Proceedings of Symposia in Pure Mathematics, vol. 34, 1979
American Mathematical Society, Providence
(Relations between Combinatorics and other parts of Mathematics)
[1978. Columbus], p. 163-179.

A COMBINATORIAL APPROACH TO THE MEHLER FORMULAS FOR HERMITE POLYNOMIALS

Dominique Foata and Adriano M. Garsia

Abstract. The multilinear expansion of the Mehler formula found by Slepian is proved combinatorially by using the partitional complex method. The combinatorial structures involved are the so-called n-involutionary graphs. They are enumerated in two different manners : first, globally, then, as the exponential of their connected components. lt is shown that these two enumerations make up the left and right side members of the formula to be proved.

1. Introduction. The Hermite polynomials $H_n(z)$ $(n \geq 1)$ can be defined by means of their exponential generating function

$$1+\sum_{m\geq 1}\frac{u^m}{m!}\,H_m(z)=\exp(uz-z^2/2). \tag{1}$$

Formula (1) yields

$$H_m(z)=\sum_{0\leq k\leq m}(-1)^k z^{m-2k}\frac{m!}{(2!)^k k!\,(1!)^{m-2k}(m-2k)!}\quad (m\geq 1). \tag{2}$$

The following bilinear extension of (1) is known as Mehler Formula

$$1+\sum_{m\geq 1}\frac{u^m}{m!}H_m(z_1)H_m(z_2) = (1-u^2)^{-1/2}\exp\Bigl(\frac{2uz_1z_2-u^2(z_1^2+z_2^2)}{(1-u^2)}\Bigr). \tag{3}$$

Multilinear extensions have been found by Carlitz [2]. Then, Slepian [8] proved the following formula that contains all the other extensions as special cases. Let $n\geq 2$ be a fixed integer and $\rho=(\rho_{ij})_{(1\leq i;1\leq j)}$ be a real symmetric matrix with all diagonal elements ρ_{ii} $(1\leq i\leq n)$ equal to 1. Denote by $\rho^{-1}=(\rho_{ij}^{-1})$ the inverse of ρ and by δ_{ij} the Kronecker symbol. In the next formula the matrix $(\nu_{ij})_{(1\leq i,j\leq n;i\neq j)}$ runs over the set of all

AMS(MOS) subject classifications (1970). Primary 05A15, 33A30; secondary 05030.
The present text was TEX-reformated in september 2021; new pagination : 163-178.

symmetric matrices with non-negative integral entries and for $1 \le j \le n$ the j-th row sum of that matrix is denoted by

$$s_j = \sum_{1 \le i \le n, i \ne j} \nu_{ij}.$$

Now Slepian's formula reads

$$\sum_{(\nu_{ij})} \frac{\prod\limits_{1 \le i < j \le n} \rho_{ij}^{\nu_{ij}}}{\prod\limits_{1 \le i < j \le n} \nu_{ij}!} H_{s_1}(z_1) \cdots H_{s_n}(z_n) = (\det \rho)^{-1/2} \exp\Big(-\frac{1}{2} \sum_{i,j} (\rho_{ij}^{-1} - \delta_{ij}) z_i z_j\Big). \tag{4}$$

Note that with $n = 2$ and $\rho = \binom{1\ u}{u\ 1}$ Slepian's formula reduces to Mehler formula.

A combinatorial proof of Mehler formula was presented by one of the authors during the Combinatorics Conference at Oberwolfach in May 1977 and was later published in [5]. The combinatorial model used was the so-called *bicolored involutionary graph*, while the computational model was the *partitional complex.* Richard Askey [11] drew the attention of the authors to Slepian's formula [8], and insisted that it should also receive a combinatorial proof.

The purpose of the present paper is to make up such a proof. Although the counting is more elaborate, the method developed in [5] applies. The left-hand side of (4) appears to be the exponential generating function for the so-called *n-involutionary graphs.* As for the right-hand side, it can be written as the exponential of the series

$$\frac{1}{2} \log \frac{1}{\det \rho} + \frac{1}{2} \sum_{i,j} (\delta_{ij} - \rho_{ij}^{-1}) z_i z_j, \tag{5}$$

and (4) has the usual form of the exponential formula that has been studied in several combinatorial contexts, namely,

$$1 + \sum_{m \ge 1} \frac{1}{m!} \mu\{Y_m^{(+)}\} = \exp \sum_{m \ge 1} \mu\{Y_m\}.$$

The problem then consists of finding a sequence of combinatorial structures $(Y_m^{(+)})_{(m \ge 1)}$ (the *n-involutionary graphs*) and a function μ defined on the union of the $Y_m^{(+)}$'s with the property that

(i) the left-hand side of (4) is the exponential generating function for μ over the sets $Y_m^{(+)}$;

(ii) μ satisfies certain compatibility properties (of being multiplicative and degree-preserving);

(iii) the argument of exp, namely (5), is the exponential generating function for μ over *connected* n-involutionary graphs.

The n-involutionary graphs and the function μ are introduced in section 2. Property (i) above follows from the well-known combinatorial interpretation of the Hermite polynomials in terms of involutions of finite sets (see Proposition 1). The properties of the partitional complex are recalled in section 3 and applied to the sequence $(Y_m^{(+)})$ of the n-involutionary graphs. For every $m \geq 2$ the set Y_m occurring in the exponential formula is the collection of the connected n-involutionary graphs. In section 4 the latter structures are shown to be either labeled cycles (in the graph-theoretic sense) with colored edges, or labeled paths again with colored edges ending with two colored loops. There remains to show that

$$\frac{1}{2}\log\frac{1}{\det\rho}$$

and

$$+\frac{1}{2}\sum_{i,j}(\delta_{ij}-\rho_{ij}^{-1})z_iz_j$$

are the exponential generating functions for μ over the cycles, and the paths, respectively. This is achieved by means of the Jacobi identity

$$\log\frac{1}{\det\rho}=\mathrm{Tr}\,\log\frac{1}{\rho}$$

and the following formula for the inverse

$$\rho^{-1}=\sum_{m\geq 0}(\mathrm{I}-\rho)^m.$$

In the whole paper each entry ρ_{ij} of the matrix ρ will be regarded as an indeterminate and formula (4) as an identity in the formal power series algebra in the variables ρ_{ij} $(1 \leq i < j \leq n)$ with coefficients in the polynomial ring $\mathbf{Q}[z_1, z_2, \ldots, z_n]$.

There is another combinatorial proof of (4) that is based on the algebraic methods developed in [3] for obtaining a non-commutative version. It is the intention of the authors to publish that proof separately.

2. Involutionary graphs. In the left-hand side of (4) taking a first summation with respect to $m \geq 0$ and then summing over all matrices (ν_{ij}) with $\sum_{i<j}\nu_{ij}=m$ yields

$$(6)\qquad \sum_{m\geq 0}\frac{1}{m!}\sum\Big\{\frac{m!}{\prod_{i<j}\nu_{ij}!}\prod_{i<j}\rho_{ij}^{\nu_{ij}}H_{s_1}(z_1)\cdots H_{s_n}(z_n) : \sum_{i<j}\nu_{ij}=m\Big\}.$$

The multinomial coefficient $m!/\prod_{i<j}\nu_{ij}!$ counts the sequences $(N_{ij})_{(i<j)}$ of subsets of the interval $[m]=\{1,2,\ldots,m\}$ with the property that $|N_{ij}|=\nu_{ij}$, $N_{ij}\cap N_{i'j'}=\emptyset$ if $(i,j)\neq(i',j')$) and the disjoint union $\sum_{i<j}N_{ij}$ is equal to $[m]$. By convention, if $i<j$, let $N_{ji}=N_{ij}$ and for $i\le j\le n$ let

$$S_j=\sum_{1\le i\le n, i\neq j}N_{ij}.$$

Note that

$$S_i\cap S_j=N_{ij}\quad (i\neq j). \tag{7}$$

By using the above relations formula (6) is transformed into

$$\sum_{m\ge 0}\frac{1}{m!}\sum\prod_{i<j}\rho_{ij}^{|N|_{ij}}H_{|S_1|}(z_1)\cdots H_{|S_n|}(z_n), \tag{8}$$

the second summation being extended over all sequences $(N_{ij})_{(i<j)}$ of disjoint subsets of $[m]$ with union equal to $[m]$.

Now let S be a finite set and $\mathfrak{I}_S$ be the set of all involutions of S. If σ is an element of $\mathfrak{I}_S$ let $t(\sigma)$ (resp. $f(\sigma)$) be the number of transpositions (resp. fixed points) of σ. Remember that σ^2 is the identity map and accordingly σ only has transpositions (cycles of length 2) and fixed points. It is readily verified that

$$\frac{m!}{(2!)^k k!\,(1!)^{m-2k}(m-2k)!}$$

occurring in (2) is equal to the number of involutions σ of a set S of cardinality m, having $t(\sigma)=k$ transpositions and $f(\sigma)=m-2k$ fixed points. In other words,

$$H_m(z)=H_{|S|}(z)=\sum\{(-1)^{t(\sigma)}z^{f(\sigma)}:\sigma\in\mathfrak{I}_S\}. \tag{9}$$

Thus, (8) may be rewritten in the form

$$\sum_{m\ge 0}\frac{1}{m!}\sum\prod_{i<j}\rho_{ij}^{|N|_{ij}}\prod_{1\le i\le n}(-1)^{t(\sigma_i)}z_i^{f(\sigma_i)}, \tag{10}$$

where the second summation is extended over all sequences $((N_{ij})_{(i<j)},\sigma_1,\ldots,\sigma_n)$ having the following properties

(i) $(N_{ij})_{(i<j)}$ is a sequence of disjoint subsets of $[m]$ with union equal to $[m]$; let $N_{ji} = N_{ij}$ for $i < j$ and $S_j = \sum\limits_{1 \le i \le n, i \ne j} N_{ij}$;

(ii) σ_i is for each $i = 1, \dots, n$ an involution of S_i.

A sequence $((N_{ij})_{(i<j)}, \sigma_1, \dots, \sigma_n)$ with the two properties (i) and (ii) will be called an n-*involutionary m-graph.* Note that, because of (7), the set N_{ij} is the only part of $[m]$ common to the domains of the involutions σ_i and σ_j.

The n-involutionary m-graph $((N_{ij})_{(i<j)}, \sigma_1, \dots, \sigma_n)$ may be viewed as an undirected graph (in the usual sense) with m vertices labeled 1, 2, … , m and edges and loops colored 1, 2, … , n. The vertices are the m elements of $[m]$. Furthermore, there is an edge colored i between vertices v and v' whenever $\sigma_i(v) = v'$ (and, accordingly, $\sigma_i(v') = v$). Also, a loop colored i is about vertex v whenever $\sigma_i(v) = v$. Each vertex v belongs to a unique component N_{ij} $(i < j)$. As the involutions σ_i and σ_j are the only ones in the sequence $(\sigma_1, \dots, \sigma_n)$ whose domains contain v, the following properties hold

(i) each vertex has valency 2 ;

(ii) any two edges or loops incident to the same vertex have different colors.

For instance, the following figure shows an n-involutionary m-graph with $n = 4$ colors and $m = 13$ vertices. The labels of the vertices have been circled to avoid any confusion with the colors of the edges and loops.

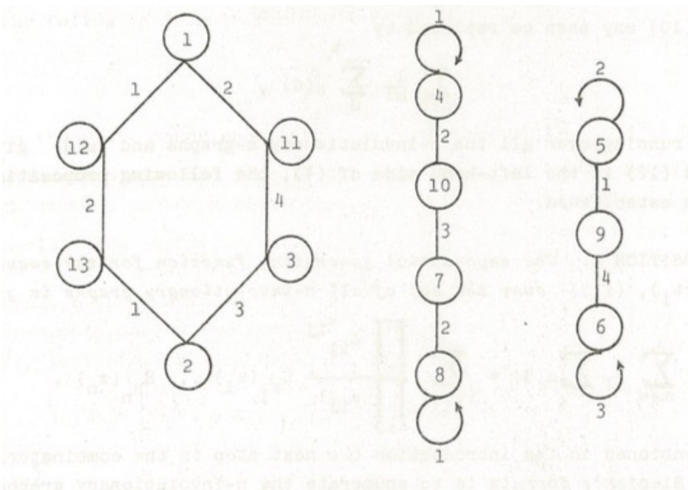

Figure 1 : a 4-involutionary 13-graph

The corresponding sequence $((N_{ij})_{(i<j)}, \sigma_1, \dots, \sigma_n)$ is then $N_{12} = \{1, 4, 5, 8, 12, 13\}$; $N_{13} = \{2\}$; $N_{14} = \{9\}$; $N_{23} = \{7, 10\}$; $N_{24} = \{11\}$; $N_{34} = \{3, 6\}$.

$S_1 = \{1, 2, 4, 5, 8, 9, 12, 13\}$; $S_2 = \{1, 4, 5, 7, 8, 10, 11, 12, 13\}$; $S_3 = \{2, 3, 6, 7, 10\}$; $S_4 = \{3, 6, 9, 11\}$; and

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 4 & 5 & 8 & 9 & 12 & 13 \\ 12 & 13 & 4 & 9 & 8 & 5 & 1 & 2 \end{pmatrix};$$
$$\sigma_2 = \begin{pmatrix} 1 & 4 & 5 & 7 & 8 & 10 & 11 & 12 & 13 \\ 11 & 10 & 5 & 8 & 7 & 4 & 1 & 13 & 12 \end{pmatrix};$$
$$\sigma_3 = \begin{pmatrix} 2 & 3 & 6 & 7 & 10 \\ 3 & 2 & 6 & 10 & 7 \end{pmatrix}; \quad \sigma_4 = \begin{pmatrix} 3 & 6 & 9 & 11 \\ 11 & 9 & 6 & 3 \end{pmatrix}.$$

Going back to the general case of an n-involutionary m-graph $G = ((N_{ij})_{(i<j)}, \sigma_1, \ldots, \sigma_n)$ note that N_{ij} is the set of all vertices incident to an edge or loop colored i and an edge or loop colored j (in short, *vertices incident to colors i, j*). Let $n_{ij} = n_{ij}(G)$ be the number of those vertices. The number of edges colored i is the number of transpositions within the involution σ_i; let $t_i = t_i(G)$ be that number. Finally, the number of loops colored i is the number of fixed points within σ_i; let $f_i = f_i(G)$ be that number. The monomial

$$\prod_{i<j} \rho_{ij}^{|N|_{ij}} \prod_{1 \le i \le n} (-1)^{t(\sigma_i)} z^{f(\sigma_i)}$$

occurring in formula (10) may be rewritten as

$$\mu(G) = \prod_{i<j} \rho_{ij}^{|N|_{ij}} \prod_{1 \le i \le n} (-1)^{t_i} z^{f_i}. \tag{11}$$

Formula (10) may then be replaced by

$$\sum_{m \ge 0} \frac{1}{m!} \sum_G \mu(G), \tag{12}$$

with G running over all the n-involutionary m-graphs and $\mu(G)$ given by (11). As (12) is the left-hand side of (4), the following proposition has then been established,

Proposition 1. *The exponential generating function for the sequence $((N_{ij}, (t_i), (f_i))$ over the set of all n-involutionary graphs is given by*

$$\sum_{m \ge 0} \frac{1}{m!} \sum_G \mu(G) = \sum_{(\nu_{ij})} \frac{\prod_{1 \le i<j \le n} \rho_{ij}^{\nu_{ij}}}{\prod_{1 \le i<j \le n} \nu_{ij!}} H_{s_1}(z_1) \cdots H_{s_n}(z_n).$$

As mentioned in the Introduction the next step in the combinatorial proof of Slepian's formula is to enumerate the n-involutionary graphs as the exponential of their *connected* components. A few basic properties of the exponential formula are first recalled.

3. The exponential formula. Let $Y = (Y_m)_{(m \ge 1)}$ be a sequence of finite sets. For each $m \ge 1$, each y in Y_m and each finite subset I of $\mathbf{N} \setminus \{0\}$

of cardinality m let (y, I) denote an indeterminate. All the variables (y, I) are assumed to commute. For each $m \geq 1$ the *partitional complex of* Y *of degree* m is defined to be the set of ali the monomials

$$(y_1, I_1)(y_2, I_2) \cdots (y_r, I_r)$$

$(1 \leq r \leq m)$ with the property that $(I_1, I_2, \ldots, I_r)$ is a partition of the interval $[m] = \{1, 2, \ldots, m\}$. The partitional complex of Y of degree m will be denoted by $Y_m^{(+)}$.

Let Ω be an algebra of polynomials with rational coefficients, $\overline{\Omega}$ be an algebra of formal power series with coefficients in Ω and μ be a map that sends each monomial in the variables (Y, I) to a monomial of $\overline{\Omega}$.

The map μ is said to be *multiplicative* if the following two conditions hold

(i) $\mu(ww') = \mu(w)\,\mu(w')$ for every pair of monomials w, w' in the variables (y, I);

(ii) $\mu(y, I) = \mu(y)$ for every variable (y, I).

Furthermore, μ is said to be *degree-preserving* if for each $m \geq 1$ and w in $Y_m^{(+)}$ the degree of the monomial $\mu(w)$ (in $\overline{\Omega}$) is m.

In the next Proposition the notations $\mu\{Y_m^{(+)}\}$ and $\mu\{Y_m\}$ will stand for $\sum\{\mu(w) : w \in Y_m^{(+)}\}$ and $\sum\{\mu(w) : w \in Y_m\}$, respectively.

Proposition 2 (the exponential formula)**.** *If μ is multiplicative and degree-preserving, the following identity holds*

$$1 + \sum_{m \geq 1} \frac{1}{m!}\, \mu\{Y_m^{(+)}\} = \exp \sum_{m \geq 1} \frac{1}{m!}\, \mu\{Y_m\}. \tag{13}$$

Proof. For each monomial $w = (y_1, I_1) \cdots (y_r, I_r))$ let $\gamma(w) = y_1 y_2 \cdots y_r$. The abstract exponential formula (see, *e.g.*, [4] p. 64) reads

$$1 + \sum_{m \geq 1} \frac{1}{m!} \sum\{\gamma(w) : w \in Y_m^{(+)}\} = \exp \sum_{m \geq 1} \frac{1}{m!} \sum\{\gamma(w) : w \in Y_m\} \tag{14}$$

in the algebra of formal power series with indeterminates y in $\bigcup_{m \geq 1} Y_m$. As μ is multiplicative and degree-preserving, it can be extended to a continuous homomorphism of the latter formal power series algebra into $\overline{\Omega}$. Hence, applying μ to both members of (14) yields (13). ∎

4. Connected involutionary graphs. An n-colored m graph is an undirected graph with m vertices labeled 1, 2, … , m with edges and loops colored 1, 2, … , n with the further property that each vertex has valency 2. When any two edges or loops incident to the same vertex have different colors, the graph is then n-involutionary, as defined in Section 2. The purpose of this section is to characterize the connected n-colored graphs. The connected n-colored m-graphs with $m = 1, 2, 3$ vertices are shown in Figure 2. In the five examples of Figure 2 the indices i, j, k, ℓ are colors taken from $[n]$ and, in the last one, $v_1 v_2 v_3$ is a permutation of 123.

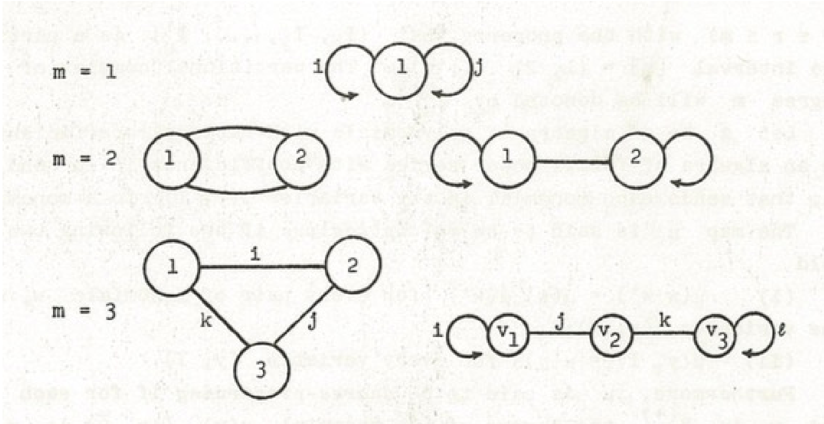

Figure 2 : connected n-colored graphs with $m \leq 3$ vertices

The connectedness is a binary relation defined on the vertex set in the usual way. Let G be an n-colored m-graph. A *path* of G is a sequence $(e_1, e_2, \ldots, e_k)$ of edges with the property that either $k = 1$, or $k \geq 2$ and for each $i = 1, 2, \ldots, k-1$ the two edges e_i and e_{i+1} are incident to exactly one vertex. If $k = 1$ and $e_1 = \{v_1, v_1'\}$ the path is said to go from v_1 to v_1' (or from v_1' to v_1. If $k \geq 2$, $e_m = \{v_m, v_m'\}$ and v_2 (resp. v_m') is not incident to e_2 (resp. e_{m-1}), the path is said to go from v_1 to v_m'. If $v_1 = v_m'$, the path is a cycle.

By convention, if e_1 and e_2 are two edges incident to the same vertices v and v', the pairs (e_1, e_2) and (e_2, e_1) are also called cycles. Two vertices v and v' are *equivalent*, if, either $v = v'$ or $v \neq v'$ and there exists a path going from v to v'. The equivalence classes are the connected components of the graph. A graph is *connected*, if it has only one connected component.

Proposition 3. *There are two kinds of connected n-colored m-graphs*

(i) the cycles with m edges, m vertices and no loops, called the n-colored m-cycles;

(ii) the paths with $(m-1)$ edges, m vertices ending with two loops, called the n-colored m-paths..

The proof of Proposition 3 was written in a former version of the paper, but has been omitted here. It only involves mere techniques of Graph Theory. The reader should be able to work it out by himseilf.

Now an n-colored m-graph G is an unordered collection of its connected components $\{G_1, G_2, \dots, G_r\}$. For $1 \le i \le r$ let I_i be the set of all vertices of the component G_i. If $\operatorname{card} I_i = m_i$, let $\omega_i : I_i \to [m_i]$ be the unique increasing map of I_i onto $[m_i]$. Let y_i be the graph obtained from G_i by replacing each vertex label v by $\omega_i(v)$. Clearly, y_i is an n-colored m-graph and G is completely characterized by the monomial

$$(y_1, I_1), (y_2, I_2) \cdots (y_r, I_r).$$

For each $m \ge 1$ let Y_m be the *set of connected n-colored m-graphs* and let $Y = (Y_m)_{(m \ge 1)}$. Then, the following proposition holds.

Proposition 4. *The set of n-colored m-graphs is the partitional complex $Y_m^{(+)}$ of Y of degree m.*

The function μ was defined in (11) for the n-involutionary graphs. To extend its definition to the n-colored graphs let $B = (b(i,j))_{(1 \le i,j \le n)}$ be a given matrix of n^2 commuting variables. For each n-colored m-graph G the number of vertices incident to color i twice is denoted by $n_{ii} = n_{ii}(G)$. As for the other functions n_{ij} $(i < j)$, t_i and f_i they keep the same meanings as in (11). Next, let

$$\mu(G) = \prod_{i \le j} (-b(i,j))^{n_{ij}} \prod_i (-1)^{t_i} z^{f_i}. \tag{15}$$

Finally, when every variable $b(i,i)$ $(1 \le i \le n)$ is mapped to zero, the notation $\overline{\mu}(G)$ will be used. It is further assumed that ρ_{ij} is equal to $-b(i,j)$ when $i < j$. Consequently, the two definitions of μ given in (11) and (15) coincide for the n-involutionary graphs. Also

$$\overline{\mu}(G) = \mu(G) \tag{16}$$

when G is n-involutionary.

Now let Ω be the algebra of polynomials with rational coefficients in the variables $z_1, z_2, \dots, z_n$. Also let $\overline{\Omega}$ be the algebra of formal power series with coefficients in Ω in the variables $b(i,j)$ $(1 \le i,j \le n)$. Both functions μ and $\overline{\mu}$ are multiplicative and degree-preserving (at least when their domains are restricted to $\bigcup_{n \ge 1} Y_n^{(+)}$). Accordingly, the next proposition is a straightforward consequence of Propositions 2 and 4.

Proposition 5. *The following identities*

$$1 + \sum_{m \ge 1} \frac{1}{m!} \mu\{Y_m^{(+)}\} = \exp \sum_{m \ge 1} \frac{1}{m!} \mu\{Y_m\} \tag{17}$$

and

$$1+\sum_{m\geq 1}\frac{1}{m!}\,\overline{\mu}\{Y_m^{(+)}\}=\exp\sum_{m\geq 1}\frac{1}{m!}\,\overline{\mu}\{Y_m\} \tag{18}$$

hold in $\overline{\Omega}$.

By (16) the series $1+\sum_{m\geq 1}(1/m!)\,\overline{\mu}\{Y_m^{(+)}\}$ is the exponential generating function for μ (or $\overline{\mu}$) over the n-involutionary graphs. By Proposition 1 it is then equal to the left-hand side of (4). Thus, there remains to prove that the argument of exp in (4) (see formula (5)) is the exponential generating function for $\overline{\mu}$ over the connected n-colored graphs. But when G is an *n-colored m-cycle*, the following relations hold $\sum_i t_i = \sum_{i\leq j} n_{ij} = m$ and $f_i = 0$ for every i. Therefore,

$$\mu(G)=\prod_{i\leq j} b(i,j)^{n_{ij}}. \tag{19}$$

When G is an n-colored m-path ending with two loops colored k, ℓ, then $\sum_i t_i = m-1$, $\sum_{i\leq j} n_{ij} = m$. Therefore,

$$\mu(G)=-z_k z_\ell\prod_{i\leq j} b(i,j)^{n_{ij}}. \tag{20}$$

This suggests to prove that

$$\frac{1}{2}\log\frac{1}{\det\rho}\qquad\text{(resp. }\frac{1}{2}\sum_{k,\ell}(\delta_{k,\ell}-\rho^{-1}_{k,\ell})z_k z_\ell) \tag{21}$$

is the exponential generating function for μ over the n-involutionary cycles (resp. n-involutionary paths). As might be expected, the oriented or directed structures corresponding to the cycles and paths will be more suitable for enumeration purposes.

5. Oriented colored cycles. Let G be an n-colored m-cycle. When $m=2$ and the two edges have different colors (*i.e.*, G is an n-involutionary 2-cycle) or when $m=3$, two *oriented* n-colored m-cycles may be constructed out of G. Each of them corresponds to a clockwise (resp. counterclockwise) reading of the vertex labels and the edge colors in succession. For instance, the first (resp. second) oriented cycle corresponds to reading vertex labels and edge colors by starting with vertex 1 and going toward the smaller (resp. greater) vertex adjacent to it. Let $v_1x_1v_2x_2\cdots v_mx_m$ be the first reading ($v_1=1$, the color of the edge joining with the smaller vertex v_2 adjacent to it). Then, $x_1x_mv_m\cdots x_2v_2x_1$ is the second reading. As noted before, if $m=2$ ans $x_1\neq x_2$, or if $m\geq 3$, the two readings are different. It will be convenient to write these two readings as circular biwords

$$c=\begin{bmatrix}x_1x_2\cdots x_m\\ v_1v_2\cdots v_m\end{bmatrix}\quad\text{and}\quad c'=\begin{bmatrix}x_m\cdots x_2x_1\\ v_m\cdots v_2v_1\end{bmatrix}$$

By circular it is meant that the biletter $\left[\begin{smallmatrix} x_1 \\ v_1 \end{smallmatrix}\right]$ is to be regarded as the biletter following (resp. preceding) $\left[\begin{smallmatrix} x_m \\ v_m \end{smallmatrix}\right]$ in c (resp. c'). The two circular biwords c and c' will then be called the two *oriented n-colored m-cycles* associated with the n-colored m-cycle G. By convention, for each color x there exists one oriented n-colored 1-cycle denoted by $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$. There corresponds no n-colored 1-cycle to it. Finally, to the n-colored 2-cycle represented in Figure 3 there corresponds

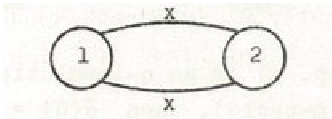

Figure 3 : an n-colored 2-cycle

only one oriented n-colored 2-cycle, namely $\left[\begin{smallmatrix} x & x \\ 1 & 2 \end{smallmatrix}\right]$.

For each color word $w = x_1 x_2 \cdots x_m$ and each n-colored m-cycle $c = \begin{bmatrix} w \\ \sigma \end{bmatrix} = \begin{bmatrix} x_1 x_2 \cdots x_m \\ v_1 v_2 \cdots v_m \end{bmatrix}$ define

$$\beta(w) = \beta(\sigma) = b(x_1, x_2)\, b(x_2, x_3)\, \cdots\, b(x_{m-1}, x_m)\, b(x_m, x_1). \tag{22}$$

The monomial $\beta(\sigma)$ records the occurrences of pairs of adjacent colors in the oriented cycle c. When every variable $b(i,i)$ ($1 \le i \le n$) is zero, write $\overline{\beta}(w)$ in place of $\beta(w)$.

For each $m \ge 2$ denote by C_m (resp. D_m, resp. C'_m, resp. D'_m) the set of the (non-oriented) n-colored m-cycles (resp. (non-oriented) n involutionary m-cycles, resp. oriented n-colored m-cycles, resp. oriented n-involutionary m-cycles). Also let C'_1 be the set of the biletters $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$ with $1 \le x \le n$. The next proposition follows immediately from the very definitions of μ in (15)) and β in (22).

Proposition 6. *For $m \ge 2$ (resp. $m \ge 3$) there is a one-to-one correspondence $G \mapsto \{c, c'\}$ between n-involutionary m-cycles (resp. n-colored m-cycles) and unordered pairs of oriented n-involutionary m-cycles (resp. of oriented n-colored m-cycles) with the property that, if the matrix β is symmetric, then*

$$\beta(G) = \beta(c) = \beta(c').$$

Corollary 7. *If $B = (b(i,j))_{(1 \le i, \supset \le n)}$ is symmetric, then for every $m \ge 2$ the following identities hold*

$$\mu\{C_m\} = \frac{1}{2}\beta\{C'_m\}$$

$$\overline{\mu}\{C_m\} = \mu\{D_m\} = \frac{1}{2}\beta\{D'_m\} = \frac{1}{2}\overline{\beta}\{C'_m\}.$$

Moreover, the exponential generating function for $\overline{\mu}$ (or μ) over the n-involutionary cycles is given by

$$\sum_{m\ge 1}\frac{1}{m!}\overline{\mu}\{D_m\}=\sum_{m\ge 1}\frac{1}{m!}\overline{\mu}\{C_m\}=\frac{1}{2}\sum_{m\ge 1}\frac{1}{m!}\overline{\beta}\{C'_m\}. \tag{23}$$

Proof. When G (resp. c) is an n-involutionary m-cycle (resp. an oriented n-involutionary m-cycle), then $\overline{\mu}(G)=\mu(G)$ (resp. $\overline{\beta}(c)=\beta(c)$). Accordingly, Corollary 7 follows immediately from the previous proposition. ∎

6. Directed colored paths. For each $m\ge 1$ a *directed n-colored m-path p* is a word $x_1v_1x_2v_2\cdots x_mv_mx_{m+1}$ with $w=x_1x_2\cdots x_mx_{m+1}$ a color word of length $(m+1)$ and $\sigma=v_1v_2\cdots v_m$ a permutation of $1\,2\,\cdots\,m$.

The pairs of adjacent colors in p will be recorded by the function α defined by

$$\begin{aligned}\alpha(p)&=\alpha(w)\\ &=-z_{x_1}b(x_1,x_2)b(x_2,x_3)\cdots b(x_{m-1},x_m)b(x_m,x_{m+1})z_{x_{m+1}}.\end{aligned} \tag{24}$$

The notation $\overline{\alpha}$ will be used whenever every variable $b(i,i)$ $(1\le i\le n)$ is mapped to zero.

Let G be an n-colored m-path. If $m=1$ and the two loops have the same colors, say, x, there corresponds to G only one directed n-colored m-path, namely $x\,1\,x$. If m and the loop colors are x and y with $x\neq y$, there corresponds to G two directed n-colored m-paths $x\,1\,y$ and $y\,1\,x$. When $m\ge 2$, every n-colored m-path G gives rise to two directed n-colored m-paths : if $x_1v_1x_2v_2\cdots x_mv_mx_{m+1}$ is the sequence of loop or edge colors and vertex labels when read from the smaller endnode v_1 to the greater endnode v_m, then the second directed path associated with G is $x_{m+1}v_mx_m\cdots v_2x_2v_1x_1$. To Proposition 6 dealing with cycles there corresponds the following proposition for paths.

Proposition 8. *For every $m\ge 1$ (resp. $m\ge 2$) there is a one-to-one correspondence $G\mapsto\{p,p'\}$ between n-involutionary m-paths (resp. n-colored m-paths) and unordered pairs of directed n-involutionary m-paths (resp. directed n-colored m-paths) with the property that, if the matrix B is symmetric, then*

$$\mu(G)=\alpha(p)=\alpha(p').$$

Proof. The proposition is a straightforward consequence of the above comment on directed n-colored paths, relation (20) and definition (24). ∎

For every $m\ge 1$ let P_m (resp. R_m, resp. P'_m, resp. R'_m) be the set of n-colored m-paths (resp. n-involutionary m-paths, resp. directed n-colored m-paths, resp. directed n-involutionary m-paths).

Corollary 9. *If $B = (b(i,j))_{1\le i,1\le j)}$ is symmetric, then for every $m \ge 2$ the following identities hold*

$$\mu\{P_m\} = \frac{1}{2}\alpha\{P'_m\}$$
$$\overline{\mu}\{P_m\} = \overline{\mu}\{R_m\} = \frac{1}{2}\overline{\alpha}\{R'_m\} = \frac{1}{2}\overline{\alpha}\{P'_m\}.$$

Moreover, the exponential generating function for $\overline{\mu}$ (or μ) over the n-involutionary paths is given by

$$\sum_{m\ge1}\frac{1}{m!}\overline{\mu}\{R_m\} = \sum_{m\ge1}\frac{1}{m!}\overline{\mu}\{P_m\} = \frac{1}{2}\sum_{m\ge1}\frac{1}{m!}\overline{\alpha}\{P'_m\}. \tag{25}$$

Proof. Again for every $m \ge 1$ the identities $\overline{\mu}\{P_m\} = \overline{\mu}\{R_m\}$ and $\overline{\alpha}(P'_m\} = \overline{\alpha}\{R'_m\}$ follow from the fact that each directed or non-directed n-colored m-path p is involutionary if and only if no vertex is incident to the same color twice, that is, if and only if no variable $b(i,i)$ occurs in $\mu(p)$ (resp. $\alpha(p)$). Formula (25) is then a consequence of Proposition 8. ∎

7. The counting. The combinatorial proof of (4) will now follow from the next two propositions.

Proposition 10. *The exponential generating function for β over the oriented n-colored cycles is given by*

$$\sum_{m\ge1}\frac{1}{m!}\beta\{C'_m\} = \log\frac{1}{\det(I-B)} \tag{26}$$

Proof. As seen in section 5 each oriented n-colored m-cycle c is a circular biword $\begin{bmatrix} x_1x_2\cdots x_m \\ v_1v_2\cdots v_m \end{bmatrix}$ with $w = x_1x_2\cdots x_m$ a color word of length m and $\sigma = v_1v_2\cdots v_m$ a permutation of length m. Moreover, by (21) $\beta(c) = \beta(w)$. As σ is a permutation, the m cyclic rearrangements of $\begin{pmatrix} x_1x_2\cdots x_m \\ v_1v_2\cdots v_m \end{pmatrix}$ are all distinct. Accordingly, with X^m denoting the set of all n^m color words

$$\begin{aligned} m\,\beta\{C'_m\} &= \sum\{\beta(w) : w \in X^m,\ \sigma \in \mathfrak{S}_m\} \\ &= m!\,\beta\{X^m\}, \end{aligned}$$

and

$$\beta\{C'_m\} = (m-1)!\,\beta\{X^m\}. \tag{27}$$

On the other hand, with Tr denoting the trace of a matrix Jacobi's identity (see *e.g.* [7] p. 159, or [6]) reads

$$\log \frac{1}{\det(I-B)} = \mathrm{Tr}\,\log \frac{1}{1-B} = \mathrm{Tr} \sum_{m\geq 1} \frac{B^m}{m}. \tag{28}$$

But the (i,j)-entry in the matrix B is

$$\sum_{x_2,\ldots,x_m} b(i,x_2)b(x_2,x_3)\cdots b(x_m,j),$$

so that

$$\begin{aligned}\mathrm{Tr}\,B^m &= \sum_i \sum_{x_2,\ldots,x_m} b(i,x_2)b(x_2,x_3)\cdots b(x_m,i)\\ &= \sum\{\beta(w) : w\in X^m\} = \beta\{X^m\}.\end{aligned}$$

Thus, Jacobi's identity may be written

$$\log \frac{1}{\det(I-B)} = \sum_{m\geq 1} \frac{1}{m!}\beta\{X^m\}. \tag{29}$$

Finally, (27) and (29) yield (26). ∎

The analogous result for paths is stated next.

Proposition 11. *The exponential generating function for α over the directed n-colored paths is given by*

$$\sum_{m\geq 1} \frac{1}{m!}\alpha\{P'_m\} = -\sum_{k,\ell}((I-B)^{-1}_{k,l} - \delta_{k,\ell})z_k z_\ell. \tag{30}$$

Proof.(*) As seen in section 6 each directed n-colored m-path is a pair $\rho=(w,\sigma)$ with w in X^m and σ in $\mathfrak{S}_n$. From (24) it follows that

$$\alpha\{P'_m\} = m!\sum_{x_1,\ldots,x_{m+1}} -z_{x_1} b(x_1,x_2)b(x_2,x_3)\cdots b(x_{m-1},x_m)b(x_m,x_{m+1})z_{x_{m+1}}.$$

For $1\leq k,\ell\leq m$ let $B_{k,\ell}$ denote the (k,ℓ)-entry in the matrix B^m. Then,

$$\alpha\{P'_m\} = m!\sum_{k,\ell} -z_k\, B^m_{k,\ell}\, z_\ell.$$

Hence,

$$\begin{aligned}\sum_{m\geq 1}\frac{1}{m!}\alpha\{P'_m\} &= -\sum_{k,\ell} z_k z_\ell \sum_{m\geq 1} B^m_{k,\ell}\\ &= -\sum_{k,\ell} z_k z_\ell ((I-B)^{-1} - I)_{k\ell}\\ &= -\sum_{k,\ell} z_k z_\ell ((I-B)^{-1}_{k\ell} - \delta_{k\ell}). \quad ∎\end{aligned}$$

(*) We owe the germ of this proof to Ed Bender.

Now the exponential generating function for μ over the connected n-involutionary graphs is the sum of the two series

$$\sum_{m\geq 1}\frac{1}{m!}\mu\{D_m\} \qquad (n\text{−involutionary cycles}) \tag{31}$$

and

$$\sum_{m\geq 1}\frac{1}{m!}\mu\{R_m\} \qquad (n\text{−involutionary paths}). \tag{32}$$

When the matrix B is symmetric and every diagonal entry $b(i,i)$ $(1\leq i\leq n)$ is zero, write

$$(I-B)=\rho=(\rho_{ij})_{(1\leq i,j\leq n)}.$$

Under this hypothesis the series (31) is equal to

$$\frac{1}{2}\sum_{m\geq 1}\frac{1}{m!}\overline{\beta}\{C'_m\} \qquad \text{(by (23))}$$

and the series (32) to

$$\frac{1}{2}\sum_{m\geq 1}\frac{1}{m!}\overline{\alpha}\{P'_m\} \qquad \text{(by (25))}.$$

By using (26) and (30) we then deduce

$$\sum_{m\geq 1}\frac{1}{m!}\mu\{D_m\}=\frac{1}{2}\log\frac{1}{\det\rho}$$

and

$$\sum_{m\geq 1}\frac{1}{m!}\mu\{R_m\}=-\frac{1}{2}\sum_{k,\ell}(\rho^{-1}_{k,\ell}-\delta_{k\ell})z_k z_\ell.$$

In view of (5), (18) and (21) this completes the proof of Slepian's formula.

REFERENCES

1. R. Askey, Personai communication.

2 .L. Carlitz, Some extensions of the Mehler formula, Collectanea Math. 21 (1970), 117-130.

3. P. Cartier and D. Foata, Problèmes combinatoires de commutation et réarrangements, Lecture Notes in Math. no. 85, Springer-Verlag, Berlin, 1969.

4. D. Foata, La série génératrice exponentielle dans les problèmes d'énumération, Presses de l'Université de Montréal, Montréal, 1974.

5. ——————, A combinatorial proof of the Mehler formula, J. of Combinatorial Theory, series A 24 (1978), (to appear shortly).

6. D. M. Jackson, The combinatorial interpretation of the Jacobi identity from Lie algebras, J. of Combinatorial Theory, series A 23 (1977), 233-256.

7. W. Miller, Jr., Symmetry Groups and Their Applications, Academic Press, New York and London, 1972.

8. D. Slepian, On the symmetrized Kronecker power of a matrix and extensions of Mehler's formula for Hermite polynomials, SIAM J. Math. Anal. 3 (1972), 606-616.

Both authors at :

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, SAN DIEGO
LA JOLLA, CALIFORNIA 92093

The first author on leave 1977-78 from :

DÉPARTEMENT DE MATHÉMATIQUE
UNIVERSITÉ DE STRASBOURG
7, RUE RÉNE-DESCARTES
67084 STRASBOURG, FRANCE