,1) & c_n(0,2) & \cdots & \cdot & c_n(0,n-2) & c_n(0,n-1) \\ \cdot & \cdot & c_n(1,2) & \cdots & \cdot & c_n(1,n-2) & c_n(1,n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & c_n(n-2,n-1) \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \end{pmatrix};$$

the dots "$\cdot$" along and below the diagonal referring to null entries.

(STS3) for each $n \ge 3$, the following relation holds:

$$c_n(m,k) - c_n(m,k+1) = c_{n-1}(m,k) \quad (m < k).$$

Record the rightmost columns of the triangles C_2, C_3, C_4, C_5, $\ldots$, read from top to bottom, namely, $c_2(0,1)$; $c_3(0,2)$, $c_3(1,2)$; $c_4(0,3)$, $c_4(1,3)$, $c_4(2,3)$; $c_5(0,4)$, $c_5(1,4)$, $c_5(2,4)$, $c_5(3,4)$; $\ldots$ as skew-diagonals of an infinite matrix $H = (h_{i,j})_{i,j\ge 0}$, as shown next:

$$\text{(4.4)} \qquad H := \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{pmatrix} c_2(0,1) & c_3(1,2) & c_4(2,3) & c_5(3,4) & c_6(4,5) & \cdots \\ c_3(0,2) & c_4(1,3) & c_5(2,4) & c_6(3,5) & & \\ c_4(0,3) & c_5(1,4) & c_6(2,5) & & & \\ c_5(0,4) & c_6(1,5) & & & & \\ c_6(0,5) & & & & & \\ \vdots & & & & & \end{pmatrix} \end{array},$$

In an equivalent manner, the entries of H are defined by:

$$h_{i,j} = c_{i+j+2}(j, i+j+1). \tag{4.5}$$

The next theorem has been proved in [FH14]. For short, write:

$$M_{n,m,k}(x,y,z) := \frac{x^{n-k-1}}{(n-k-1)!}\,\frac{y^{k-m-1}}{(k-m-1)!}\,\frac{z^m}{m!}. \tag{4.6}$$

Theorem 4.1. *The three-variable generating function for the Seidel triangle sequence* $(C_n = (c_n(m,k)))_{n\ge 1}$ *is equal to*

$$\sum_{1\le m+1\le k\le n-1} c_n(m,k) M_{n,m,k}(x,y,z) = e^x H(x+y,z), \tag{4.7}$$

where $H(x,y)$ *is the exponential generating function attached to the matrix* H *defined in* (4.4).

This theorem is now used for calculating the generating function for another triangle sequence $(C_n = (c_n(m,k)_{(0\le m<k\le n-1)})$ $(n \ge 1)$, called *dual Seidel triangle sequence*, which obeys rules (ST1) and (ST2) (unchanged), plus the new rule:

$$\text{(ST3}'\text{)} \qquad c_n(m,k) - c_n(m,k+1) = (-1)^{n-1} c_{n-1}(m,k) \quad (m < k).$$

Theorem 4.2. *The three-variable generating function for the dual Seidel triangle sequence* $(C_n = (c_n(m,k)))$ $(n \ge 1)$ *is equal to*

$$\begin{aligned} \sum_{1\le m+1\le k\le n-1} & c_n(m,k) M_{n,m,k}(x,y,z) \\ &= \cos(x) H(x+y,z) + \sin(x) H(-x-y,-z), \end{aligned} \tag{4.8}$$

where $H(x,y)$ *is still the exponential generating function attached to the matrix* H *defined in* (4.4).

Proof. Let

$$\epsilon(n) := \begin{cases} 1, & \text{if } n \equiv 0, 1 \pmod 4; \\ -1, & \text{if } n \equiv 2, 3 \pmod 4; \end{cases} \tag{4.9}$$

$$b_n(m,k) := \epsilon(n) c_n(m,k). \tag{4.10}$$

Since $\epsilon(n)\epsilon(n-1) = (-1)^{n-1}$ we have

$$b_n(m,k) - b_n(m,k+1) = b_{n-1}(m,k) \quad (m < k).$$

Thus, the sequence $(B_n := (b_n(m,k))$ $(n \geq 1)$ is a Seidel triangle sequence. Let $H^{(b)} = (h_{i,j}^{(b)})$ be the H-matrix to be associated with (B_n) to make up identity (4.7). By (4.5) its entries are equal to:

$$h_{i,j}^{(b)} = b_{i+j+2}(j, i+j+1) = \epsilon(i+j+2) h_{i,j}. \tag{4.11}$$

and identity (4.7) becomes

$$\sum_{1 \leq m+1 \leq k \leq n-1} b_n(m,k)\ M_{n,m,k}(x,y,z) = e^x H^{(b)}(x+y, z). \tag{4.12}$$

Therefore, by using identity (4.10), and letting $I := \sqrt{-1}$,

$$\sum_{1 \leq m+1 \leq k \leq n-1} I^{n-2} \epsilon(n) c_n(m,k)\ M_{n,m,k}(x,y,z) = e^{Ix} H^{(b)}(Ix + Iy, Iz). \tag{4.13}$$

Now, $I^{n-2}\epsilon(n) = -1$ if n is even, and equal to $-I$ if n is odd. Sorting out real and imaginary parts yields the two identities:

$$\sum_{n \text{ even}} \sum_{1 \leq m+1 \leq k \leq n-1} c_n(m,k)\ M_{n,m,k}(x,y,z) = -\Re\, e^{Ix} H^{(b)}(Ix + Iy, Iz); \tag{4.14}$$

$$\sum_{n \text{ odd}} \sum_{1 \leq m+1 \leq k \leq n-1} c_n(m,k)\ M_{n,m,k}(x,y,z) = -\Im\, e^{Ix} H^{(b)}(Ix + Iy, Iz). \tag{4.15}$$

Let $H^{(b)}(Ix + Iy, Iz) =: \alpha + I\beta$. Then, by (4.14)–(4.15),

$$\begin{aligned} \sum_{1 \leq m+1 \leq k \leq n-1} c_n(m,k)\ M_{n,m,k}(x,y,z) &= -\Re\, e^{Ix}(\alpha + I\beta) - \Im\, e^{Ix}(\alpha + I\beta) \\ &= -\Re(\cos(x) + I\sin(x))(\alpha + I\beta) - \Im(\cos(x) + I\sin(x))(\alpha + I\beta) \\ &= -\cos(x)\alpha + \sin(x)\beta - \cos(x)\beta - \sin(x)\alpha \\ &= -\cos(x)(\alpha + \beta) - \sin(x)(\alpha - \beta). \qquad (4.16) \end{aligned}$$

As $H(x,y)=\sum_{i,j} h_{i,j}\dfrac{x^i}{i!}\dfrac{y^j}{j!}$, by definition, relation (4.11) implies

$$H^{(b)}(Ix,Iy)=\sum_{i,j} I^{i+j}\epsilon(i+j+2)h_{i,j}\frac{x^i}{i!}\frac{y^j}{j!}.$$

Now, $I^{i+j}\epsilon(i+j+2)=-1$ if $i+j$ is even, and equal to $-I$ if $i+j$ is odd. Sorting out real and imaginary parts yields the two identities:

$$\Re H^{(b)}(Ix,Iy)=-\sum_{i,j,\,i+j\,\text{even}} h_{i,j}\frac{x^i}{i!}\frac{y^j}{j!};$$
$$\Im H^{(b)}(Ix,Iy)=-\sum_{i,j,\,i+j\,\text{odd}} h_{i,j}\frac{x^i}{i!}\frac{y^j}{j!}.$$

Therefore,

$$\Re H^{(b)}(Ix,Iy)+\Im H^{(b)}(Ix,Iy)=-H(x,y);$$
$$\Re H^{(b)}(Ix,Iy)-\Im H^{(b)}(Ix,Iy)=-H(-x,-y);$$

and by (4.16)

$$\sum_{1\le m+1\le k\le n-1} c_n(m,k)\ M_{n,m,k}(x,y,z)$$
$$=\cos(x)H(x+y,z)+\sin(x)H(-x-y,-z).$$

Finally, (4.10) and (4.11) imply that $h_{i,j}=c_{i+j+2}(i,i+j+1)$, which are then the entries of the matrix H displayed in (4.4). ∎

4.2. *Generating function for the upper triangles* (Up_n). The trick is to fabricate a *dual* Seidel triangle sequence with the triangles Up_n, find the corresponding matrix H and use identity (4.8). Let

$$(4.17)\quad c_n(m,k):=(-1)^{n+1}a_{n+1}(m+1,k+1)\quad (1\le m+1\le k\le n-1).$$

Then, rule $(R\,1)$ of Proposition 3.3 for the $a_n(m,k)$'s implies that rule (ST3′) holds for the $c_n(m,k)$'s. Therefore, the sequence of triangles $C_1:=(\cdot)$, $C_n:=(-1)^{n+1}\mathrm{Up}_{n+1}$ $(n\ge 2)$ is a dual Seidel Triangle Sequence. It remains to determine its corresponding matrix H. Taking (4.4) and (4.17) into account we then have:

$$H=\begin{array}{c} \\ 0\\ 1\\ 2\\ 3\\ 4\\ \vdots\end{array}\!\!\begin{array}{c}\begin{array}{cccccc}0&1&2&3&4&\cdots\end{array}\\ \begin{pmatrix} -a_3(1,2)&a_4(2,3)&-a_5(3,4)&a_6(4,5)&-a_7(5,6)&\cdots\\ a_4(1,3)&-a_5(2,4)&a_6(3,5)&-a_7(4,6)&&\\ -a_5(1,4)&a_6(2,5)&-a_7(3,6)&&&\\ a_6(1,5)&-a_7(2,6)&&&&\\ -a_7(1,6)&&&&&\\ \vdots&&&&&\end{pmatrix}\end{array}$$

But by Rule 3 of Proposition 3.2 the rightmost column of each triangle Up_n $(n \geq 3)$, from top to bottom, has zero entries if n is odd, and is equal to $(P_{n-1}(1), P_{n-1}(2), \ldots, P_{n-1}(n-2))$ if n is even, that is, $a_n(m, n-1) = P_{n-1}(m)$ $(m = 1, 2, \ldots, n-2)$. Therefore,

$$H = \begin{pmatrix} \cdot & P_3(2) & \cdot & P_5(4) & \cdot & P_7(6) \cdots \\ P_3(1) & \cdot & P_5(3) & \cdot & P_7(5) & \\ \cdot & P_5(2) & \cdot & P_7(4) & & \\ P_5(1) & \cdot & P_7(3) & & & \\ \cdot & P_7(2) & & & & \\ P_7(1) & & & & & \\ \vdots & & & & & \end{pmatrix} = \begin{pmatrix} \cdot & 2 & \cdot & 4 & \cdot & 32 \cdots \\ 0 & \cdot & 8 & \cdot & 64 & \\ \cdot & 4 & \cdot & 80 & & \\ 0 & \cdot & 64 & & & \\ \cdot & 32 & & & & \\ 0 & & & & & \\ \vdots & & & & & \end{pmatrix},$$

and this matrix H can be obtained from $\Gamma^{P^{(\tan)}}$, displayed in (4.3), by chopping off the top row. Hence,

$$H(x, y) = \frac{\partial}{\partial x} \Gamma^{P^{(\tan)}}(x, y) = \frac{\partial}{\partial x} \sec(x + y) \cos(x - y) = \frac{\sin(2y)}{\cos^2(x + y)}.$$

Making use of identity (4.8) of Theorem 4.2, together with the definition of $c_n(m, k)$ shown in (4.17), we conclude that

$$\sum_{1 \leq m+1 \leq k \leq n-1} (-1)^{n+1} a_{n+1}(m+1, k+1) \frac{x^{n-k-1}}{(n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!} = \frac{(\cos x - \sin x) \sin(2z)}{\cos^2(x + y + z)}. \tag{4.18}$$

that is,

$$\sum_{2 \leq m+1 \leq k \leq n-1} a_n(m, k) \frac{x^{n-k-1}}{(n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^{m-1}}{(m-1)!} = \frac{(\cos x + \sin x) \sin(2z)}{\cos^2(x + y + z)}.$$

4.3. *Generating function for the lower triangles* (Low_n). For $n \geq 2$ let

$$c_n(m, k) := a_{n+1}(n - m, n - k) \quad (1 \leq m + 1 \leq k \leq n - 1), \tag{4.19}$$

Rule $(R2)$ of Proposition 3.3 for the $a_n(m, k)$'s implies that rule (ST3′ holds for the $c_n(m, k)$'s. Therefore, the sequence of triangles $C_n)$ $(n \geq 1)$ is a dual Seidel Triangle Sequence, whose first elements are displayed in Table 4.1.

$$C_1=(\cdot);\quad C_2=\begin{array}{cc}\cdot&1\\ &\cdot\end{array};\quad C_3=\begin{array}{ccc}\cdot&1&0\\ &\cdot&0\\ &&\cdot\end{array};\quad C_4=\begin{array}{cccc}\cdot&0&1&1\\ &\cdot&3&3\\ &&\cdot&1\\ &&&\cdot\end{array};\quad C_5=\begin{array}{ccccc}\cdot&2&2&1&0\\ &\cdot&6&3&0\\ &&\cdot&1&0\\ &&&\cdot&0\\ &&&&\cdot\end{array};$$

$$C_6=\begin{array}{cccccc}\cdot&0&2&4&5&5\\ &\cdot&6&12&15&15\\ &&\cdot&20&21&21\\ &&&\cdot&15&15\\ &&&&\cdot&5\\ &&&&&\cdot\end{array};\quad C_7=\begin{array}{ccccccc}\cdot&16&16&14&10&5&0\\ &\cdot&48&42&30&15&0\\ &&\cdot&62&42&21&0\\ &&&\cdot&30&15&0\\ &&&&\cdot&5&0\\ &&&&&\cdot&0\\ &&&&&&\cdot\end{array};$$

Table 4.1. Dual Seidel Triangle Sequence for the lower triangles

The underlying H-matrix is simply $\Gamma^{P^{(\sec)}}$ shown in (4.3), whose generating function is equal to $\sec^2(x+y)\,\cos(x-y)$. By Theorem 4.2 we then get:

$$\sum_{1\le m+1\le k\le n-1} a_{n+1}(n-m,n-k)\frac{x^{n-k-1}}{(n-k-1)!}\,\frac{y^{k-m-1}}{(k-m-1)!}\,\frac{z^m}{m!} = \frac{(\cos x+\sin x)\cos(x+y-z)}{\cos^2(x+y+z)};$$

that is,

$$(4.20)\qquad \sum_{2\le k+1\le m\le n-1} a_n(m,k)\frac{x^{k-1}}{(k-1)!}\,\frac{y^{k-m-1}}{(k-m-1)!}\,\frac{z^{n-m-1}}{(n-m-1)!} = \frac{(\cos x+\sin x)\cos(x+y-z)}{\cos^2(x+y+z)}.$$

4.4. *Generating function for the rightmost columns and diagonals*

By Rule (2) of Proposition 3.2 each entry $a_{2n}(k,2n-1)$ of the rightmost column Col_{2n} of the matrix A_{2n} is equal to $(E_{2n-1}(k)$ $(1\le k\le 2n-2)$. As $E_{2n-1}(2n-1)=0$ when $n\ge 1$, we have:

$$\begin{aligned}\sum_{1\le k\le 2n-2} a_{2n}(k,2n-1)\frac{x^{k-1}}{(k-1)!}\,\frac{y^{2n-k-1}}{(2n-k-1)!} &= \sum_{1\le k\le 2n-1} E_{2n-1}(k)\frac{x^{k-1}}{(k-1)!}\,\frac{y^{2n-k-1}}{(2n-k-1)!}\\ &= \Gamma^{E^{(\tan)}}(y,x)=\sec(x+y)\,\cos x,\end{aligned}$$

by (1.12).

By Rule (1) of Proposition 3.2 each entry $a_{2n+1}(k,k)$ is equal to $E_{2n}(2n+1-k)$ $(1 \le k \le 2n)$. Hence,

$$\begin{aligned}
&\sum_{1\le k\le 2n} a_{2n+1}(k,k)\frac{x^{k-1}}{(k-1)!}\frac{y^{2n-k}}{(2n-k)!}\\
&\qquad = \sum_{1\le k\le 2n} E_{2n}(2n+1-k)\frac{x^{k-1}}{(m-1)!}\frac{y^{2n-k}}{(2n-k)!}\\
&\qquad = \sum_{1\le m\le 2n} E_{2n}(m)\frac{x^{2n-m}}{(2n-m)!}\frac{y^{m-1}}{(m-1)!}\\
&\qquad = \sec(x+y)\,\sin x,
\end{aligned}$$

by (1.17). As the generating functions for the rightmost columns Col_{2n-1} and the diagonals Diag_{2n} are null, identity (1.23) is proved.

5. Marginal distributions. The row sums

Identities (3.4) and (3.5) have been proved *combinatorially*, in our previous paper [FH13] for the first one, and by Entringer [En66] for the second. The purpose of this section is to prove them *analytically*: assuming that the sequence of matrices $A_n = (a_n(m,k)$ $(n \ge 2)$ has a generating function given by formulas (1.21)–(1.23) of Theorem 1.2, derive that the row and column sums of the matrices are equal to: $a_n(m,\bullet) = P_n(m)$ and $a_n(\bullet,k) = E_n(k)$ if n is odd, and $E_n(n+1-k)$ is n is even. This derivation can also be seen as a method for calculating the *marginal distributions* of the pair $(\mathbf{grn},\mathbf{L})$, when their joint generating function has the form given in (1.21)–(1.23). The formal Laplace transform already used in section 2 will be the main tool.

5.1. *The upper triangles* Up_n. Apply the Laplace transform to the left-hand side of identity (1.21) three times with respect to (x,u), (y,t), (z,s), respectively. We get

$$\sum_{2\le m+1\le k\le n-1} \frac{1}{s^m}\frac{1}{t^{k-m}}\frac{1}{u^{n-k}}a_n(m,k),$$

which becomes

$$\sum_{2\le m+1\le k\le n-1} \frac{1}{s^m}\frac{1}{u^n}a_n(m,k), \tag{5.1}$$

when $t \leftarrow u$ and $s \leftarrow su$. Apply the Laplace transform to the right-hand side of (1.21) three times with respect to (x,u), (y,t), (z,s), respectively, and let $t \leftarrow u$, $s \leftarrow su$. With $r = x+y$ we get:

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{(\sin x+\cos x)\,\sin(2z)}{\cos^2(x+y+z)}e^{-xu-yu-szu}dx\,dy\,dz$$

$$= \int_0^\infty \int_0^\infty \int_0^r \frac{(\sin x + \cos x)\ \sin(2z)}{\cos^2(r+z)} e^{-ru-szu}\, dx\, dr\, dz$$

$$= \int_0^\infty \int_0^\infty \frac{(1-\cos r + \sin r)\sin(2z)}{\cos^2(r+z)} e^{-ru-szu}\, dr\, dz. \tag{5.2}$$

5.2. *The lower triangles* Low_n. With identity (1.22) apply the Laplace transform to its left-hand side three times with respect to (x,s), (y,t), (z,u), respectively. We get

$$\sum_{2\le k+1\le m\le n-1} \frac{1}{s^k}\frac{1}{t^{m-k}}\frac{1}{u^{n-m}} a_n(m,k),$$

which becomes

$$\sum_{2\le k+1\le m\le n-1} \frac{1}{s^m}\frac{1}{u^n} a_n(m,k), \tag{5.3}$$

when $s \leftarrow su$ and $t \leftarrow su$. Apply the Laplace transform to the right-hand side of (1.22) three times with respect to (x,s), (y,t), (z,u), respectively, and let $s \leftarrow su$, $t \leftarrow su$. With $r = x+y$ we get:

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{(\sin x + \cos x)\ \cos(x+y-z)}{\cos^2(x+y+z)} e^{-xsu-ysu-zu} dx\, dy\, dz$$

$$= \int_0^\infty \int_0^\infty \int_0^r \frac{(\sin x + \cos x)\ \cos(r-z)}{\cos^2(r+z)} e^{-rsu-zu} dx\, dr\, dz$$

$$= \int_0^\infty \int_0^\infty \frac{(1-\cos r + \sin r)\ \cos(r-z)}{\cos^2(r+z)} e^{-rsu-zu} dr\, dz$$

$$= \int_0^\infty \int_0^\infty \frac{(1-\cos z + \sin z)\ \cos(r-z)}{\cos^2(r+z)} e^{-ru-szu}\, dr\, dz \tag{5.4}$$

5.3. *The diagonal* Diag_n *and the rightmost column.* The Laplace transform of the left-hand side of (1.23) with respect to (x,s) and (y,u) leads to

$$\sum_{1\le m\le n-1} \frac{1}{s^m}\frac{1}{u^{n-m}} (a_n(m,m)+a_n(m,n))$$

and then to

$$\sum_{1\le m\le n-1} \frac{1}{s^m}\frac{1}{u^n} (a_n(m,m)+a_n(m,n)), \tag{5.5}$$

with the change $s \leftarrow su$. The Laplace transform of the right-hand side of (1.23) yields

$$\int_0^\infty \int_0^\infty \frac{\sin x + \cos x}{\cos(x+y)} e^{sx-uy}\, dx\, dy$$

and with the changes $x \leftarrow z$, $y \leftarrow r$, $s \leftarrow su$

$$\int_0^\infty \int_0^\infty \frac{\sin z + \cos z}{\cos(r+z)} e^{-ru-szu}\, dr\, dz. \tag{5.6}$$

5.4. *Altogether.* By (5.1)—(5.6) we then get (as $a_n(n,k)=0$)

$$(5.7)\ \sum_{1\le k,m\le n} \frac{1}{s^m}\frac{1}{u^n} a_n(m,k) = \int_0^\infty\int_0^\infty (F_1(z,r)+F_2(z,r))\, e^{-zsu-ru} dr\, dz,$$

where

$$F_1(z,r) = \frac{(1-\cos r)\ \sin(2z)}{\cos^2(r+z)} + \frac{\sin z\ \cos(r-z)}{\cos^2(r+z)} + \frac{\sin z}{\cos(r+z)};$$

$$F_2(z,r) = \frac{\sin r\ \sin(2z)}{\cos^2(r+z)} + \frac{(1-\cos z)\ \cos(r-z)}{\cos^2(r+z)} + \frac{\cos z}{\cos(r+z)}.$$

By making those two summations, then using formulas (1.12) and (1.13), this leads to:

$$F_1(z,r) = \frac{\sin(2z)}{\cos^2(r+z)} = \frac{\partial}{\partial r}\sec(r+z)\ \cos(r-z) = \frac{\partial}{\partial r}\Gamma^{P^{(\tan)}}(r,z);$$

$$F_2(z,r) = \frac{\cos(r-z)}{\cos^2(r+z)} = \Gamma^{P^{(\sec)}}(r,z).$$

The expansion of $\Gamma^{P^{(\tan)}}(r,z)$ (resp. of $\Gamma^{P^{(\sec)}}(r,z)$) can be obtained by rewriting (1.3) with the substitutions $f_{n+1}(m) \leftarrow P_{2n+1}$ (resp. $f_{n+1}(m) \leftarrow P_{2n}$), $x \leftarrow r$, $y \leftarrow z$, so that

$$\Gamma^{P^{(\tan)}}(r,z) = \sum_{n\ge 0}\sum_{1\le m\le 2n+1} P_{2n+1}(m)\frac{r^{2n+1-m}}{(2n+1-m)!}\frac{z^{m-1}}{(m-1)!};$$

$$\Gamma^{P^{(\sec)}}(r,z) = \sum_{n\ge 1}\sum_{1\le m\le 2n-1} P_{2n}(m)\frac{r^{2n-1-m}}{(2n-1-m)!}\frac{z^{m-1}}{(m-1)!}.$$

Hence,

$$\frac{\partial}{\partial r}\Gamma^{P^{(\tan)}}(r,z) = \sum_{n\ge 1}\sum_{1\le m\le 2n} P_{2n+1}(m)\frac{r^{2n-m}}{(2n+1-m)!}\frac{z^{m-1}}{(m-1)!}.$$

Thus,

$$\begin{aligned}
&\sum_{1\le k,m\le n} \frac{1}{s^m}\frac{1}{u^n} a_n(m,k)\\
&\quad = \int_0^\infty\int_0^\infty \Big(\frac{\partial}{\partial r}\Gamma^{P^{(\tan)}}(r,z) + \Gamma^{P^{(\sec)}}(r,z)\Big) e^{-zsu-ru} dr\, dz\\
&\quad = \int_0^\infty\int_0^\infty \Big(\sum_{1\le m\le n} P_n(m)\frac{r^{n-m-1}}{(n-m-1)!}\frac{z^{m-1}}{(m-1)!}\Big) e^{-zsu-ru} dr\, dz\\
&\quad = \sum_{1\le m\le n}\frac{1}{(su)^m}\frac{1}{u^{n-m}} P_n(m)
\end{aligned}$$

and

$$\sum_{1\le k\le n} a_n(m,k) = a_n(m,\bullet) = P_n(m) \quad (1\le m\le n). \quad \square$$

6. Marginal distributions. The column sums

Again, apply the Laplace transform to (1.21) and (1.22) in the same way, but making the reduction of variables in a different manner.

6.1. *The upper triangles* Up_n. Apply the Laplace transform to the left-hand side of identity (1.21) three times with respect to (x,u), (y,t), (z,s), respectively. We get

$$\sum_{2\le m+1\le k\le n-1} \frac{1}{s^m}\frac{1}{t^{k-m}}\frac{1}{u^{n-k}}a_n(m,k),$$

which becomes

$$\sum_{2\le m+1\le k\le n-1} \frac{1}{t^k}\frac{1}{u^n}a_n(m,k), \tag{6.1}$$

when $t\leftarrow tu$ and $s\leftarrow tu$. Apply the Laplace transform to the right-hand side of (1.21) three times with respect to (x,u), (y,t), (z,s), respectively, and let $t\leftarrow tu$, $s\leftarrow tu$. We get:

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{(\sin x+\cos x)\,\sin(2z)}{\cos^2(x+y+z)}e^{-xu-(y+z)tu}dx\,dy\,dz.$$

With the change of variables $X=x$, $Y=y$, $Z=y+z$, the integral becomes:

$$\int_0^\infty\int_0^\infty\int_{Y=0}^{Y=Z} \frac{(\sin X+\cos X)\,\sin(2Z-2Y)}{\cos^2(X+Z)}e^{-Xu-Ztu}\,dX\,dY\,dZ$$

$$= \int_0^\infty\int_0^\infty \frac{(\sin X+\cos X)\,\sin^2 Z}{\cos^2(X+Z)}e^{-Xu-Ztu}\,dX\,dZ. \tag{6.2}$$

6.2. *The lower triangles* Low_n. With identity (1.22) apply the Laplace transform to its left-hand side three times with respect to (x,s), (y,t), (z,u), respectively. We get

$$\sum_{2\le k+1\le m\le n-1} \frac{1}{s^k}\frac{1}{t^{m-k}}\frac{1}{u^{n-m}}a_n(m,k),$$

which becomes

$$\sum_{2\le k+1\le m\le n-1} \frac{1}{t^k}\frac{1}{u^n}a_n(m,k), \tag{6.3}$$

when $s\leftarrow tu$ and $t\leftarrow u$. Apply the Laplace transform to the right-hand side of (1.22) three times with respect to (x,s), (y,t), (z,u), respectively, and let $s\leftarrow tu$, $t\leftarrow u$. We get:

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{(\sin x+\cos x)\,\cos(x+y-z)}{\cos^2(x+y+z)}e^{-xtu-(y+z)u}dx\,dy\,dz.$$

With the change of variables $X=y+z$, $Y=y-z$, $Z=x$ the integral becomes, the jacobian of the transformation being equal to 1/2:

$$\int_0^\infty \int_0^\infty \int_{Y=-X}^{Y=X} \frac{(\sin Z + \cos Z)\ \cos(Z+Y)}{\cos^2(X+Z)}\ \frac{1}{2}\, e^{-Ztu-Xu}\, dY\, dX\, dZ$$

$$= \int_0^\infty \int_0^\infty \frac{(\sin Z + \cos Z)\,(\sin(Z+X) - \sin(Z-X))}{2\ \cos^2(X+Z)} e^{-Ztu-Xu} dX\, dZ$$

$$= \int_0^\infty \int_0^\infty \frac{(\sin Z + \cos Z)\sin X\ \cos Z}{\cos^2(X+Z)} e^{-Ztu-Xu}\, dX\, dZ. \tag{6.4}$$

6.3. *The diagonal and the rightmost column.* By § 5.3 we also have:

$$\sum_{1\le m\le n-1} \frac{1}{t^m}\frac{1}{u^n}\, a_n(m,m) = \int_0^\infty \int_0^\infty \frac{\sin Z}{\cos(X+Z)} e^{-Xu-tZu}\, dX\, dZ; \tag{6.5}$$

$$\sum_{1\le m\le n-1} \frac{1}{t^m}\frac{1}{u^n} a_n(m,n) = \int_0^\infty \int_0^\infty \frac{\cos Z}{\cos(X+Z)} e^{-Xu-tZu}\, dX\, dZ. \tag{6.6}$$

6.4. *The first* $(n-1)$ *columns.* Note that the ultimate column of each matrix A_{2n+1} $(n \ge 1)$ has zero entries. Then,

$$\sum_{1\le k,m\le n-1} \frac{1}{t^k}\frac{1}{u^n} a_n(m,k) \tag{6.7}$$
$$= \int_0^\infty \int_0^\infty (G_1(X,Z) + G_2(X,Z)) e^{-Ztu-Xu} dX\, dZ,$$

where

$$\begin{aligned}
G_1(X,Z) &= \frac{\cos X\ \sin^2 Z}{\cos^2(X+Z)} + \frac{\sin X\ \cos^2 Z}{\cos^2(X+Z)} + \frac{\sin Z}{\cos(X+Z)}\\
&= \frac{\cos Z\ \sin(X+Z)}{\cos^2(X+Z)}\\
&= \frac{\partial}{\partial X}\cos Z\ \sec(X+Z) = \frac{\partial}{\partial X}\Gamma^{E^{(\tan)}}(X,Z)\\
&= \sum_{n\ge 1}\sum_{1\le k\le 2n} E_{2n+1}(k)\frac{X^{2n-k}}{((2n-k)!}\frac{Z^{k-1}}{(k-1)!}; \qquad (6.8)\\
G_2(X,Z) &= \frac{\cos X\ \sin^2 Z}{\cos^2(X+Z)} + \frac{\sin X\ \sin Z\ \cos Z}{\cos^2(X+Z)}\\
&= \sin Z\frac{\sin(X+Z)}{\cos^2(X+Z)} = \frac{\partial}{\partial X}\Gamma^{E^{(\sec)}}(Z,X)\\
&= \frac{\partial}{\partial X}\Big(\sum_{1\le m\le 2n} E_{2n}(m)\frac{Z^{2n-m}}{(2n-m)!}\frac{X^{m-1}}{(m-1)!}\Big)
\end{aligned}$$

$$= \sum_{2\le m\le 2n} E_{2n}(m)\frac{Z^{2n-m}}{(2n-m)!}\frac{X^{m-2}}{(m-2)!}$$

$$= \sum_{1\le k\le 2n-1} E_{2n}(2n+1-k)\frac{Z^{k-1}}{(k-1)!}\frac{X^{2n-k-1}}{(2n-k-1)!}. \tag{6.9}$$

Let

$$E_n^*(k) := \begin{cases} E_n(k), & \text{if } n \text{ odd;}\\ E_n(n+1-k), & \text{if } n \text{ even.}\end{cases}$$

Then, by (6.8) and (6.9),

$$G_1(X,Z)+G_2(X,Z) = \sum_{n\ge 2}\sum_{1\le k\le n-1} E_n^*(k)\frac{X^{n-k-1}}{(n-k-1)!}\frac{Z^{k-1}}{(k-1)!}$$

and

$$\sum_{1\le m,k\le n-1}\frac{1}{t^k}\frac{1}{u^n}a_n(m,k) = \sum_{1\le k\le n-1}\frac{1}{t^k}\frac{1}{u^n}E_n^*(k),$$

so that

$$a_n(\bullet,k) = E_n^*(k) \qquad (1\le k\le n-1). \quad \square \tag{6.10}$$

16.5. *The rightmost column.* As the rightmost column of each matrix A_{2n+1} has zero entries, identity (6.6) may be rewritten as:

$$\sum_{1\le k\le 2n-1}\frac{1}{t^k}\frac{1}{u^{2n}}a_{2n}(k,2n) = \int_0^\infty\int_0^\infty \frac{\cos Z}{\cos(X+Z)}e^{-Ztu-Xu}dX\,dZ. \tag{6.11}$$

As $\cos Z/\cos(X+Z) = \Gamma^{(\tan)}(Z,X)$, the derivation of $a_{2n}(m,2n) = E_{2n-1}(m)$ $(1\le m\le 2n-1)$ is straightforward by using the current technique. Hence, $a_{2n}(\bullet,2n) = E_{2n-1}$ holds. To show that $E_{2n}(1) = E_{2n}$ and then

$$a_{2n}(\bullet,2n) = E_{2n}(1) \tag{6.12}$$

the traditional first-order recurrence of the Entringer numbers must be used.

Appendix I

In the following table the row sums $\sum f_{n+1}^{\ell,0}(m)$ and $\sum f_n^{\ell,1}(m)$ for $1\le m\le 2n+1$, written in the middles of the triangles, are the first values of the ℓ-tangent numbers $E_{2n+1}^{(\ell)}$ for $n=1,2,3,4$. Above each ℓ-tangent number $E_{2n+1}^{(\ell)}$ is written the corresponding generating function, as defined in (1.1). Under each triangle $f^{\ell,0}$ and $f^{\ell,1}$ appears the generating function for the triangle, as defined in (1.3).

$$\tan x$$

$$E^{(1)}_{2n+1} = E_{2n+1}$$

$$f^{1,0} = \begin{array}{c} 1 \\ 1\quad 1\quad 0 \\ 5\quad 5\quad 4\quad 2\quad 0 \\ 61\quad 61\quad 56\quad 46\quad 32\quad 16\quad 0 \\ \cdots \end{array} \qquad \begin{array}{c} 1 \\ 2 \\ 16 \\ 272 \\ \cdots \end{array} \qquad \begin{array}{c} 1 \\ 0\quad 2\quad 0 \\ 0\quad 4\quad 8\quad 4\quad 0 \\ 0\quad 32\quad 64\quad 80\quad 64\quad 32\quad 0 \\ \cdots \end{array} = f^{1,1}$$

$$\sec(x+y)\ \cos y \qquad\qquad \sec(x+y)\ \cos(x-y)$$

$$\sec(x)\tan x$$

$$E^{(2)}_{2n+2} = E_{2n+2}$$

$$f^{2,0} = \begin{array}{c} 1 \\ 2\quad 2\quad 1 \\ 16\quad 16\quad 14\quad 10\quad 5 \\ 272\quad 272\quad 256\quad 224\quad 178\quad 122\quad 61 \\ \cdots \end{array} \qquad \begin{array}{c} 1 \\ 5 \\ 61 \\ 1385 \\ \cdots \end{array} \qquad \begin{array}{c} 1 \\ 1\quad 3\quad 1 \\ 5\quad 15\quad 21\quad 15\quad 5 \\ 61\quad 183\quad 285\quad 327\quad 285\quad 183\quad 61 \\ \cdots \end{array} = f^{2,1}$$

$$\sec^2(x+y)\ \cos y \qquad\qquad \sec^2(x+y)\ \cos(x-y)$$

$$\sec^2(x)\tan x$$

$$E^{(3)}_{2n+1}$$

$$f^{3,0} = \begin{array}{c} 1 \\ 3\quad 3\quad 2 \\ 33\quad 33\quad 30\quad 24\quad 16 \\ 723\quad 723\quad 690\quad 624\quad 528\quad 408\quad 272 \\ \cdots \end{array} \qquad \begin{array}{c} 1 \\ 8 \\ 136 \\ 3968 \\ \cdots \end{array} \qquad \begin{array}{c} 1 \\ 2\quad 4\quad 2 \\ 16\quad 32\quad 40\quad 32\quad 16 \\ 272\quad 544\quad 752\quad 852\quad 752\quad 544\quad 272 \\ \cdots \end{array} = f^{3,1}$$

$$\sec^3(x+y)\ \cos y \qquad\qquad \sec^3(x+y)\ \cos(x-y)$$

$$\sec^3(x)\tan x$$

$$E^{(4)}_{2n+1}$$

$$f^{4,0} = \begin{array}{c} 1 \\ 4\quad 4\quad 3 \\ 56\quad 56\quad 52\quad 44\quad 33 \\ 1504\quad 1504\quad 1448\quad 1336\quad 1172\quad 964\quad 723 \\ \cdots \end{array} \qquad \begin{array}{c} 1 \\ 11 \\ 241 \\ 8651 \\ \cdots \end{array} \qquad \begin{array}{c} 1 \\ 3\quad 5\quad 3 \\ 33\quad 55\quad 65\quad 55\quad 33 \\ 723\quad 1205\quad 1555\quad 1685\quad 155\quad 1205\quad 723 \\ \cdots \end{array} = f^{4,1}$$

$$\sec^4(x+y)\ \cos y \qquad\qquad \sec^4(x+y)\ \cos(x-y)$$

The two sum refinements of $E^{(\ell)}_{2n+1}$ for $\ell = 1, 2, 3, 4$

Appendix II

In the following table the first Entringer-Poupard matrices are displayed, together with their row sums (the Poupard numbers) and column sums (the Entringer numbers), and then the total value of their entries (secant/tangent numbers).

$$A_2 = \begin{array}{c|cc|c} & 1 & 2 & \\ \hline 1 & \cdot & 1 & 1 \\ 2 & \cdot & \cdot & 0 \\ \hline & 0 & 1 & 1 = E_2 \end{array}; \quad A_3 = \begin{array}{c|ccc|c} & 1 & 2 & 3 & \\ \hline 1 & \cdot & \cdot & \cdot & 0 \\ 2 & 1 & 1 & \cdot & 2 \\ 3 & \cdot & \cdot & \cdot & 0 \\ \hline & 1 & 1 & 0 & 2 = E_3 \end{array}; \quad A_4 = \begin{array}{c|cccc|c} & 1 & 2 & 3 & 4 & \\ \hline 1 & \cdot & \cdot & \cdot & 1 & 1 \\ 2 & \cdot & & 2 & 1 & 3 \\ 3 & \cdot & 1 & \cdot & \cdot & 1 \\ 4 & \cdot & \cdot & \cdot & \cdot & 0 \\ \hline & 0 & 1 & 2 & 2 & 5 = E_4 \end{array};$$

$$A_5 = \begin{array}{c|ccccc|c} & 1 & 2 & 3 & 4 & 5 & \\ \hline 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 2 & 1 & 1 & 2 & \cdot & \cdot & 4 \\ 3 & 3 & 3 & 2 & \cdot & \cdot & 8 \\ 4 & 1 & 1 & \cdot & 2 & \cdot & 4 \\ 5 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \hline & 5 & 5 & 4 & 2 & 0 & 16 = E_5 \end{array}; \quad A_6 = \begin{array}{c|cccccc|c} & 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 5 & 5 \\ 2 & \cdot & \cdot & 2 & 4 & 4 & 5 & 15 \\ 3 & \cdot & 1 & \cdot & 8 & 8 & 4 & 21 \\ 4 & \cdot & 3 & 6 & \cdot & 4 & 2 & 15 \\ 5 & \cdot & 1 & 2 & 2 & \cdot & \cdot & 5 \\ 6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \hline & 0 & 5 & 10 & 14 & 16 & 16 & 61 = E_6 \end{array};$$

$$A_7 = \begin{array}{c|ccccccc|c} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ \hline 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 2 & 5 & 5 & 10 & 8 & 4 & \cdot & \cdot & 32 \\ 3 & 15 & 15 & 10 & 16 & 8 & \cdot & \cdot & 64 \\ 4 & 21 & 21 & 20 & 14 & 4 & \cdot & \cdot & 76 \\ 5 & 15 & 15 & 12 & 6 & 16 & \cdot & \cdot & 64 \\ 6 & 5 & 5 & 4 & 2 & \cdot & 16 & \cdot & 32 \\ 7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \hline & 61 & 61 & 56 & 46 & 32 & 16 & 0 & 272 = E_7 \end{array};$$

$$A_8 = \begin{array}{c|cccccccc|c} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ \hline 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 61 & 61 \\ 2 & \cdot & \cdot & 10 & 20 & 28 & 32 & 32 & 61 & 183 \\ 3 & \cdot & 5 & \cdot & 40 & 56 & 64 & 64 & 56 & 285 \\ 4 & \cdot & 15 & 30 & \cdot & 76 & 80 & 80 & 46 & 327 \\ 5 & \cdot & 21 & 42 & 62 & \cdot & 64 & 64 & 32 & 285 \\ 6 & \cdot & 15 & 30 & 42 & 48 & \cdot & 32 & 16 & 183 \\ 7 & \cdot & 5 & 10 & 14 & 16 & 16 & \cdot & \cdot & 61 \\ 8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \hline & 0 & 61 & 122 & 178 & 224 & 256 & 272 & 272 & 1385 = E_8 \end{array};$$

$$A_9 = \begin{array}{c|ccccccccc|c} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \\ \hline 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 2 & 61 & 61 & 122 & 112 & 92 & 64 & 32 & \cdot & \cdot & 544 \\ 3 & 183 & 183 & 122 & 224 & 184 & 128 & 64 & \cdot & \cdot & 1088 \\ 4 & 285 & 285 & 280 & 178 & 236 & 160 & 80 & \cdot & \cdot & 1404 \\ 5 & 327 & 327 & 312 & 282 & 224 & 128 & 64 & \cdot & \cdot & 1664 \\ 6 & 285 & 285 & 264 & 222 & 160 & 256 & 32 & \cdot & \cdot & 1404 \\ 7 & 183 & 183 & 168 & 138 & 96 & 48 & 272 & \cdot & \cdot & 1088 \\ 8 & 61 & 61 & 56 & 46 & 32 & 16 & \cdot & 272 & \cdot & 544 \\ 9 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \hline & 1385 & 1385 & 1324 & 1202 & 1024 & 800 & 544 & 272 & 0 & 7936 = E_9 \end{array}$$

Table of the Entringer-Poupard matrices A_n for $n = 2, 3, \ldots, 9$.

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Dominique Foata
Institut Lothaire
1, rue Murner
F-67000 Strasbourg, France
`foata@unistra.fr`

Guo-Niu Han
I.R.M.A. UMR 7501
Université de Strasbourg et CNRS
7, rue René-Descartes
F-67084 Strasbourg, France
`guoniu.han@unistra.fr`