ON THE ROOK POLYNOMIALS OF FERRERS RELATIONS

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1. INTRODUCTION

A quasi-permutation on a set I is a relation on I that is contained in a bijective relation. Formally $Q \subset I \times I$ is a quasi-permutation if and only if

$$(i,j),\ (i',j') \in Q \implies (i,j) = (i',j') \quad \text{or} \quad i \neq i' \text{ and } j \neq j'.$$

For Q finite the weight $\lambda(Q)$ of Q is the number of elements $(i,j) \in Q$ and for any relation $R \subset I \times I$ we shall denote $Q_k(R)$ the set of the quasi-permutations of weight k contained in R. Then assuming R finite, the rook polynomial $\rho(R)$ of R is the generating function

$$1 + \sum_{0<k} t^k \operatorname{card} Q_k(R).$$

We refer the reader to the last two chapters of Riordan's book on combinatorial analysis [4] for the general theory of rook polynomials and their applications. Riordan gives several theorems stating conditions for two relations to have the same rook polynomial or, as we shall say, to be rook-equivalent. Our theorem 3 belongs to this type. It generalizes the obvious fact that any relation is rook-equivalent with its transpose.

The case when R is a Ferrers relation plays a role in the applications since it relates to the Laguerre polynomials, the Eulerian polynomials, the Shanks polynomials, the Poussin polynomials and more generally to the so-called "Newcomb's problem for arbitrary specification". Our main result (theorem 11) states that a cross-section ("minimal set of representatives") of the Ferrers relations with respect to the rook-equivalence is provided by those relations which are strictly decreasing, i.e. which correspond to partitions into unequal parts.

In the last section we effectively compute the rook-equivalent decreasing Ferrers relations for those relations which are total preorders deprived for an arbitrary subset of its equivalence classes (theorem 19). Since this family is closed under complementation, one might apply Riordan's theory of "complementary boards" to deduce non-trivial identities by using the rather explicit expressions for the rook polynomial that are given in our property 5.

2. A GENERAL PROPERTY OF ROOK EQUIVALENCE

We use the standard notation $[n]$ for the ordered set $\{1,2,\dots,n\}$ $([0] = \emptyset)$. Thus we can say in short that α is a (m,n)-injection if and only if it is a map sending each pair $(i,j) \in [m]\times[m]$ onto the pair $(\alpha_1(i), \alpha_2(j)) \in [n] \times [n]$ where α_1 and α_2 are two injections of $[m]$ into $[n]$. We write then $\alpha = \alpha_1 \times \alpha_2$, $A_i = \alpha_i([m])$ $(i = 1,2)$.

Definition 1.

The (m,n)-injection $\alpha : [m]\times[m] \rightarrow [n]\times[n]$ is compatible with the relation $R \subset [n] \times [n]$ if and only if there exist subsets $\bar{A}_i \subset [n]\setminus A_i$ $(i=1,2)$ such that $R \cap (A_1 \times [n]) = R \cap (A_1 \times A_2) \cup (A_1 \times \bar{A}_2)$ and symmetrically $R \cap ([n] \times A_2) = R \cap (A_1 \times A_2) \cup (\bar{A}_1 \times A_2)$.

Definition 2.

With the same notations, the α-transpose of R is the relation

$$R' = R \setminus (A_1 \times A_2) \cup \alpha(\tilde{S}) \subset [n] \times [n]$$

where $\tilde{S}$ is the ordinary transpose

$\tilde{S} = \{(i,j) \in [m] \times [m] : (j,i) \in S\} \subset [m] \times [m]$ of the inverse image

$$S = \alpha^{-1}(R \cap (A_1 \times A_2)) \subset [m] \times [m] .$$

THEOREM 3.

If the injection α is compatible with the relation R, then R and its α-transpose R' are rook-equivalent.

Proof.

Note that reciprocally R is the α-transpose of R'. Accordingly, if suffices to construct an injective map of $\underset{\sim}{Q}(R)$ into $\underset{\sim}{Q}(R')$.

Consider any given quasi-permutation Q contained in R. The inverse image $P = \alpha^{-1}(Q \cap (A_1 \times A_2))$ is a quasi-permutation contained in $S = \alpha^{-1}(R \cap (A_1 \times A_2)) \subset [m] \times [m]$.

Let B_1 and B_2 be the least subsets of $[m]$ that satisfy $P \subset B_1 \times B_2$. Since P is a quasi-permutation, we have $\lambda(P) = \underset{\sim}{\mathrm{Card}}\, B_1 = \underset{\sim}{\mathrm{Card}}\, B_2$.

We define a (m,m)-injection $\sigma = \sigma_1 \times \sigma_2$ by the following two conditions where $i, i' = 1,2$ and $i' \neq i$.

(1) The restriction of σ_i to $[m] \setminus B_i$ is the unique order-preserving bijection of this set onto $[m] \setminus B_{i'}$;

(2) For each $(k,k') \in P$ we set $\sigma_1(k) = k'$ and $\sigma_2(k') = k$.

Clearly $\sigma(P)$ is a quasi-permutation contained in the transpose $\tilde{S}$ of S.

We now extend σ to a (n,n)-injection $\tau = \tau_1 \times \tau_2$ by letting

$$\tau_i(j) = j \quad \text{for any} \quad j \in [n] \setminus A_i ;$$

$$\tau_i(j) = \alpha_i(\sigma_i(\alpha_i^{-1}(j))) \qquad \text{for any} \quad j \in A_i .$$

Because of the compatibility condition $R \setminus (A_1 \times A_2)$ is invariant under τ. Thus $Q' = \tau(Q)$ is a quasi-permutation contained in R'. Finally $Q \longrightarrow Q'$ is injective because in view of our canonical choice of σ, this injection, hence Q itself, are determined by Q' without ambiguity.

3. APPLICATION TO FERRERS RELATIONS

Let $\mathbb{P} = \mathbb{N} \setminus \{0\}$ denote the set of all positive integers. A Ferrers relation is a relation $F(\varphi) \subset \mathbb{P} \times \mathbb{P}$ that is defined by a non-increasing map $\varphi : \mathbb{P} \longrightarrow \mathbb{N}$ such that the sum $\sum_i \varphi(i)$ is finite and the condition

$$F(\varphi) = \{(i,j) \in \mathbb{P} \times \mathbb{P} : \ j \leq \varphi(i)\} .$$

Its transpose $\tilde{F}(\varphi)$ is another Ferrers relation $F(\tilde{\varphi})$ where $\tilde{\varphi} : \mathbb{P} \longrightarrow \mathbb{N}$ is defined by

$$\begin{aligned} \tilde{\varphi}(i) &= \mathrm{Max}\,\{ j \in \mathbb{P} : \ \varphi(j) \geq i \} \qquad && \text{if} \quad i \leq \varphi(i) \\ &= \qquad 0 && \text{otherwise} \end{aligned}$$

We shall say shortly that φ is <u>decreasing</u> if and only if it is so on the non-trivial part of its domain, i.e. if and only if $\varphi(i) > 0$ implies $\varphi(i) > \varphi(i+1)$. Then $F(\varphi)$ will be called a <u>decreasing</u> Ferrers relation.

In the next two lemmas we consider a fixed decreasing map φ and for convenience we set

$$\begin{aligned} p &= \tilde{\varphi}(1) \quad (= \mathrm{Max}\{ i \in \mathbb{P} : \ \varphi(i) > 0 \}) \\ x_i &= \varphi(i) \qquad\qquad (i \in [p]) ; \\ r_0 &= 1 ; \\ r_k &= \sum_{(k)} \prod_{1 \leq j \leq k} (x_{i_j} - k + j) \qquad\qquad (k \in [p]) \\ r_k &= 0 \quad \text{for} \quad \mathfrak{p} > 0 \end{aligned}$$

where $\sum_{(k)}$ indicates a summation over the set $[p]^{(k)}$ of the $\binom{p}{k}$ strictly increasing sequences $(i_1 < i_2 < \cdots < i_k)$ of length k with elements in $[p]$.

Lemma 4. The rook polynomial of $F(\varphi)$ is

$$\rho(F(\varphi)) = \sum_{0 \le k \le p} t^k r_k .$$

Proof. For any relation $R \subset [p] \times \mathbb{P}$ the set $Q_k(R)$ of the quasi-permutations of weight k contained in R is empty for $k > p$; and for $k \le p$ it is the disjoint union over all sequences (of length k) $I \subset [p]^{(k)}$ of the sets $Q_k((R \cap I) \times \mathbb{P})$.

Further, when $R = F(\varphi)$, each restriction $F(\varphi) \cap (I \times \mathbb{P})$ is rook-equivalent with a decreasing Ferrers relation $F(\varphi')$ where $\varphi'(j) = \varphi(i_j)$ $(I = (i_1, i_2, \ldots, i_k))$ and $\tilde{\varphi}'(1) = k$.

Thus it suffices to consider the special case of $k = p$. Then $r_p = (x_1 - p + 1)(x_2 - p + 2) \ldots (x_{p-1} - 1) x_p$ and the equality of this quantity with $\underset{\sim\sim}{\text{Card}}\, Q_p(F(\varphi))$ is trivial since this last number is the number of injections $\eta : [p] \longrightarrow \mathbb{P}$ such that $\eta(i) \le \varphi(i)$ identically.

Q.E.D.

PROPERTY 5. For each $k \in [p]$ one has the identity

$$r_k = \sum_{0 \le j \le k} a_j\, S(p-j\, ,\, p-k) \qquad (0 \le k \le p)$$

where the $S(i,j)$'s are Stirling numbers of the second kind* and where a_j $(0 \le j \le p)$ are the symmetric functions $a_0 = 1$ and $a_j = \sum_{(j)} y_{i_1} y_{i_2} \cdots y_{i_j}$ $(j \in [p])$ in the variables $y_i = x_i - p + i$ $(i \in [p])$.

Proof. Let $p' = p - 1$; $x'_i = x + 1$ $(i \in [p'])$ and denote by

*We make the usual convention that $S(0,0) = 1$ and $S(i,j) = 0$ if exactly one of i and j is zero.

primed letters the quantities defined with respect to p' and the x'_i's in the same manner as the corresponding quantities (r, y or a) were defined with respect to p and the x_i's.

Thus, by definition

$$r_k = (x_1 - k+1) r'_{k-1} + r'_k = (y_1 + p - k) r'_{k-1} + r'_k$$

and

$$y_{i+1} = x_{i+1} - p + i + 1 = x'_i - p' + i = y'_i \qquad (i \in [p']).$$

Using induction on p, the first relation gives

$$r_k = (y_1 + p - k) \sum_j a'_j S'(p'-j, p'-k+1) + \sum_j a'_j S'(p'-j, p'-k) =$$

$$= \sum_j y_1 a'_j S(p'-j, p'-k+1) + \sum_j a'_j ((p-k) S(p'-j, p'-k+1) + S(p'-j, p'-k))$$

that is

$$r_k = \sum_j (y_1 a'_{j-1} + a'_j) S(p-j, p-k)$$

because of the classical identity

$$S(p-j, p-k) = (p-k) S(p-j-1, p-k) + S(p-j-1, p-k-1)$$

and the result follows since the second relation implies the identity $a_j = y_1 a'_{j-1} + a'_j$.

Q.E.D.

Corollary 6. The Ferrers relations $F(\varphi)$ and $F(\varphi')$ defined by two decreasing maps φ and φ' are rook-equivalent only if $\varphi = \varphi'$.

Proof. Let $\tilde{\varphi}(1) = p \geq \tilde{\varphi}'(1) = p'$. If $p \neq p'$, t^p has a positive coefficient in $\rho(F(\varphi))$ and a zero coefficient in $\rho(F(\varphi'))$. Thus we can assume $p = p'$.

Because of the strictly decreasing character of the sequence

$\underset{\sim}{x} = (x_1, x_2, \cdots, x_p)$, the sequence $\underset{\sim}{y} = (y_1, y_2, \cdots, y_p)$ is non-increasing. Its members are positive because $y_p = x_p > 0$.

Since the correspondence $\underset{\sim}{x} \longrightarrow \underset{\sim}{y}$ is injective for each given p, it follows that $\underset{\sim}{x} \longrightarrow \{y_1, y_2, \ldots, y_p\}$ is also injective. Now the set $\{y_1, y_2, \cdots, y_p\}$ is unequivocally determined by the symmetric functions a_k and the result follows from lemma 5.

Q.E.D.

Observation 7. Because of the well-known orthogonality relations between the Stirling numbers of the first kind $s(i,j)$ and those of the second kind $S(i,j)$, the formula of lemma 5 is equivalent with

$$a_k = \sum_j r_j\, s(p-j, p-k).$$

As a side remark we may note that the formula of Property 6 does not depend upon the fact that the x_i's form a decreasing sequence of positive integers. By taking all the y_i's equal to 0 (resp. to 1) and by using a straightforward computation this formula gives directly the known identities

$$S(p, p-k) = \sum_{(k)} i_1 (i_2 - 1) \ldots (i_k - k + 1)$$

$$S(p+1, p+1-k) = \sum_{0 \le j \le k} \binom{p}{j} S(p-j, p-k).$$

We now return to our main argument. To simplify notations, each Ferrers relation $F(\varphi)$ is considered as a relation in $[n] \times [n]$ where $n = \sum_i \varphi(i)$.

Definition 8.

Let $\varphi: \mathbb{P} \longrightarrow \mathbb{N}$ be non-increasing. The element $(k, k') \in \mathbb{P} \times \mathbb{P}$ is <u>admissible</u> if and only if the following inequalities hold

(1) $$0 \le \varphi(k) - k' \le \tilde{\varphi}(k') - k \le \varphi(k-1) - k'$$

where by convention $\varphi(k-1) - k' = +\infty$ for $k = 1$.

Definition 8 bis.

If (k,k') is admissible, the (k,k')-transform of φ is the map $\varphi' = \Theta(k,k';\varphi)$ defined by $\varphi'(i) = k'-k+\tilde{\varphi}(k'-k+i)$ if $k \le i \le \tilde{\varphi}(k')$ and $\varphi'(i) = \varphi(i)$ otherwise.

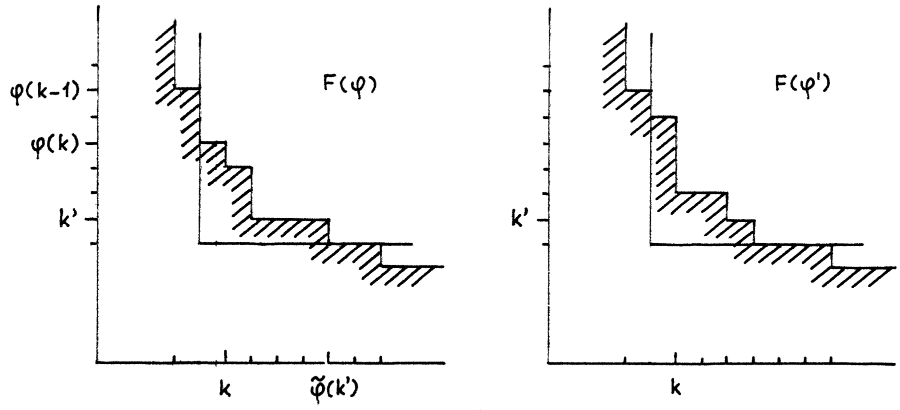

Fig. 1.

As can be seen from figure 1, where (k,k') is admissible, the relation $F(\varphi')$ is obtained from $F(\varphi)$ by transposing the set $F(\varphi) \cap B$ where $B = \{(i,j) \in \mathbb{P}\times\mathbb{P} : i \ge k,\ j \ge k'\}$ and leave $F(\varphi)\setminus B$ invariant. As shown in lemma 5 below, the admissibility conditions will insure that the relation $F(\varphi')$ obtained from $F(\varphi)$ is still a Ferrers relation having the same rook polynomial as $F(\varphi)$.

Lemma 9.

Let φ, (k,k') and $\varphi' = \Theta(k,k';\varphi)$ as above and assume $\sum \varphi(i) \le n$. Then φ' is a non-increasing map such that the Ferrers relations $F(\varphi)$ and $F(\varphi')$ are rook-equivalent.

Proof.

Using inequalities (1) and the fact that φ is non-increasing, we have $k' \le \varphi(i) \le \varphi(k)$ for each i such that $k \le i \le \tilde{\varphi}(k')$. Accordingly the set $F(\varphi) \cap B = \{(i,j) \in F(\varphi) : i \ge k,\ j \ge k'\}$ is entirely contained in $[k, \tilde{\varphi}(k')] \times [k', \varphi(k)]$.

Let $m = \tilde{\varphi}(k') - k + 1$ and consider the (m,n)-injection α such that $\alpha(i,j) = (k-1+i, k'-1+j)$ identically.

Keeping the same notations as in section 2 we have

$$A_1 = \alpha_1([m]) = [k, \tilde{\varphi}(k')] \quad \text{and} \quad A_2 = \alpha_2([m]) = [k', k'-1+m] .$$

Again using inequalities (1) we get $k'-1+m = \tilde{\varphi}(k') + k' - k \geq \varphi(k) \geq k'$. Hence $F(\varphi) \cap B$ is a subset of $A_1 \times A_2$. Therefore $\alpha^{-1}(F(\varphi) \cap (A_1 \times A_2))$ is the Ferrers relation $F(\psi) \subset [m] \times [m]$ where $\psi(i) = \varphi(k-1+i) - k'+1$ for each $i \in [m]$.

For the same reason $F(\varphi) \setminus (A_1 \times A_2)$ is the Ferrers relation $F(\bar{\varphi})$ where $\bar{\varphi}(i) = k'-1$ or $= \varphi(i)$ depending upon $k \leq i \leq k-1+m = \tilde{\varphi}(k')$ or not. It follows that $F(\bar{\varphi}) \cap (A_1 \times [n]) = A_1 \times [k'-1]$.

Because of inequalities (1) we have

$F(\bar{\varphi}) \cap ([n] \times A_2) = [k-1] \times A_2$ since either $[k-1] = \phi$ or $\varphi(k-1) \geq \tilde{\varphi}(k') + k' - k = k'-1+m$.

Thus the compatibility condition is satisfied and $F(\varphi)$ is rook-equivalent with its α-transpose $F' = F(\bar{\varphi}) \cup \alpha(F(\tilde{\psi}))$.

Now since $\bar{\varphi}(i) = k'-1$ for $k \leq i \leq k-1+m$ we have

$$F' = \{(i,j) : j \leq \bar{\varphi}(i) + \psi'(i)\} \quad \text{where} \quad \psi'(i) = \tilde{\psi}(i-k+1) \text{ if } k \leq i \leq k-1+m ;$$
$$= 0 \text{ otherwise.}$$

Direct computation shows that $\bar{\varphi} + \psi'$ is in fact the (k,k')-transform φ' of φ .

Further, inequalities (1) imply $\varphi'(k-1) \geq \varphi'(k)$ if $k > 1$. Since both $\bar{\varphi}$ and ψ' (for $i \geq k$) are non-increasing, this establishes that φ' is also non-increasing, and that, accordingly, $F' = F(\varphi')$ is a Ferrers relation.

Q.E.D.

Remark 10.

Consider any non-increasing map φ. It follows from definition 4 that the pair $(1, \varphi(1))$ is always admissible (with respect to φ) since inequalities (1) reduce to

$$0 \le \varphi(1) - \varphi(1) \le \tilde{\varphi}(\varphi(1)) - 1 .$$

Let $m = \tilde{\varphi}(\varphi(1))$. The $(1,\varphi(1))$ -transform φ' of φ satisfies

$$\begin{aligned} \varphi'(1) &= \varphi(1) + m - 1 \\ \varphi'(i) &= \varphi(i) - 1 && \text{for } 1 \le i \le m; \\ \varphi'(i) &= \varphi(i) && \text{otherwise} . \end{aligned}$$

In particular $\varphi = \varphi'$ if and only if $m = 1$, that is, if and only if $\varphi(1) > \varphi(2)$ since $m = \tilde{\varphi}(\varphi(1))$ is characterized by $\varphi(1) = \varphi(2) = \cdots$ $\cdots = \varphi(m) > \varphi(m+1)$.

We shall say that $\bar{\varphi}$ is a transform of the non-increasing map φ if and only if $\bar{\varphi}$ can be obtained by a succession of (k,k') -transformations (in which, of course, the admissibility conditions are satisfied).

We now come to our main theorem.

THEOREM 11.

Each Ferrers relation is rook-equivalent with exactly one decreasing Ferrers relation.

Proof. In view of Corollary 6, it suffices to show that any Ferrers relation has a transform which is decreasing.

Consider a Ferrers relation $F(\varphi)$ which is not decreasing and let j be the least value such that $\varphi(j) = \varphi(j+1)$. Setting $k' = \varphi(j)$ and $j' = \tilde{\varphi}(k') = \mathrm{Max}\{ i \in [n] : \varphi(i) = \varphi(j)\}$ we have $j < j'$, hence

$$0 = \varphi(j) - k' < j' - j = \tilde{\varphi}(k') - j .$$

proving that $k = \mathrm{Min}\{ i \in [n] : \varphi(i) - k' < \tilde{\varphi}(k') - i \}$ is positive.

Because of the minimal character of k, the pair (k, k') is admissible. Thus φ admits a (k,k')-transform $\varphi' = \Theta(k, k'; \varphi)$. Further φ' precedes φ in the sense that the sequence $(\varphi'(1), \varphi'(2), \dots, \varphi'(n))$ precedes the sequence $(\varphi(1), \varphi(2), \dots, \varphi(n))$ in lexicographic order because by construction

$$\varphi'(i) = \varphi(i) \quad \text{for } i < k \quad \text{and}$$

$$\varphi'(i) = \tilde{\varphi}(k') + k' - k > \varphi(i) .$$

Using induction on this order, it concludes the proof.

Q.E.D.

The subsequent results require some further definitions.

For $a > 0$ and $b \geq 0$ we shall denote $\eta_{a,b}$ the non-increasing map such that $\eta_{a,b}(i) = b$ or 0 depending upon $i \leq a$ or not. Let φ be a non-increasing map. The transpose of $\tilde{\varphi} + \eta_{a,b}$ will be called the (a,b)-translate of φ and denoted $\Delta(a, b; \varphi)$.

From the geometrical point of view the graph of $\Delta(a,b;\varphi)$ is obtained from $F(\varphi)$ by first considering $F(\varphi)$ as a subset of $\mathbb{P} \times [a]$ and then making a b-length translation of $F(\varphi)$ to the right.

In particular if $a \geq \varphi(1)$, the $(a-b)$-translate of φ is the non-increasing map φ' such that

$$\varphi'(i) = a \qquad \text{for } i \leq b \quad \text{and}$$

$$\varphi'(i) = \varphi(i-b) \quad \text{for } i > b .$$

From the proof of Theorem 11 it follows that the unique decreasing map $\hat{\varphi}$ in which the Ferrers relations $F(\varphi)$ and $F(\hat{\varphi})$ are rook-equivalent is a transform of φ; we shall then say that $\hat{\varphi}$ is the decreasing transform of φ.

The following lemma is a special case of a theorem in Riordan ([4] p. 181, theorem 3).

Lemma 12. Let $\bar{\varphi}$ (resp. $\bar{\varphi}'$) be a transform of φ (resp. φ') Then

(i) $\bar{\varphi} + \eta_{a,b}$ is a transform of $\varphi + \eta_{a,b}$ if we have $a \geq \tilde{\varphi}(1)$;

(ii) $\Delta(a,b;\bar{\varphi})$ is a transform of $\Delta(a,b;\varphi)$ if the inequality $a \geq \varphi(1)$ holds;

(iii) $\Delta(a,b;\bar{\varphi}) + \bar{\varphi}'$ is a transform of $\Delta(a,b;\varphi) + \varphi'$ if the inequalities $a \geq \varphi(1)$ and $b \geq \tilde{\varphi}'(1)$ both hold.

Proof.

For proving (i) and (ii), is suffices to consider the case when $\bar{\varphi} = \Theta(k,k';\varphi)$. Direct verification shows that

(i) if $a \geq \tilde{\varphi}(1)$, the pair $(k, k'+b)$ is admissible for $\varphi_1 = \varphi + \eta_{a,b}$ and then we have $\bar{\varphi}_1 = \Theta(k, k'+b; \varphi_1)$;

(ii) if $a \geq \varphi(1)$, the pair $(k+b, k')$ is admissible for $\varphi_2 = \Delta(a,b;\varphi)$ and we get $\bar{\varphi}_2 = \Theta(k+b, k'; \varphi_2)$.

Now let $\psi = \Delta(a,b;\varphi) + \varphi'$ and $\bar{\psi} = \Delta(a,b;\bar{\varphi}) + \bar{\varphi}'$. We already know from part (ii) that $\bar{\varphi}_1 = \Delta(a,b;\bar{\varphi})$ is a transform of $\varphi_1 = \Delta(a,b;\varphi)$. Note also that $\varphi_1(i) = a$ for $i \leq b$ and $\tilde{\varphi}_1(i) = 0$ for $i > a$; this implies

$$\tilde{\psi}(i) = \tilde{\varphi}'(i-a) \quad \text{for} \quad i > a.$$

To prove (iii) it suffices to consider the case when $\bar{\varphi}' = \Theta(k,k';\varphi')$. Since $b \geq \tilde{\varphi}'(1)$, we necessarily have $k \leq b$ and then $\varphi_1(k) = a$. It then follows that

$$\begin{aligned} \psi(k) - (k'+a) &= \varphi_1(k) + \varphi'(k) - (k'+a) \\ &= \varphi'(k) - k'. \end{aligned}$$

We also have

$$\tilde{\psi}(k'+a) - k = \tilde{\varphi}'(k') - k \qquad \text{and}$$

$$\psi(k-1) - (k'+a) = \varphi'(k-1) - k'.$$

These last three equations show that the pair $(k, k'+a)$ is admissible for ψ and then we have $\tilde{\psi} = \Theta(k, k'+a; \psi)$.

Q.E.D.

We consider now a very special case needed in the study of Newcomb's problem.

Definition 13.

Let $\underset{\sim}{d} = (d_1, d_2, \ldots, d_r)$ be a sequence of $r > 0$ positive integers and set $\bar{d}_s = d_1 + d_2 + \cdots + d_s \quad (1 \le s \le r)$.

The special map of type $\underset{\sim}{d}$, denoted by $\underset{\sim}{\text{Spec}}\ \underset{\sim}{d}$, is the mapping φ from $\mathbb{P}$ into $\mathbb{N}$ that is defined by the following conditions:

$$\varphi(i) = 0 \qquad \text{for} \qquad i > \bar{d}_r$$

$$\varphi(\bar{d}_r) = 1 \qquad \text{and}$$

$$\varphi(i) = \varphi(i+1) + 2 \quad \text{or} \quad = \varphi(i+1) + 1 \quad \text{depending upon}$$

$$i \in \{\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_{r-1}\} \qquad \text{or not for} \qquad 1 \le i < \bar{d}_r.$$

The definition implies that any special map is decreasing. One computes easily that the value at 1 of $\underset{\sim}{\text{Spec}}\ \underset{\sim}{d} = \varphi$ is $\bar{d}_r + r - 1$ and that $\tilde{\varphi}(1) = d_r$.

For $\underset{\sim}{a} = (a_1)$ the Ferrers relation $F(\underset{\sim}{\text{Spec}}\,(d))$ is a "triangular board" in Riordan's terminology (see [4] p. 213). Then its rook polynomial is given by Riordan's formula ([4] p. 214) which involves the Stirling numbers $S(i,j)$ as it could be shown directly from our property 5, since it corresponds to the case when $y_i = 1$ identically.

We now give two results which show that "translation" preserves the "special" character of a Ferrers relation.

Lemma 14. Let $\psi = \underset{\sim}{\text{Spec}}(d_1, d_2, \dots, d_r)$ and $q \le r$. The special map $\bar{\psi} = \underset{\sim}{\text{Spec}}(d_1+1, d_2+1, \dots, d_q+1, d_{q+1}, \dots, d_r)$ is the decreasing transform of $\Delta(d,q;\psi)$ where $d = 1+\psi(1)$.

Proof.

Let $\varphi = \Delta(d,q;\psi)$; we have $\varphi(i) = d$ for $1 \le i \le q$ and $\varphi(i) = \psi(i-q)$ for $q < i \le \bar{d}_r + q$. On the other hand $\bar{\psi}(i) = \psi(i-q)$ for $\bar{d}_q + q \le i \le \bar{d}_r + q$.

Consequently, if $q = 1$, it is readily verified that $\Delta(d,1;\psi) = \bar{\psi}$ and the lemma is proved in this case.

If $q > 1$, we construct the $(1,\varphi(1))$-transform φ' of φ. We have $\varphi'(1) = \varphi(1) + q - 1$ $(= d')$, $\varphi'(i) = \varphi(1) - 1 = \psi(1)$ for $1 < i \le q$ and $\varphi'(i) = \varphi(i)$ for $i > q$.

We now distinguish two cases:

(i) suppose $d_1 > 1$, and let $\psi' = \underset{\sim}{\text{Spec}}(d_1 - 1, d_2, \dots, d_r)$; one can verify that φ' is obtained by the following two transformations. We first let $\psi'' = \Delta(\psi(1), q; \psi')$; then we have $\varphi' = \Delta(d', 1; \psi'')$. On the other hand let $\bar{\psi}' = \underset{\sim}{\text{Spec}}(d_1 - 1 + 1, d_2 + 1, \dots, d_q + 1, d_{q+1}, \dots, d_r)$; as we have $d' = 1 + \psi'(1) + q = 1 + \bar{\psi}'(1)$, we get $\bar{\psi} = \Delta(d', 1; \bar{\psi}')$.

By induction on $d_1 + d_2 + \dots + d_r$ the special map $\bar{\psi}'$ is a transform of ψ'' and by lemma 12 $\bar{\psi} = \Delta(d', 1; \bar{\psi}')$ is a transform of $\varphi' = \Delta(d', 1; \psi'')$. This concludes the proof in this case.

(ii) suppose $d_1 = 1$ and construct the $(2, \varphi'(2))$-transform φ'' of φ'. Using the same device as above, let $\psi' = \underset{\sim}{\text{Spec}}(d_2, d_3, \dots, d_r)$; we successively form $\psi'' = \Delta(1 + \psi'(1), q-1; \psi')$ and $\psi''' = \Delta(d'-1, 1; \psi'')$ where $d' = \psi(1) + q$. Then we obtain $\varphi'' = \Delta(d', 1; \psi''')$. On the other hand, let $\bar{\psi}' = \underset{\sim}{\text{Spec}}(d_2+1, d_3+1, \dots, d_q+1, d_{q+1}, \dots, d_r)$; we obtain $\bar{\psi} = \Delta(d', 1; \chi)$ where $\chi = \Delta(d'-1, 1; \bar{\psi}')$.

Again by induction and using lemma 12 as above, we verify that $\bar{\psi}$ is a transform of φ'', hence of φ.

Q.E.D.

Lemma 15.

Let be ψ as above, $q \geq r \geq 0$, $d = \psi(1) + q - r + 1$ and $\varphi = \Delta(d, q; \psi)$. Then the decreasing transform of φ is $\bar{\psi} = \underset{\sim}{\text{Spec}}(d'_1, d'_2, \dots, d'_q)$ where $d'_i = 1$ for $i \leq q-r$ and $d'_i = 1 + d_{i-q+r}$ for $i > q-r$.

Proof.

Lemmas 14 and 15 coincide for $q = r$. Suppose $q > r$, and take the $(1, \varphi(1))$ -transform φ' of φ. As in the proof of lemma 14, we verify that φ' is obtained from ψ by the two successive transformations. First let $\psi'' = \Delta(\psi(1)+q-r, q-1; \psi)$; then $\varphi' = \Delta(\bar{\psi}(1), 1; \psi'')$. By induction on q a transform of $\psi'' = \Delta(\psi(1) + (q-1) - r+1, q-1; \psi)$ is given by $\bar{\psi}'' = \underset{\sim}{\text{Spec}}(d''_1, d''_2, \dots, d''_{q-1})$ where $d''_i = 1$ for $i \leq q-1-r$ and $d''_i = 1 + d_{i-q+1+r}$ for $i > q-1-r$. Therefore it follows from lemma 12 that $\bar{\psi} = \Delta(\bar{\psi}(1), 1; \bar{\psi}'')$ is a transform of $\varphi' = \Delta(\bar{\psi}(1), 1; \psi'')$ (hence of φ).

Q.E.D.

It is to be noted that for $r = 0$, i.e. for $\varphi = \eta_{q,q}$, one has simply $\bar{\psi} = \sum(1, 1, \dots, 1)$. This is a special case of Riordan's formula (34) ([4] p. 211).

4. APPLICATION TO NEWCOMB'S PROBLEM

Let $\gamma : [n] \longrightarrow [p]$ be an order-preserving surjection and J a subset of $[p]$. Then the relation $\bar{R}(\gamma, J) \subset [n] \times [n]$ is defined by the conditions $(i,j) \in \bar{R}(\gamma, J)$ if and only if $\gamma(i) < \gamma(j)$ or $\gamma(i) = \gamma(j) \in J$. Thus for $J = [p]$ the relation $\bar{R}(\gamma, J)$ is simply the total preorder induced by γ, and $\bar{R}(\gamma, J)$ is the complement of a total preorder when $J = \emptyset$. Both extreme cases occur naturally in Newcomb's problem that involves the determination of the rook polynomial of $\bar{R}(\gamma, [p])$ or $\bar{R}(\gamma, \emptyset)$. In the generalization of this problem studied by one of us ([1] & [2]) it appears just as natural to consider the rook polynomial of $\bar{R}(\gamma, J)$ for any subset J of $[p]$. This is the purpose of this last section.

Keeping the same notations we let $\pi : [p] \longrightarrow \mathbb{P}$ be the map defined by

$$\pi(j) = \underset{\sim\sim}{\text{card}}\, \gamma^{-1}(j) \qquad (j \in [p])$$

and prove the following property (Cf. Riordan [4], Ex. 4, p. 185).

Property 16. For any permutation σ of $[p]$, the relations $\bar{R}(\gamma, J)$ and $\bar{R}(\sigma\gamma, \sigma J)$ are rook-equivalent.

Proof.

It suffices to verify the property in the special case where σ is the transposition exchanging two consecutive values q and $q+1$ of $[p]$.

Set

$$m = \pi(q) + \pi(q+1)\,;$$

$$a' = \text{Min}\,\{i \in [n] : \gamma(i) = q\}\,;$$

$$a = \text{Max}\,\{i \in [n] : \gamma(i) = q+1\}\,;$$

and define the (m,n)-injection α by $\alpha(i,j) = (a+1-i,\ a'+j-1)$.

The verification of the compatibility condition is trivial and one sees that $\bar{R}(\sigma\gamma, \sigma J)$ is the α-transpose of $\bar{R}(\gamma, J)$.

Q.E.D.

Observe now that the relation

$$R(\gamma, J) = \{(i,j) \in [n] \times [n] : (i, n+1-j) \in \bar{R}(\gamma, J)\}$$

is obviously a Ferrers relation which is rook-equivalent to $\bar{R}(\gamma, J)$. We let $\varphi = \varphi(\gamma, J)$ denote the non-increasing map such that $R(\gamma, J)$ is the Ferrers relation $F(\varphi)$. According to property 16 we can assume that J is the interval $[p']$ with $0 \leq p' \leq p$ and that the two restrictions of π respectively to $[p']$ and $[p] \setminus [p']$ are <u>non-increasing</u>.

Throughout this section φ will designate the non-increasing map $\varphi = \phi(\gamma, J)$ with $\gamma : [n] \longrightarrow [p]$ and $J = [p']$ $(0 \leq p' \leq p)$.

We first consider the case when $J = [p]$, that is when $\bar{R}(\gamma, [p])$ is the <u>total preorder</u> induced by γ. With our conventions the map $\pi : [p] \longrightarrow \mathbb{P}$ is then non-increasing and we can define the non-increasing map $\tilde{\pi}$ its transpose, as defined in the beginning of section 3.

Lemma 17.

The decreasing transform of the defining map $\varphi = \phi(\gamma, [p])$ of the Ferrers relation $R(\gamma, [p])$ is the special map

$$\psi = \underset{\sim}{\mathrm{Spec}}(\tilde{\pi}(q), \tilde{\pi}(q-1), \dots, \tilde{\pi}(1))$$

where $q = \pi(1)$.

Proof.

For $p = 1$, we have on the one hand

$\pi(1) = n = q$, $\tilde{\pi}(1) = \tilde{\pi}(2) = \dots = \tilde{\pi}(n) = 1$, $\tilde{\pi}(n+1) = 0$.

On the other hand, $\varphi = \eta_{n,n}$ since $R(\gamma, [p]) = [n] \times [n]$. Thus the result is covered by Lemma 15 and we can use induction on $p \geq 2$.

Define $\pi' : [p-1] \longrightarrow \mathbb{P}$ by letting $\pi'(i) = \pi(i+1)$ for each $i \in [p-1]$. Thus let $r = \pi'(1)$ $(= \pi(2))$; we have

(1) $$r \leq q.$$

Moreover, let

$$d'_1 = \tilde{\pi}'(r),\ d'_2 = \tilde{\pi}'(r-1), \dots, d'_r = \tilde{\pi}'(1)$$

and $\psi' = \underset{\sim}{\mathrm{Spec}}(\underset{\sim}{d'})$; then, by the induction hypothesis, ψ' is the decreasing transform of $\varphi' = \phi(\gamma', [p-1])$ where γ' is the unique order-preserving surjection of $[n']$ $(n' = n - \pi(1))$ onto $[p-1]$ that satisfies

$\pi'(j) = \underset{\sim}{\text{card}}\ \gamma'^{-1}(j)$ $(j \in [p-1])$ identically.

As we know, the value of ψ' at 1 is

$$\psi'(1) = d'_1 + d'_2 + \cdots + d'_r + r - 1 = \sum \tilde{\pi}'(i) + r - 1 .$$

On the other hand

$$\sum \tilde{\pi}'(i) = \sum \pi'(i) = \sum \pi(i+1) = n - \pi(1) = n - q ;$$

hence

(2) $$n = \psi'(1) + q - r + 1 .$$

We also note that the sequence $(\tilde{\pi}(q), \tilde{\pi}(q-1), \ldots, \tilde{\pi}(1))$ is the sequence obtained when putting $q-r$ elements equal to 1 in front of $\underset{\sim}{d}'$ and increasing by 1 each term of $\underset{\sim}{d}'$. In other words

$$\psi = \underset{\sim}{\text{Spec}}\,(1, \ldots, 1, 1+d'_1, \ldots, 1+d'_r)$$

where the 1's are repeated $(q-r)$ times in the sequence.

Furthermore φ is seen to be the (n,q)-translate of φ', i.e.

(3) $$\varphi = \Delta(n, q; \varphi') .$$

As $n \geq \varphi'(1)$, the conditions of lemma 12 are fulfilled and we conclude that φ is a transform of $\bar{\varphi} = \Delta(n, q; \psi')$.

In view of (1), (2) and (3) we can apply lemma 15 by taking ψ' instead of ψ, $d = n$ and $\bar{\varphi}$ in place of φ and we deduce that ψ is a transform of $\bar{\varphi}$, hence of φ.

Q.E.D.

We now study the case of <u>complements of total preorders</u>. With the same notations as above, the set J is assumed to be empty. In the following lemma we characterize the decreasing Ferrers relation that is rook-equivalent to $R(\gamma,\phi)$. We can assume $p>1$, since otherwise $R(\gamma,\phi)=\bar{R}(\gamma,\phi)=\phi$. We set $b=\pi(1)-\pi(2)$ and $r=\pi(2)$. Since π is non-increasing, we can consider the transpose $\tilde{\pi}$ of π. We then have $r=\mathrm{Max}\{i:\tilde{\pi}(i)>1\}$.

Lemma 18.

The decreasing transform of the defining map $\varphi=\dot{\phi}(\gamma,\phi)$ of $R(\gamma,\phi)$ is $\psi+\eta_{m,b}$ where $m=n-\pi(1)$ and where ψ is the special map $\underset{\sim}{\mathrm{Spec}}\,(\tilde{\pi}(1)-1,\tilde{\pi}(2)-1,\dots,\tilde{\pi}(r)-1)$.

Proof.

Instead of $\phi(\gamma,\phi)$ and $R(\gamma,\phi)$ we will also write $\phi(\pi)$ and $R(\pi)$. Since the lemma is trivial for $n\le 2$, we can use induction on $n\ge 3$.

We distinguish two cases.

Case 1: $b=\pi(1)-\pi(2)>0$.

Let $n'=n-b$ and define $\pi':[p]\longrightarrow\mathbb{P}$ by letting $\pi'(1)=\pi(1)-b=\pi(2)$; $\pi'(i)=\pi(i)$ otherwise. By construction we have $b'=\pi'(1)-\pi'(2)=0$ and then

$$\tilde{\pi}'(i)=\tilde{\pi}(i)\quad\text{for}\quad 1\le i\le r;$$
$$=0\quad\text{for}\quad i>r.$$

Thus since $b'=0$, the induction hypothesis implies that the special map ψ defined in the theorem is a transform of the defining map $\varphi'=\phi(\pi')$ of the Ferrers relation $R(\pi')$. Now from the definition of $R(\pi)$ and $R(\pi')$ we have that

$$\varphi=\varphi'+\eta_{m,b}$$

where $m=n-\pi(1)$. As $m=n-\pi(1)=n-b-r=\sum\{\tilde{\pi}(i)-1:1\le i\le r\}=\tilde{\psi}(1)$, the conditions of lemma 12 are satisfied and $\psi+\eta_{m,b}$ is indeed a transform of φ.

Case 2: $b = \pi(1) - \pi(2) = 0$.

By hypothesis we have $\pi(1) = \pi(2) = \ldots = \pi(s) = r$ where $s \geq 2$ and either $s = p$ or $s < p$ and $\pi(s) > \pi(s+1)$. Define maps π' and π'' of $[p]$ into $\mathbb{P}$ by letting $\pi'(s) = \pi(s) - 1$; $\pi'(i) = \pi(i)$ for $i \neq s$;

$\pi''(1) = \pi(1) - 1$; $\pi''(i) = \pi(i)$ for $i \neq 1$.

Thus $\sum \pi'(i) = \sum \pi''(i) = n-1$. The map π' is non-increasing and $\tilde{\pi}'(r) = r-1$, $\tilde{\pi}'(i) = \tilde{\pi}(i)$ for $i < r$ $(= \pi(1) = \pi'(1))$. Hence by the induction hypothesis $\varphi' = \phi(\pi')$ has for transform

$\psi' = \underset{\sim}{\mathrm{Spec}}\,(\tilde{\pi}(1)-1, \tilde{\pi}(2)-1, \ldots, \tilde{\pi}(r-1)-1, \tilde{\pi}(r)-2)$

where eventually $\tilde{\pi}(r) - 2 = 0$.

Now by property 16, $R(\pi')$ and $R(\pi'')$ that is $F(\varphi')$ and $F(\varphi'')$ (where $\varphi'' = \phi(\pi'')$) are rook-equivalent. Further $\varphi = \phi(\pi)$ is equal to $\varphi'' + \eta_{m,1}$. Thus observing that $\tilde{\psi}'(1) = m-1$, we can again conclude from lemma 12 that the special map ψ defined in the lemma is a transform of φ since it is equal to $\psi' + \eta_{m,1}$.

Q.E.D.

We now come to the general case when J is not necessarily equal to $[p]$ or empty.

Theorem 19.

Let $\varphi = \phi(\gamma, J)$ be the defining map of the Ferrers relation $R(\gamma, J)$ where $\gamma : [n] \longrightarrow [p]$ is an order-preserving surjection and $J = [p']$ with $0 \leq p' \leq p$.

Moreover let

$$\pi'(i) = \underset{\sim}{\mathrm{card}}\, \gamma^{-1}(i) \quad \text{for} \quad i \in [p']$$
$$= 0 \quad \text{for} \quad i > p',$$
$$\pi''(i) = \underset{\sim}{\mathrm{card}}\, \gamma^{-1}(p'+i) \quad \text{for} \quad i \in [p-p']$$
$$= 0 \quad \text{for} \quad i > p-p'.$$

Then the decreasing transform of φ is

$$\psi + \eta_{m,b}$$

where $m = n - \pi''(1)$, $b = \pi''(1) - \pi''(2)$ and

$\psi = \underset{\sim}{\text{Spec}}\,(\tilde{\pi}'(q), \ldots, \tilde{\pi}'(1), \tilde{\pi}''(1)-1, \ldots, \tilde{\pi}''(r)-1)$

with $q = \pi'(1)$ and $r = \pi''(2)$.

Proof.

The case $J = [p]$ or $J = \emptyset$ has been considered in lemmas 17 and 18. We shall then assume $1 \leq p' < p$, and will also use the following notations:

$$n' = \pi'(1) + \cdots + \pi'(p'), \quad p'' = p - p', \quad n'' = n - n',$$

$$\gamma'(i) = \gamma(i) \quad \text{for} \quad i \in [n'] \quad \text{and} \quad \gamma''(i) = \gamma(n'+i) \quad \text{for} \quad i \in [n''].$$

Let us define the two maps φ' and φ'' as follows:

$$\begin{aligned} \varphi'(i) &= \varphi(i) - n'' \quad \text{for} \quad i \in [n'] \\ &= 0 \quad \text{for} \quad i > n' \\ \varphi''(i) &= \varphi(i+n') \quad \text{for} \quad i \in \mathbb{P}. \end{aligned}$$

By construction we have

$$\varphi = \Delta(n'', n'; \varphi'') + \varphi'.$$

Moreover we clearly have $\varphi' = \phi(\gamma', [p'])$ and $\varphi'' = \phi(\gamma'', \emptyset)$ if $p'' > 1$ and $\varphi'' = 0$ if $p'' = 1$. According to lemma 17 the decreasing transform of φ' is then the special map

$$\bar{\varphi}' = \underset{\sim}{\text{Spec}}\,(\tilde{\pi}'(q), \tilde{\pi}'(q-1), \ldots, \tilde{\pi}'(1))$$

where $q = \pi'(1)$. In the same manner lemma 18 asserts that if $p'' > 1$, the decreasing transform of φ'' is $\bar{\varphi}'' = \psi'' + \eta_{m'',b}$ where $m'' = n'' - \pi''(1)$ and $\psi'' = \underset{\sim}{\text{Spec}}\,(\tilde{\pi}''(1)-1, \tilde{\pi}''(2)-1, \ldots, \tilde{\pi}''(r)-1)$ with $r = \pi''(2)$. If $p'' = 1$, we let $\bar{\varphi}'' = \varphi'' = 0$.

Now since $n'' > \varphi''(1)$ and $n' = \tilde{\varphi}'(1)$, the conditions of lemma 12 are fulfilled and we can conclude that

$$\bar{\varphi} = \Delta(n'', n'; \bar{\varphi}'') + \bar{\varphi}'$$

is a transform of $\varphi = \Delta(n'', n'; \varphi'') + \varphi'$. As one has
$n' = \pi(1) + \dots + \pi(p') = \tilde{\pi}'(q) + \dots + \tilde{\pi}'(1)$, the value of $\bar{\varphi}'$ at n' is equal to 1. On the other hand the value of $\bar{\varphi}''$ at 1 is equal to 0 if $p'' = 1$ and if $p'' > 1$, we have

$$\begin{aligned}\bar{\varphi}''(1) &= b + \sum_{1 \le i \le r} (\tilde{\pi}''(i) - 1) + r - 1 \\ &= n'' - 1 .\end{aligned}$$

Accordingly we have $\bar{\varphi}(n') = n'' + 1$ and $\bar{\varphi}(n'+1) = n'' - 1$ if $p'' > 1$ and $\bar{\varphi}(n'+1) = 0$ if $p'' = 1$. As $\bar{\varphi}'$ and ψ'' are special, this shows, in the case when $p'' > 1$, that $\bar{\varphi}$ is equal to $\eta_{m,b} + \psi$ where
$m = n' + m'' = n - \pi''(1)$, $b = \pi''(1) - \pi''(2)$
and $\psi = \underset{\sim}{\mathrm{Spec}}\, (\tilde{\pi}'(q), \dots, \tilde{\pi}'(1), \tilde{\pi}''(1) - 1, \dots, \tilde{\pi}''(r) - 1)$
with $r = \pi''(2)$. If $p'' = 1$, one has $\bar{\varphi} = \eta_{n', n''} + \psi'$ where
$\psi' = \underset{\sim}{\mathrm{Spec}}\, (\tilde{\pi}'(q), \dots, \tilde{\pi}'(1))$. In fact, in this last case we can also write
$\bar{\varphi} = \eta_{m,b} + \psi$ since then $r = \pi''(2) = 0$ and ψ is reduced to
$\psi = \underset{\sim}{\mathrm{Spec}}\, (\tilde{\pi}'(q), \dots, \tilde{\pi}'(1))$.

Q.E.D.

Remark 20.

The sequence $(\tilde{\pi}'(q), \dots, \tilde{\pi}'(1), \tilde{\pi}''(1) - 1, \dots, \tilde{\pi}''(r) - 1)$ just defined is unimodal, namely it is the juxtaposition of a non-decreasing sequence (of length q) and a non-increasing sequence (of length r). Thus the decreasing transform of a map $\varphi = \phi(\gamma, J)$ is apart from the function $\eta_{m,b}$ a special map $\psi = \underset{\sim}{\mathrm{Spec}}(d_1, \dots, d_s)$ whose defining sequence $(d_1, \dots, d_s)$ is unimodal. Conversely, let $\underset{\sim}{d} = (d_1, \dots, d_s)$ be an unimodal sequence of positive integers and b be a non-negative integer. Then form the decreasing map $\psi = \eta_{m,b} + \underset{\sim}{\mathrm{Spec}}\, (\underset{\sim}{d})$. It is readily verified that ψ is the decreasing transform of (at least) one non-increasing map φ of the form $\phi(\gamma, [p'])$.

A Ferrers relation $R(\gamma, [p'])$ in fact depends on four parameters n, p, γ and p'. Let us write $R(n,p,\gamma,p')$ instead of $R(\gamma,p')$. Thus it is easily proved that $F(\psi)$ is rook-equivalent to exactly one Ferrers relation $R(n,p,\gamma,p')$ such that $p \geq 2p'$.

Example 21.

Let $n = 14$, $p = 6$, $p' = 3$; $\pi(1) = 4$, $\pi(2) = 3$, $\pi(3) = 2$, $\pi(4) = 2$, $\pi(5) = 2$, $\pi(6) = 1$. The sequence of positive values of $\varphi = \phi(\gamma, [3])$ where γ is determined by the map π just defined, is $(14,14,14,14,10,10,10,7,7,3,3,1,1)$. The decreasing transform of φ is $\bar{\varphi} = \underset{\sim}{\mathrm{Spec}}(1,2,3,3,2,1) + \eta_{12,0}$ and the sequence of positive values of $\bar{\varphi}$ is then $(17,15,14,12,11,10,8,7,6,4,3,1)$.

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