DOMINIQUE FOATA and VOLKER STREHL

EULER NUMBERS AND VARIATIONS OF PERMUTATIONS

RIASSUNTO. — Usando le proprietà di una famiglia di gruppi di trasformazioni si ottiene la distribuzione congiunta dei massimi locali, dei minimi locali, degli aumenti, delle discese nei gruppi simmetrici.

Si trova una nuova espressione per i numeri di Eulero e la connessione con classi di permutazioni aventi notevoli proprietà geometriche, come le permutazioni alternanti e quelle di André.

In two recent papers ([9], [13]) we have introduced a family of transformation groups $(G_n)_{n>0}$ which have the following property: G_n acts on the $n!$ elements of the symmetric group $\mathfrak{S}_n$ and the number of its orbits is equal to the n-th tangent or secant number, according as n is odd or even. The purpose of this paper is to make use of the properties of these groups to study the joint distribution of various characteristics attached to sequences of numbers, such as peaks (local maxima), troughs (local minima), rises, descents. For instance, the permutation $w = 315642$ has two peaks 3 and 6, one trough 1, one rise 5 and two descents 4 and 2. Such a study is not new. It certainly goes back to Kermack and McKendrick ([15], [16]), and more recently to David and Barton [3]. What we shall get here will be a refinement of their results using an entirely different set-up.

In section 1 we recall the basic properties of the groups G_n. Let X_p, X_f, X_s, X_r, X_d be five commutative variables and w a permutation. We denote by $p(w), f(w), s(w), r(w), d(w)$ the number of peaks, troughs of the first kind (for a further classification of the troughs will be introduced), troughs of the second kind, rises and descents (respectively) in the permutation w. The so-called *variation* of w is then the monomial

$$V(w) = X_p^{p(w)}\, X_f^{f(w)}\, X_s^{s(w)}\, X_r^{r(w)}\, X_d^{d(w)}\,.$$

In section 2 we compute the polynomial $\Sigma V(w)$ over any orbit of G_n all elements of which have k peaks and obtain the formula

$$\Sigma V(w) = X_p^k (X_f + X_s)^{k-1} (X_r + X_d)^{n-2k+1}\,,$$

(see theorem 7). This formula implies in particular that the number of rises (resp. descents) has a binomial distribution over such an orbit. More remarkably, it provides with a new expression for the tangent or secant number D_n namely

$$D_n = \Sigma \{ X^{f(w)} (1 - X)^{s(w)} Y^{r(w)} (1 - Y)^{d(w)} : w \in \mathfrak{S}_n \}$$

where X and Y are two commutative variables (corollary 9). Let $P_n = \Sigma \{ V(w) : w \in \mathfrak{S}_n \}$ $(n > 0)$. We then establish in section 3 an identity for the exponential generating function of the P_n's. We conclude the paper by a survey of some related coefficients and show that the *André permutations* introduced in previous paper ([12], [9]) form natural cross-sections of the orbits of G_n. For a study of the tangent or secant numbers D_n see [14], [17] (§ 63) or [18].

1. The rearrangement group G_n

As made in [13] the construction of G_n is based upon a remarkable factorization of the elements of $\mathfrak{S}_n$. Let $w : i \to x_i$ $(1 \leq i \leq n)$ be a permutation of the interval $[n] = \{1, 2, \cdots, n\}$, that we identify with the *word* $x_1 x_2 \cdots x_n$ in the n distinct letters $x_1, x_2, \cdots, x_n$ taken out of $[n]$. A *factorization of length* q $(q > 0)$ of w is any sequence $(w_1, w_2, \cdots, w_q)$ of words (some of then possibly empty) such that the juxtaposition product $w_1 w_2 \cdots w_q$ is equal to w. The next lemma was proved in [13].

1. Lemma. *Let* $w = x_1 x_2 \cdots x_n$ $(n > 0)$ *be a permutation and* x *be one of the letters* x_i $(1 \leq i \leq n)$. *Then* w *has a unique factorization* (w_1, w_2, x, w_4, w_5) *of length* 5, *called its* x-*factorization which is characterized by the two properties*

(i) w_1 *(resp.* w_5*) is empty or its last (resp. first) letter is less than* x.

(ii) w_2 *(resp.* w_4*) is empty or all its letters are greater than* x.

For instance, for $x = 4$ the x-factorizations of 78314562 and 71654823 are (7831, e, 4, 56, 2) and (71, 65, 4, 8, 23), respectively, the empty word being denoted by e.

Now let (w_1, w_2, x, w_4, w_5) be the x-factorization of a permutation w containing the letter x. We define

$$\varphi_x(w) = w_1 w_4 x w_2 w_5 . \tag{1}$$

As (w_1, w_4, x, w_2, w_5) is trivially the x-factorization of $\varphi_x(w)$; we have

$$\varphi_x \varphi_x(w) = w . \tag{2}$$

Thus φ_x is an involution acting on $\mathfrak{S}_n$. The rearrangement group G_n is then the subgroup of the permutation group acting on $\mathfrak{S}_n$, which is generated by the set $\{\varphi_x : 1 \leq x \leq n\}$. The group G_n has the properties listed in the next proposition (see [13]).

2. Proposition. (i) *The group* G_n $(n > 0)$ *is abelian, of order* 2^{n-1}. *It consists of the* 2^{n-1} *distinct involutions*

$$\varphi_X = \prod_{x \in X} \varphi_x \qquad (X \subset [n-1]) ,$$

with the operation

$$\varphi_X \varphi_Y = \varphi_{X \Delta Y} \qquad (X , Y \subset [n - 1]) ,$$

the symbol φ_0 *being the identity.*

(ii) *The number of orbits of* G_n *is equal to the secant or tangent number of rank* n, *namely the coefficient* D_n *of* $u^n/n!$ *in the series expansion of* $\tan u + \sec u = 1 + \sum_{n<0} (u^n/n!) D_n$.

To state the other properties of the group G_n it will be convenient to describe the action of φ_x on the *linear ordering* $w(\cdot , \cdot)$ of $[n]$ associated to the word w. Let z , z' belong to w with $z \neq z'$. Then

(3) $w(z , z')$ iff w admits a factorization of the form $(w_1 , z , w_3 , z' , w_5)$.

We next define $\mu_w(z , z') = \mu_w(z' , z)$ as the minimum letter of the unique factor of w whose endpoints are z , z' or z' , z. For instance, with $w = 71654823$, and $z = 6 , z' = 8$, the factor in question is 6548 whose minimum letter is 4. Then $\mu_w(6 , 8) = \mu_w(8 , 6) = 4$. Let $(w_1 , w_2 , x , w_4 , w_5)$ be the x-factorization of w. As φ_x permutes only the words w_2 and w_4, we see that φ_x reverses the ordering $w(\cdot , \cdot)$ for exactly those pairs (z , z') such that $z \neq z'$, z a letter of $w_2 x$, and z' of $x w_4$. Clearly

(4) z is a letter of $w_2 x$ and z' of $x w_4$ with $z \neq z'$ iff $w(z , z')$ and $\mu_w(z , z') = x$.

Hence φ_x can also be defined by

(5) $$\text{if } w(z , z'), \text{ then } \begin{cases} \varphi_x(w)(z', z) & \text{if } \mu_w(z , z') = x \\ \varphi_x(w)(z , z') & \text{otherwise.} \end{cases}$$

Let us also recall the following property (see [13]).

3. PROPOSITION. *The function* μ *is an orbit-characteristic, i.e.* $w' = \varphi_X(w)$ *for some* $X \subset [n - 1]$ *iff* $\mu_w = \mu_{w'}$. *Moreover, if* $\{\mu_w\}$ *denotes the range of* μ_w *and* $|\mu_w|$ *the cardinality of that range, the orbit containing* w *consists of the* $2^{|\mu_w|}$ *distinct elements*

$$\varphi_X(w) \qquad \textit{with} \quad X \subset \{\mu_w\} .$$

2. THE VARIATION OF A PERMUTATION

Let us now introduce the notions of peak, trough, rise and descent of a permutation.

4. DEFINITION. Let $w = x_1 x_2 \cdots x_n$ $(n > 0)$ be a permutation and $1 \leq i \leq n$. Then x_i is a *peak* of w if (i) either $i = 1$ or $1 < i$ and $x_{i-1} < x_i$; (ii) either $i = n$ or $i < n$ and $x_i > x_{i+1}$. Moreover x_i is a *trough* of w if $1 < i < n$, $x_{i-1} > x_i$ and $x_i < x_{i+1}$. Finally x_i is a *rise* (resp. *descent*) of w if (i) either $i = 1$ (resp. $i = n$) or $1 < i$ and $x_{i-1} < x_i$ (resp. $i < n$ and $x_i > x_{i+1}$); (ii) $i < n$ and $x_i < x_{i+1}$ (resp. $1 < i$ and $x_{i-1} > x_i$).

Thus, a permutation can only start with a peak $(x_1 > x_2)$ or a rise $(x_1 < x_2)$ and end with a peak $(x_{n-1} < x_n)$ or a descent $(x_{n-1} > x_n)$. These four notions are closely related to that of x-factorization, as we show in the next proposition.

5. PROPOSITION. *Let (w_1, w_2, x, w_4, w_5) be the x-factorization of a permutation w. Then, the following holds*

(i) *x is a peak if and only if w_2 and w_4 are empty;*
(ii) *x is a trough if and only if w_2 and w_4 are non-empty;*
(iii) *x is a rise if and only if w_2 is empty and w_4 non-empty;*
(iv) *x is a descent if and only if w_2 is non-empty and w_4 empty.*

Proof. Let $x = x_i$ for some i $(1 \leq i \leq n)$. The proposition follows from the property of the x-factorization given in lemma 1. We have indeed w_2 (resp. w_4) $= e$ (the empty word) iff the letter preceding (resp. following) x in w (if any) is smaller than x.

Q.E.D.

Thus, two different kinds of trough can be defined by comparing the non-empty words w_2 and w_4 in a suitable manner. Of course, any total order on the set of all words, e.g. the lexicographic order, will permit such a comparison. But in order to preserve the symmetry between the two kinds of trough, we introduce the notion of factor-ordering relation. Let $\mathscr{F}$ be the set of all finite subsets of positive integers. Let also $\mathscr{F}'$ be the set of all pairs (I, J) of finite subsets of positive integers such that $I \cap J = \emptyset$. Then, a *factor-ordering relation* is any binary relation on $\mathscr{F} \times \mathscr{F}$, whose restriction to $\mathscr{F}'$ is antisymmetric and total. In other words, R is factor-ordering if for any I, J in $\mathscr{F}$ such that $I \cap J = \emptyset$, there is exactly one pair (I, J) or (J, I) that belongs to R. For example, the relation R defined by

$$(6) \quad \begin{cases} \text{(i)} & \emptyset \, R \, I \quad \text{if } I \text{ is any finite subset;} \\ \text{(ii)} & I \, R \, J \quad \text{if } \max I \leq \max J, \quad \text{when } I \text{ and } J \text{ are non-empty,} \end{cases}$$

is factor-ordering.

If v is a word whose letters are all distinct, say $v = z_1 z_2 \cdots z_m$, its *content* is the set $Cv = \{z_1, z_2, \cdots, z_m\}$.

6. DEFINITION. Let R be a factor-ordering relation. Moreover, let (w_1, w_2, x, w_4, w_5) be the x-factorization of a permutation w and x a trough of w. Then, the trough x is of the *first* (resp. *second*) *kind* if $Cw_2 \; R \; Cw_4$ (resp. $Cw_4 \; R \; Cw_2$).

For instance, the permutation $w = 7\ 8\ 3\ 1\ 4\ 5\ 6\ 2$ has one trough, namely 1. With respect to the factor-ordering relation (6), this trough is of the second kind.

Let us now introduce five commutative variables X_p ("p" for "peak"), X_f ("f" for "first"), X_s ("s" for "second"), X_r ("r" for "rise") and X_d ("d" for "descent"). For any w in $\mathfrak{S}_n$ and x in $[n]$, we let

$V_x(w) = X_p$, X_f, X_s, X_r or X_d, according as x is a peak, trough of the first kind, trough of the second kind, rise or descent in w. The *variation* $V(w)$ is then defined by

$$V(w) = \prod_{1 \leq x \leq n} V_x(w) . \tag{7}$$

For instance, if we keep the relation (6), and take $w = 7\,8\,3\,1\,4\,5\,6\,2$, we have

$$V(w) = X_r X_p X_d X_s X_r X_r X_p X_d = X_p^2 X_s X_r^3 X_d^2 .$$

7. THEOREM. *Let w belong to $\mathfrak{S}_n (n > 0)$ and have k peaks. Then, the generating polynomial of the peaks, troughs of either kind, rises, and descents in the orbit Θ of G_n containing w is given by*

$$\Sigma \{V(w) : w \in \Theta\} = X_p^k (X_f + X_s)^{k-1} (X_r + X_d)^{n-2k+1} .$$

Theorem 7 is based upon the following lemma.

8. LEMMA. *Let $1 \leq x \leq n$ and $w \in \mathfrak{S}_n$. Then*

(i) *$\varphi_x(w) = w$ if and only if x is a peak of w;*

(ii) *x is a trough of the first kind of w if and only if x is a trough of the second kind of $\varphi_x(w)$;*

(iii) *x is a rise of w if and only if x is a descent of $\varphi_x(w)$;*

(iv) *$V_y(w) = V_y \varphi_x(w)$ for any $y \in [n] \setminus \{x\}$.*

Proof. First (i), (ii) and (iii) are immediate consequences of proposition 5 and the definition of φ_x given in (1). Let (w_1, w_2, y, w_4, w_5) (resp. $(w_1', w_2', y, w_4', w_5')$) be the y-factorization of w (resp. $w' = \varphi_x(w)$). Then, part (iv) of the lemma is established if we show that $Cw_2 = Cw_2'$ and $Cw_4 = Cw_4'$. But it follows from (4) that the two cartesian products $Cw_2 \times Cw_4$ and $Cw_2' \times Cw_4'$ are given by

$$Cw_2 \times Cw_4 = \{(z, z') : y, z, z' \text{ distinct, } w(z, z') \;\&\; \mu_w(z, z') = y\}$$

and

$$Cw_2' \times Cx_4' = \{(z, z') : y, z, z' \text{ distinct, } \varphi_x(w)(z, z') \;\&\; \mu_{w'}(z, z') = y\}.$$

As $y \neq x$ and also $\mu_{w'} = \mu_w$, we deduce from (5) the implication

$$w(z, z') \;\&\; \mu_w(z, z') = y \Rightarrow \varphi_x(w)(z, z') \;\&\; \mu_{w'}(z, z') = y , \text{ i.e.}$$

$$Cw_2 \times Cw_4 \subset Cw_2' \times Cw_4' . \quad \text{As } w = \varphi_x \varphi_x(w),$$

the inverse inclusion also holds. Hence $Cw_2 = Cw_2'$ and $Cw_4 = Cw_4'$.

Q.E.D.

A trough being necessarily between two peaks, the number of peaks in a permutation is equal to the number of troughs plus one. It follows from the previous lemma that all the permutations in the same orbit have the same peaks, say $x_1, \cdots, x_k$, the same troughs, say $x_{k+1}, \cdots, x_{2k-1}$ and the same integers occurring either as rises or descents (call them *rise-descents*), say $x_{2k}, \cdots, x_n (0 < 2k \leq n)$. Let w_0 be a permutation in the orbit Θ. Clearly x is a peak of w_0 iff we can never have $\mu_{w_0}(z, z') = x$ for any pair of distinct integers (z, z'). Hence x is a peak iff $x \notin \{\mu_{w_0}\}$. Thus $\{\mu_{w_0}\} = \{x_{k+1}, \cdots, x_n\}$. Proposition 3 asserts that Θ consists of the 2^{n-k} (here $|\mu_{w_0}| = n - k$) permutations $\varphi_X w_0$ with $X \subset \{x_{k+1}, \cdots, x_n\}$. Let us consider the group algebras of $\mathfrak{S}_n$ and G_n over the rationals. For instance, the formal sum $\Sigma\{w : w \in \Theta\}$ belongs to the group algebra of $\mathfrak{S}_n$ and the product $(1 + \varphi_{x_{k+1}}) \cdots (1 + \varphi_{x_n})$ to the group algebra of G_n. Moreover

$$\Sigma\{w : w \in \Theta\} = (1 + \varphi_{x_{k+1}}) \cdots (1 + \varphi_{x_n})(w_0).$$

Let us extend the definition of V to the group algebra of $\mathfrak{S}_n$ in a natural manner, i.e. if $\Sigma\, bw$ is a formal sum with w in $\mathfrak{S}_n$ and b rational, we let $V(\Sigma\, bw) = \Sigma\, bV(w)$. Consequently, for any formal sum $\Sigma\, b\psi$ in the group algebra of G_n we have $V(\Sigma\, b\psi) = \Sigma\, bV(\psi)$. Now

$$\Sigma\{V(w) : w \in \Theta\} = V\Sigma\{w : w \in \Theta\}$$
$$= V(1 + \varphi_{x_{k+1}}) \cdots (1 + \varphi_{x_n})(w_0)$$
$$= V_{x_1} \cdots V_{x_n}(1 + \varphi_{x_{k+1}}) \cdots (1 + \varphi_{x_n})(w_0)$$
$$= V_{x_1}(w_0) \cdots V_{x_k}(w_0)(V_{x_{k+1}}(1 + \varphi_{x_{k+1}})(w_0)) \cdots (V_{x_n}(1 + \mu_{x_n})(w_0))$$
$$= X_p^k (X_f + X_s)^{k-1} (X_r + X_d)^{n-2k+1}.$$

This proves theorem 7.

For any permutation w let $p(w), f(w), s(w), r(w), d(w)$ denote respectively the number of peaks, troughs of the first kind, troughs of the second kind, rises and descents. We have then the following evaluation for the secant (or tangent) number.

9. COROLLARY. *Let* X *and* Y *be two variables. Then the polynomial*

$$\Sigma\{X^{f(w)}(1 - X)^{s(w)} Y^{r(w)} (1 - Y)^{d(w)} : w \in \mathfrak{S}_n\} \tag{8}$$

is a constant equal to the tangent or secant number D_n.

Proof. The variation $V(w)$ of a permutation w is equal to $V(w) = X_p^{p(w)} X_f^{f(w)} X_s^{s(w)} X_r^{r(w)} X_d^{d(w)}$. Hence, the corollary follows from theorem 7 by making the substitutions $1 \leftarrow X_p$, $X \leftarrow X_f$, $1 - X \leftarrow X_s$, $Y \leftarrow X_r$, $1 - Y \leftarrow X_d$, summing over all the orbits of G_n and taking proposition 2 (ii) into account.

Q.E.D.

Note that when $X = Y = 1/2$, formula (8) reduces to $D_n = \Sigma\{2^{-n+p(w)} : w \in \mathfrak{S}_n\}$, already established in our previous paper [13]. We note that Burnside's lemma ([5] p. 191 or [8] p. 90) could be used to get the latter formula but a further extension would be needed to obtain (8) in that way.

3. The variation polynomials

In this section we establish an identity for the so-called *variation polynomials*

$$P_n = \Sigma\{V(w) : w \in \mathfrak{S}_n\} \ (n > 0).$$

From theorem 7 there exist positive integers $d_{n,k}$ $(0 < k \leq [n/2])$ (the symbol $[n/2]$ standing for the smallest integer greater than or equal to $n/2$) such that

$$P_n = \sum_{0<k\leq[n/2]} d_{n,k}\, X_p^k (X_f + X_s)^{k-1} (X_r + X_d)^{n-2k+1}. \tag{9}$$

The coefficient $d_{n,k}$ is equal to the number of orbits of G_n having k peaks. These numbers will be studied in the next section. Now we only give the formulas involving the polynomials P_n. First, note that $P_1 = X_p$, $P_2 = X_p (X_r + X_d)$. Then, we establish two recurrence formulas, that extend the two recurrence formulas for the secant and tangent numbers.

10. Theorem. *The variation polynomials* P_n *satisfy the following two equivalent recurrence relations*

$$P_1 = X_p \,;\ P_{n+1} = (X_r + X_d)\, P_n + \sum_{1\leq i\leq n-1} \binom{n-1}{i} P_i (X_f + X_s) P_{n-i} \quad (n > 0) \tag{10}$$

$$P_1 = X \,;\ 2\,P_{n+1} = 2\,(X_r + X_d)\, P_n + \sum_{1\leq i\leq n-1} \binom{n}{i} P_i (X_f + X_s) P_{n-i} \quad (n > 0). \tag{11}$$

Proof. Let $X_r + X_d = s$, $X_p = t$, $X_f + X_s = u$. We have

$$\sum_{1\leq i\leq n-1} \binom{n}{i} P_i\, u P_{n-i} = \sum_{1\leq i\leq n-1} \binom{n-1}{i} P_i\, u P_{n-i} + \sum_{1\leq i\leq n-1} \binom{n-1}{i-1} P_i\, u P_{n-i}.$$

By making the change of variables $j = n - i$ in the last summation we obtain the identity

$$\sum_{1\leq i\leq n-1} \binom{n}{i} P_i\, u P_{n-i} = 2 \sum_{1\leq i\leq n-1} \binom{n-1}{i} P_i\, u P_{n-i} \qquad (n > 0).$$

Hence, relations (10) and (11) are equivalent. Let us establish relation (11). We denote by $\mathfrak{S}_{n+1,i}$ the set of all $w = x_1 x_2 \cdots x_{n+1}$ in $\mathfrak{S}_{n+1}$ such that $x_{i+1} = 1$ $(0 \leq i \leq n\,;\ n \geq 1)$. We have

$$\begin{aligned} V\{\mathfrak{S}_{n+1,i}\} + V\{\mathfrak{S}_{n+1,n-i}\} &= \Sigma\{V(w) + V(\varphi_1 w) : w \in \mathfrak{S}_{n+1,i}\} \\ &= \Sigma\{V(1 + \varphi_1)(w) : w \in \mathfrak{S}_{n+1,i}\} \\ &= \Sigma\{V_1(1 + \varphi_1)(w)\, V'(w) : w \in \mathfrak{S}_{n+1,i}\}, \end{aligned}$$

where $V'(w) = \prod_{1<x\leq n+1} V_x(w)$ when $w \in \mathfrak{S}_{n+1}$. But if $i = 0$ or n and $w \in \mathfrak{S}_{n+1,i}$, we have

$$V_1(1+\varphi_1)(w) = X_r + X_d = s$$

for 1 in the beginning (resp. at the end) of a permutation can only be a rise (resp. descent). If $0 < i < n$, we have

$$V_1(1+\varphi_1)(w) = X_f + X_s = u,$$

for 1 is always a trough in such a permutation. Hence

$$2V\{\mathfrak{S}_{n+1}\} = \sum_{0\leq i\leq n} V\{\mathfrak{S}_{n+1,i}\} + V\{\mathfrak{S}_{n+1,n-i}\} =$$

$$= sV'\{\mathfrak{S}_{n+1,0}\} + sV'\{\mathfrak{S}_{n+1,n}\} + u\sum_{0<i<n} V'\{\mathfrak{S}_{n+1,i}\}.$$

Clearly $V'\{\mathfrak{S}_{n+1,0}\} = V'\{\mathfrak{S}_{n+1,n}\} = P_n$. When $0 < i < n$ and $w = x_1 x_2 \cdots \cdots x_{n+1} \in \mathfrak{S}_{n+1,i}$, we write $w = w' \, 1 \, w''$ so that $w' = x_1 \cdots x_i$, $w'' = x_{i+2} \cdots x_{n+1}$ and $x_{i+1} = 1$. Then $V'(w) = \prod_{1\leq j\leq i} V_{x_j}(w') \prod_{i+2\leq j\leq n+1} V_{x_j}(w'')$. If we restrict the letters of w' to belong to a given subset I of $\{2, \cdots, n+1\}$ of cardinality i, we have $\Sigma V'(w) = P_i P_{n-i}$, for the notions of peak, trough, rise and descent depend only on the mutual order of the integers and not on the integers themselves. To obtain $V'\{\mathfrak{S}_{n+1,i}\}$ we let I run over all the subsets of $\{2, \cdots, n+1\}$ of cardinality i. Thus $V'\{\mathfrak{S}_{n+1,i}\} = \binom{n}{i} P_i P_{n-i}$. Altogether $2V\{\mathfrak{S}_{n+1}\} = 2sP_n + \sum_{0<i<n} \binom{n}{i} P_i u P_{n-i}$, which is identity (11).

Q.E.D.

Another equivalent form for identities (10) and (11) is given in the next theorem.

11. THEOREM. *In the formal power series algebra in one variable v with coefficients in the algebra of polynomials in the variables X_p, X_f, X_s, X_r, X_d, the following identity holds*

$$(12) \qquad \sum_{n\geq 0} (v^n/n!) P_{n+1} = X_p \exp\Big((X_r + X_d)v + \sum_{n\geq 2} (v^n/n!)(X_f + X_s) P_{n-1}\Big).$$

Proof. As before, let $X_r + X_d = s$, $X_p = t$, $X_f + X_s = u$. By induction on n it follows from (10) that all the polynomials P_n $(n \geq 1)$ are divisible by X_p. We then define Q_n by $X_p Q_n = P_n$ $(n \geq 1)$. Relation (10) can be rewritten with the change of variables $j = n - 1 - i$

$$P_{n+1} = sP_n + \sum_{1\leq j\leq n-2} \binom{n-1}{n-1-j} P_{n-1-j} u P_{j+1} + P_{n-1} u P_1,$$

i.e.

$$Q_{n+1} = sQ_n + \sum_{1\leq j\leq n-2} \binom{n-1}{j} Q_{j+1} u P_{n-1-j} + P_{n-1} u.$$

Hence

$$(13) \qquad Q_{n+1} = P_{n-1}\,u + \sum_{1 \le j \le n-2} \binom{n-1}{j} Q_{j+1}\, P_{n-1-j}\, u + \binom{n-1}{n-1} Q_n\, s\,,$$

i.e.

$$(14) \qquad a_n = b_n + \sum_{1 \le j \le n-1} \binom{n-1}{j} a_j\, b_{n-j} \qquad (n > 0)$$

by letting $a_n = Q_{n+1}\,(n > 0)$, $b_1 = s$ and $b_n = P_{n-1}\,u\,(n > 1)$. As it is well-known (see e.g. [10] chap. 1), a binomial recurrence relation such as (14) is equivalent to the exponential identity

$$1 + \sum_{n>0} (v^n/n!)\, a_n = \exp\Big(\sum_{n>0} (v^n/n!)\, b_n\Big).$$

Consequently, (13) is equivalent to

$$1 + \sum_{n>0} (v^n/n!)\, Q_{n+1} = \exp\Big(sv + \sum_{n\ge 2} (v^n/n!)\, uP_{n-1}\Big).$$

Taking into account the relation $X_p\, Q_n = P_n\,(n > 0)$, the latter identity is equivalent to (12). Q.E.D.

4. Related coefficients

The coefficients $d_{n,k}$ (see formula (9)) count the orbits of G_n with k peaks. The first values are given in the following table

TABLE 1.

$n \backslash k$	1	2	3	4	$D_n = \Sigma d_{n,k}$
1	1				1
2	1				1
3	1	1			2
4	1	4			5
5	1	11	4		16
6	1	26	34		61
7	1	57	180	34	272
8	1	120	768	496	1385

Note that the summation by row gives the Euler numbers D_n. The most extensive table of the coefficients D_n themselves appears in Buckholtz–Knuth [4].

From Theorem 7 it follows that in any orbit of G_n with k peaks, there is one and only one permutation w with variation $V(w) = X_p^k\, X_f^{k-1}\, X_r^{n-2k+1}$, i.e. a permutation *having no descents and whose troughs are all of the first kind.* Such permutations were called *André* *permutations* in Foata-Schüzenberger [12]. For instance, the first André permutations, with respect to the factor-ordering relation (6), are given in Table 2.

TABLE 2.

$n = 1$	1						
$n = 2$	12						
$n = 3$	123	213					
$n = 4$	1234	1324	2314	2134	3124		
$n = 5$	12345	12435	13425	23415	13245	14235	34125
	24135	23145	21345	41235	31245	21435	32415
	41325	31425					

So $d_{n,k}$ also counts the number of André permutations in $\mathfrak{S}_n$ with k peaks.

The André permutations can be defined with respect ot *any* factor-ordering relation. In any case, their number in $\mathfrak{S}_n$ will be equal to the tangent or secant number D_n. Let us call *André permutations of the second kind* those permutations having no descents and whose troughs are all of the second kind with respect to the following factor-ordering relation, R, defined by

(15) (i) $\emptyset \,R\, I$

(ii) $I \,R\, J$ if $\min I \leq \min J$.

Again from theorem 7 it follows that

(16) any orbit of G_n contains exactly one André permutation of the second kind.

Let $1 \leq x \leq n$ and $w \in \mathfrak{S}_n$ $(n > 0)$. The permutation deduced from w by inserting $n + 1$ in w *just before* (resp. *after*) x is denoted by $I_{n+1,x}\, w$ (resp. $J_{x,n+1}\, w$). For example, if $w = 31452$ and $x = 4$, we have $I_{6,4}\, w = 316452$ and $J_{4,6}\, w = 314652$. The next proposition is an immediate consequence of the definition of the factor-ordering relation R in (15) and given without proof.

12. PROPOSITION. (i) *Let w be an André permutation of the second kind in $\mathfrak{S}_n$. If x is a peak (resp. rise) of w, then $I_{x,n+1}\, w$ (resp. $J_{n+1,x}\, w$) is an André permutation of the second kind in $\mathfrak{S}_{n+1}$.*

(ii) *Conversely, if $v = w' (n + 1)\, w''$ is an André permutation of the second kind in $\mathfrak{S}_{n+1}$ then $w = w'\, w''$ is an André permutation of the second kind in $\mathfrak{S}_n$ and there exists exactly one $x \in [n]$ such that $v = J_{x,n+1}\, w$ or $v = I_{n+1,x}\, w$.*

Let $A_{n,k}$ be the set of all André permutation of the second kind with k peaks. It follows from the previous proposition that $A_{n+1,k}$ is the union of the two sets $\{I_{x,n+1}\, w : w \in A_{n,k}\,,\, x \text{ a peak of } w\}$ and $\{J_{n+1,x}\, w : w \in A_{n,k-1}\,,\, x \text{ a rise of } w\}$. But a permutation w in $A_{n,k-1}$ has $n - (k - 1) - (k - 2) = n + 3 - 2k$ rises. Hence

(17) $$d_{n+1,k} = k d_{n,k} + (n + 3 - 2k)\, d_{n,k-1},$$

with $0 < k \leq [(n + 1)/2]$, $n \geq 1$ and the limit conditions $d_{1,1} = 1$ and $d_{n,k} = 0$ for $k < 0$ and $n + 1 < 2k$. The above recurrence formula was already established in [12] by using analytic methods.

The first André permutations of the second kind are given in Table 3.

TABLE 3.

$n = 1$	1					
$n = 2$	12					
$n = 3$	123	312				
$n = 4$	1234	1423	3412	4123	3124	
$n = 5$	12345	12534	14523	34512	15234	14235
	34125	45123	35124	51234	41235	31245
	51423	53412	41523	31524		

Again from (9) we see that the coefficient

$$c_{n,k} = 2^{k-1} d_{n,k} \qquad (0 < k \leq [n/2])$$

counts the number of permutations w *in* $\mathfrak{S}_n$ *with* k *peaks and* $n - 2k + 1$ *rises* (resp. *descents*). When $n - 2k + 1 = 0$, the coefficient $c_{2k-1,k}$ is the number of permutations w in $\mathfrak{S}_n$ having k peaks and $k - 1$ troughs. These permutations are called *alternating* and according to André's famous discovery ([1], [2]) we have

$$c_{2k-1,k} = D_{2k-1} \qquad \text{for any } k \geq 1.$$

There are orbits of G_{2k-1} that are only made of alternating permutations. As $c_{2k-1,k} = 2^{k-1} d_{2k-1,k}$, we have $2^{k-1} \mid D_{2k-1}$. It is in no case the best result since it can be proved analytically (see Nielsen [18] p. 258). that the 2-valuation of kD_{2k-1} is $2k - 2$. When k is odd, a grouping of the orbits of G_{2k-1} should be found in order to establish the latter result directly.

Let $1 \leq x \leq n$ and $w \in \mathfrak{S}_n$. We define

$$\begin{aligned} \psi_x(w) &= \varphi_x(w) && \text{if } x \text{ is either a rise or descent of } w \\ &= w && \text{otherwise.} \end{aligned}$$

Let H_n denote the subgroup of the permutation group acting on $\mathfrak{S}_n$ generated by $\{\psi_x : 1 \leq x \leq n\}$. Using the same techniques as above it is easy to show that H_n is an abelian group of order 2^{n-1}. Furthermore, any orbit of H_n is a subset of some orbit of G_n. Finally, $c_{n,k}$ is the number of orbits of H_n all elements of which have k peaks.

From recurrence formula (17) we obtain

$$c_{1,1} = 1$$

and

$$c_{n+1,k} = kc_{n,k} + 2(n + 3 - 2k)\, c_{n,k-1} \qquad (0 < k \leq [(n+1)/2])$$

with

$$c_{n,k} = 0 \quad \text{if } k < 0 \quad \text{or} \quad n + 1 < 2k.$$

TABLE 4.

$n \backslash k$	1	2	3	4
1	1			
2	1			
3	1	2		
4	1	8		
5	1	22	16	
6	1	52	136	
7	1	114	720	272
8	1	240	3072	3968

The coefficients $c_{n,k}$ already appeared in Foata-Schützenberger ([11], chap. 5). They also occurred in Carlitz-Scoville [7], who used them in several enumeration studies on permutations.

Now put $X_f = X_s = X_r = X_d = 1$ in formula (9). We then see that the coefficient

$$t_{n,k} = 2^{n-k} d_{n,k}$$

counts the *number of permutations in* $\mathfrak{S}_n$ *with k peaks*. The recurrence formula is $t_{1,1} = 1$ and, using the same conventions as above,

$$t_{n+1,k} = 2\,kt_{n,k} + (n+3-2\,k)\,t_{n,k-1} \qquad (0 < k \leq [(n+1)/2])\,.$$

As for the first values, we have

TABLE 5.

n \ k	1	2	3	4
1	1			
2	2			
3	4	2		
4	8	16		
5	16	88	16	
6	32	416	272	
7	64	1824	2880	272
8	128	7680	24576	7936

The distribution of the number of peaks and so the above recurrence formula for the $t_{n,k}$ was obtained by Kermack and McKendrick ([15], [16]). See also Barton and David ([3], p. 163).

Let us now mention the relation of these coefficients with the Eulerian polynomials. Let $(A_n(t))_{n>0}$ be the sequence of the Eulerian polynomials defined by their exponential generating function

$$1 + \sum_{n>0} (u^n/n!)\,A_n(t) = (1-t)/(1-t\exp((1-t)\,u))\,.$$

The first values are $A_1(t) = t$, $A_2(t) = t^2 + t$, $A_3(t) = t^3 + 4\,t^2 + t$, $A_4(t) = t^4 + 11\,t^3 + 11\,t^2 + t$. It is known (Foata-Schützenberger [11], see also Riordan [19], pp. 213–216, Carlitz [6]) that $A_n(t)$ is the generating polynomial of the number of peaks or descents over $\mathfrak{S}_n$. From formula (9) we have $A_n(t) = P_n$ when we make the substitutions $X_p \leftarrow t$, $X_d \leftarrow t$ and let $X_f = X_s = X_r = 1$. In other words

$$A_n(t) = \Sigma\, 2^{k-1} d_{n,k}\, t^k (1+t)^{n-2k+1}$$
$$= \Sigma c_{n,k}\, t^k (1+t)^{n-2k+1},$$

a result that can be proved analytically (Foata-Schützenberger [11], Carlitz-Scoville [7]).

References

[1] D. André (1879) – *Développements de sec x et de tan x*, « C. R. Acad. Sc. Paris, *88*, 965–967.

[2] D. André (1881) – *Sur les permutations alternées*, « J. Math. Pures Appl. », 7, 167–184.

[3] F. N. David and D. E. Barton (1962) – *Combinatorial Chance*, Griffin, London.

[4] T. J. Buckholtz and D. E. Knuth (1967) – *Computation of tangent, Euler and Bernoulli numbers*, « Math. Comp. », *21*, 663–688.

[5] W. Burnside (1911) – *Theory of Groups of Finite Order*, Dover, New York.

[6] L. Carlitz (1959) – *Eulerian numbers and polynomials*, « Math. Magazine », *33*, 247–260.

[7] L. Carlitz and R. Scoville (1974) – *Generalized Eulerian numbers: combinatorial applications*, « *J. für die reine und angewandte Mathematik* », *265*, 110–137..

[8] L. Comtet (1970) – *Analyse Combinatoire*, Vol. 2, Presses Universitaires de France, Paris.

[9] D. Foata (1972) – *Groupes de réarrangements et nombres d'Euler*, « C. R. Acad. Sc. Paris », *275*, 1147–1150.

[10] D. Foata (1973) – *La série génératrice exponentielle dans les problèmes d'énumération*, Presses de l'Université de Montréal, Montréal.

[11] D. Foata and M. P. Schützenberger (1970) – *Théorie géométrique des polynômes eulériens*, Lecture Notes in Math., vol. 138, Springer-Verlag, Berlin.

[12] D. Foata and M. P. Schützenberger (1973) – *Nombres d'Euler et permutations alternantes*, in *A Survey of Combinatorial Theory* (J. N. Srivastava *et al.* editors), North-Holland, Amsterdam, pp. 173–187.

[13] D. Foata and V. Strehl (1974) – *Rearrangements of the symmetric group and enumerative properties of the tangent and secant numbers*, « Math. Z. », *137*, 257–264.

[14] G. Frobenius (1910) – *Über die Bernoullischen Zahlen und die Eulerschen Polynome*, Sitzungsberichte der königliche preussischen Akademie der Wissenschaften, 809–847.

[15] W. O. Kermack and A. G. McKendrick (1937) – *Some distributions associated with a randomly arranged set of numbers*, « Proc. Roy. Soc. Edinburgh », *67*, 332–376.

[16] W. O. Kermack and A. G. McKendrick (1938) – *Some properties of points arranged on a Möbius surface*, « The Math. Gazette », *22*, 66–72.

[17] E. Netto (1901) – *Lehrbuch der Combinatorik*, Teubner, Leipzig.

[18] N. Nielsen (1923) – *Traité élémentaire des nombres de Bernoulli*, Gauthier-Villars, Paris.

[19] J. Riordan (1958) – *An Introduction to Combinatorial Theory*, J. Wiley, New York.