DISCRETE MATHEMATICS – Volume 1, No. 2 (1971) 181–186

ENUMERATING k-TREES

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Received 13 January 1971

Abstract. A functional definition of rooted k-trees is given, enabling k-trees with n labeled points to be enumerated without any calculation.

§ 1. Introduction

Let $0 < k \leq n$ and Γ be an undirected graph with n labeled vertices. A *k-chain* of Γ is a rearrangement $(x_1, x_2, \ldots, x_n)$ of the n vertices of Γ having the following property:

For each $i = k+1, k+2, \ldots, n$, the vertex x_i is adjacent to exactly k vertices of the set $\{x_1, x_2, \ldots, x_{i-1}\}$ and, moreover, these k vertices are themselves mutually adjacent.

We say that Γ is a *k-tree with n labeled points,* if it contains a k-chain. The notion of k-tree was introduced by Harary and Palmer [3]. Beineke and Pippert [1] derived a formula for the number $B_k(n)$ of k-trees with n labeled points and showed that

$$(1.1) \qquad B_k(n) = \binom{n}{k} (k(n-k)+1)^{n-k-2} .$$

This formula was also proved by Palmer [6] himself in the case $k = 2$ and Moon [4] in the general case. Finally, the late Alfréd and Catherine Rényi [8] proposed a proof by using the now traditional method of Prüfer [7].

As in many counting problems in graph theory, the *undirected* structures (here k-trees) are first transformed into *directed* structures (rooted k-trees). This is what all the above-mentioned authors have done; they have then derived a formula for the number of these directed structures, either by an induction reasoning or by exhibiting a one-to-one correspondence. Finally, formula (1.1) was obtained in an obvious manner.

The purpose of this paper is to show that with an adequate *functional* definition of rooted k-trees, *no calculation* is necessary to obtain their enumeration. We appeal only to the known counting methods used in the study of *acyclic* functions. Let X be a finite set of cardinality n ($n > 0$); then an acyclic function maps X into itself and is such that $f^n = f^{n-1}$. What is needed here is the notion of an acyclic function g that maps X into another set Y. The mth iterate g^m can no longer be defined. But what is suggested by the geometric properties of k-trees is to introduce a *surjection* $\gamma: Y \to X$ and to say that $g: X \to Y$ is (X, Y)-*acyclic* iff the composite function $\gamma \circ g$ (that maps X into X) is acyclic in the ordinary sense. In § 2 we easily enumerate the (X, Y)-acyclic functions and show in § 3 that rooted k-trees are very peculiar (X, Y)-acyclic functions and, by means of this observation, are readily enumerated.

§ 2. (X, Y)-acyclic functions

Let X be a finite set of cardinality n ($n > 0$) and f a function of X into itself. For each integer $m > 0$ and $x \in X$, we define inductively: $f^0(x) = x$ and $f^m(x) = f(f^{m-1}(x))$. If C is a non-empty subset of X, we let $A = X - C$, and denote by $F(A, C)$ the collection of all acyclic functions whose set of fixed points is C, i.e. the set of all functions $f: X \to X$ such that $f^n = f^{n-1}$ and $f(x) = x$ iff $x \in C$. Note that $F(A, C)$ is reduced to the identity map if A is empty.

Besides the disjoint subsets A and C of X we suppose given another finite set B, whose intersection with X is empty and also a *fixed surjection* γ of the set $Y = B \cup C$ onto the set $X = A \cup C$ such that $\gamma(x) = x$ for each $x \in C$ and $\gamma(B) = A$. For any function $g: X \to Y$ we can form the composite function $\gamma \circ g$:

$$X \xrightarrow{g} Y \xrightarrow{\gamma} X$$

We then say that g is an *(X, Y)-acyclic function whose set of fixed points is C* iff the composite function $\gamma \circ g$ is acyclic and belongs to the class $F(A, C)$. The collection of all these functions $g: X \to Y$ such that $\gamma \circ g \in F(A, C)$ will be denoted by $F(A, B, C)$. Furthermore, we let $\text{card}(A) = p$, $\text{card}(B) = q$, $\text{card}(C) = r$ and designate by $(a_1, a_2, \ldots, a_p)$ the *increasing* sequence of the elements of A with respect to some given total order.

Let $H(A, C)$ be the set of all maps $h: A \to X (= A \cup C)$ such that $h(a_1) \in C$. Obviously

$$\text{(2.1.)} \qquad \text{card}\,(H(A, C)) = r(p+r)^{p-1} \;,$$

which is the cardinality of $F(A, C)$ (see e.g. [2] theorem 5.9 or [5] application 1 of theorem 8). In the appendix we recall the construction of a bijection $\Phi: F(A, C) \to H(A, C)$. But what suffices to know at this stage is the fact (see [2] theorem 5.9) that we can construct a bijection $\Phi: F(A, C) \to H(A, C)$ having the following property.

Let $f \in F(A, C)$ and $h = \Phi(f)$; then, the sequence $(h(a_1), h(a_2), \ldots, h(a_p))$ is a rearrangement of the sequence $(f(a_1), f(a_2), \ldots, f(a_p))$.

In other words, it is possible to associate to any $f \in F(A, C)$ a permu tation σ_f of the set A such that

$$\text{(2.2.)} \qquad \Phi(f) = f \circ \sigma_f \;.$$

Now, let $g \in F(A, B, C)$; then since $f = \gamma \circ g$ belongs to $F(A, C)$, we can define

$$\text{(2.3.)} \qquad \Psi(g) = g \circ \sigma_{\gamma \circ g} \;.$$

Actually, the "f" and "g" occurring in the right-hand side of (2.2.) and (2.3.) represent the restrictions of the original functions f and g to the set A. There should be no confusion in using the same notation for a function and its restriction.

Theorem 1. *The function Ψ is a bijection of $F(A,B,C)$ onto the set $H(A,B,C)$ of all maps $h: A \to B \cup C$ such that $h(a_1) \in C$.*

Proof. Let $g \in F(A, B, C)$ and $f = \gamma \circ g$. Since $f \circ \sigma_f$ belongs to $H(A, C)$, we have $\gamma \circ g \circ \sigma_{\gamma \circ g}(a_1) \in C$, hence $g \circ \sigma_{\gamma \circ g}(a_1) \in C$ by definition of γ. Thus Ψ maps $F(A, B, C)$ into $H(A, B, C)$. Let us now consider the diagram for all i, j and $\gamma(c) = c$; we also extend g to a function $g: X \to Y$ by letting $g(c) = c$.

By definition, any c-rooted k-tree g with n labeled points has the following characteristic property: There exists a rearrangement $(x_{k+1}, x_{k+2}, \ldots, x_n)$ of the $n-k$ elements of A such that for each $i = k+1, k+2, \ldots, n$ we have either $g(x_i) = c$ or $g(x_i) \in \gamma^{-1}(x_{i'})$ with $i' < i$; that is to say, either $\gamma(g(x_i)) = c$ or $\gamma(g(x_i)) = x_{i'}$ with $i' < i$. But this is equivalent to saying that $\gamma \circ g$ is acyclic and only has the fixed point c. Thus, the c-rooted k-trees with n labeled points are nothing else but (X, Y)-acyclic functions whose set of fixed points is $C = \{c\}$. Accordingly, as card$(A) = n-k$, card$(B) = k(n-k)$ (since each vertex is the image under γ of exactly k faces) and card$(C) = 1$, it follows from formula (2.4) that the number of c-rooted k-trees with n labeled points is equal to $R_k(n, c) = (k(n-k)+1)^{n-k-1}$. The number $R_k(n)$ of all rooted k-trees with n labeled points is therefore equal to $R_k(n) = \binom{n}{k}(k(n-k)+1)^{n-k-1}$. Finally, formula (1.1) is obtained by noting that any k-tree with n labeled points can be "rooted" at any one of its $k(n-k)+1$ faces, i.e. there always exists a k-chain whose root is a given face.

§ 4. Appendix

We recall the construction of the bijection $\Phi: F(A, C) \to H(A, C)$ that appears in a previous paper [2]. Let $f \in F(A, C)$; if A is empty, both sets $F(A, C)$ and $H(A, C)$ are reduced to one element. Let us assume that card$(A) > 0$; then the set $A - f(A)$ is not empty. Let $(z_1, z_2, \ldots, z_s)$ denote the increasing sequence of its elements with respect to some given total order of A. By induction we now construct $s+1$ sequences $\delta_0, \delta_1, \ldots, \delta_s$ of elements belonging to $A \cup C$. First, δ_0 is the increasing sequence of the r elements of C with respect to some given total order. Then assume that the sequences $\delta_0, \delta_1, \ldots, \delta_{t-1}$ have been constructed for some t such that $1 \leqslant t \leqslant s$. We define m_t as being the smallest integer $m > 0$ such that $f^m(z_t)$ is equal to some term of the juxtaposition product $\delta_0 \delta_1 \ldots \delta_{t-1}$. It can be shown that this integer exists. We then put

$$\delta_t = (f^{m_t}(z_t), f^{m_t - 1}(z_t), \ldots, f(z_t)) \quad (1 \leqslant t \leqslant s)$$

for all i, j and $\gamma(c) = c$; we also extend g to a function $g: X \to Y$ by letting $g(c) = c$.

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$$\delta_t = (f^{m_t}(z_t), f^{m_t - 1}(z_t), \ldots, f(z_t)) \quad (1 \leqslant t \leqslant s)$$

Now let $(u_1, u_2, \ldots, u_p)$ be the juxtaposition product of the s sequences $\delta_1, \delta_2, \ldots, \delta_s$ (it can be shown that this product is of length p (= card(A))). Then $h = \Phi(f) \in H(A, C)$ is defined by

$$h(a_i) = u_i \text{ for all } i = 1, 2, \ldots, p.$$

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