Linear and Multilinear Algebra, 1979, Vol. 7, pp. 281–298
0308–1087/79/0704–0281 $04.50/0

A Matrix-analog for Viennot's Construction of the Robinson Correspondence†

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(*Received July 7, 1978*)

The notion of Viennot matrix which is here introduced provides a convenient combinatorial set-up to describe the Robinson correspondence between permutations and pairs of Young tableaux. By means of that set-up a Simon-Newcomb property is reproved in a simple manner.

1. INTRODUCTION

The *Robinson correspondence* associates with each permutation an ordered pair of Young tableaux of same shape. It was introduced by Robinson (1938) in his paper on symmetric group representation. For each partition λ of the integer n let $f(\lambda)$ denote the degree of the irreducible representation of the symmetric group $\mathfrak{S}_n$ corresponding to λ. Then, the identity

$$n! = \sum_{\lambda} f(\lambda)^2$$

is a by-product of the general theory of group representation. However, the transformation introduced by Robinson provides a combinatorial interpretation for this identity and has many interesting properties that, surprisingly enough, were only discovered recently (Schensted, 1961; Schützenberger, 1963; 1977; Knuth, 1970; Thomas, 1977; Greene, 1974). In particular,

Supported in part by N.S.F. grant MCS 77–02113.
† Paper read at the N.S.F. Conference on Finite Dimensional Linear and Multilinear Algebra, December 7–9, 1977, University of California, Santa Barbara.

as was shown by Schützenberger (1977), the transformation plays a keyrole in the combinatorial approach to the Littlewood–Richardson rule for the multiplication of Schur functions.

The present paper is self-contained and of semi-expository nature. Its purpose is to give a matrix-analog of the geometric construction found by Viennot (1977) for the Robinson correspondence. The main tool in Viennot's construction was his sequence of skeletons ("Squelettes") that characterized each permutation. The Viennot matrices that are here introduced are nothing but codings of these skeleton sequences and, as such, are already implicit in Viennot's paper. The only credit claimed by the author is to have isolated that notion and shown that it provides a convenient set-up for the construction. It is shown that each permutation matrix C can be embedded in a unique manner into a Viennot matrix D (Sections 3 and 4). Let a (resp. b) be the word of the maximum entries in each row (resp. column) of D. Then a and b are shown to be Yamanouchi words and rearrangements of each other. Furthermore, D is completely characterized by the pair (a, b). The Robinson correspondence is then the bijection that associates with C the pair (a, b). Yamanouchi words are alternate presentations of Young tableaux and do occur naturally in the present construction.

The paper is organized as follows. After establishing in Sections 2, 3 and 4 the correspondence between permutation and Viennot matrices, the basic properties on Yamanouchi words and Young tableaux are recalled in Section 5. The adjunct mapping defined in Section 6 is the transcription to Yamanouchi words of Schensted's bumping process (1961). The characterization of each Viennot matrix by its pair of maximum row and columns words (a, b) is discussed in Section 7, and the actual algorithm for the Robinson correspondence using Viennot matrices described in Section 8. Finally, it is shown in Section 9 that the use of Viennot matrices provides a simple proof of the so-called Simon–Newcomb property of the Robinson correspondence.

It is the hope of the author this paper be followed by an article describing the action of the dihedral group on the Robinson correspondence by means of the same set-up.

2. VIENNOT MATRICES

Let n be a positive integer. The group of all permutations of the set $[n] = \{1, 2, \ldots, n\}$ will be denoted by $\mathfrak{S}_n$. On several occasions each permutation σ of $[n]$ will be referred as a word $\sigma = \sigma(1)\,\sigma(2) \ldots \sigma(n)$ in the n letters $1, 2, \ldots, n$. Let $C = (c_{ij})$ be a square matrix of order n. It will be assumed that the label i of the *rows* increases *from bottom to top* (the usual labelling is from top to bottom), while the label j of the columns increases from left to right, as usual.

The permutation matrix $C=(c_{ij})$ associated with the permutation $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$ is defined by:

$$c_{ij} = 1 \quad \text{if} \quad \sigma(j) = i$$
$$= 0 \quad \text{otherwise.}$$

Let i_1, i_2, j_1, j_2 be four integers satisfying $1 \leqslant i_1 \leqslant i_2 - 1$, $1 \leqslant j_1 \leqslant j_2 - 1$. In the rectangle $[i_1, i_2] \times [j_1, j_2]$ the union of the top row and the rightmost column is called the *hook* with *ends* (i_2, j_1) and (i_1, j_2).

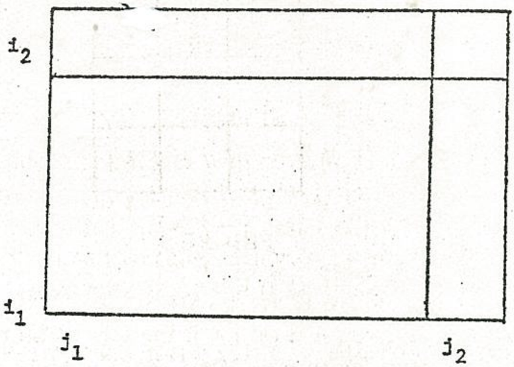

FIGURE 1

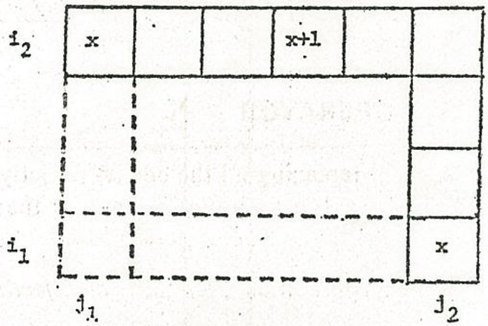

FIGURE 2

The hook consists of the points $(i_2,j_1),(i_2,j_1+1),\ldots,(i_2,j_2),(i_2-1,j_2),\ldots,(i_1+1,j_2),(i_1,j_2)$. The row $\{(i_2,j_1),(i_2,j_1+1),\ldots,(i_2,j_2)\}$ and the column $\{(i_1,j_2),(i_1+1,j_2),\ldots,(i_2,j_2)\}$ are called the *arm* and the *leg* of the hook, respectively.

A *Viennot matrix* of order n is defined to be a square matrix $D=(d_{ij})_{(1\leqslant i,j\leqslant n)}$ of order n with nonnegative integral entries having the following properties

i) The entry 1 occurs exactly once in every row and column.

ii) When read from left to right (resp. from bottom to top) the nonzero entries in every row (resp. every column) are in natural order $1, 2, 3, \ldots$.

iii) If H is a hook in $[n] \times [n]$ with ends $(i_2, j_1), (i_1, j_2)$ and if $d_{i_2 j_1} = d_{i_1 j_2} = x \geqslant 1$, then $d_{ij} = x+1$ for at least one element (i,j) of the hook H.

Notice that axiom (ii) does not imply axiom (ii), as can be verified by examining the hook with ends (2, 1) and (1, 3) in the matrix shown in Figure 3. An example of Viennot matrix is given in Figure 12.

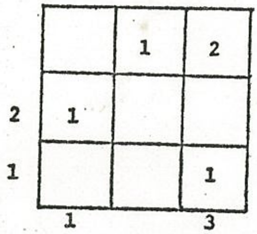

FIGURE 3

For each $n \geqslant 1$ denote by V_n (resp. S_n) the set of all Viennot (resp. permutation) matrices of order n. For $n = 0$ the sets V_n and S_n are supposed to be singletons containing the empty matrix.

The *maximum row word* (resp. *maximum column word*) of a Viennot matrix $D = (d_{ij})_{(1 \leqslant i, j \leqslant n)}$ is defined to be the word $a = a_1 a_2 \ldots a_n$ (resp. $b = b_1 b_2 \ldots b_n$) with $a_i = \max_{1 \leqslant j \leqslant n} d_{ij} (1 \leqslant i \leqslant n)$ and $b_j = \max_{1 \leqslant i \leqslant n} d_{ij} (1 \leqslant j \leqslant n)$.

3. THE ERASING OPERATOR

By definition (axiom (i)) replacing all the entries (strictly) greater than 1 in a Viennot matrix D by 0 yields a permutation matrix, that will be denoted by ERASE (D).

PROPOSITION 1 *The transformation* ERASE *is an injection of* V_n *into* S_n.

Proof Let $D = (d_{ij})_{(1 \leqslant i, j \leqslant n)}$ and $D' = (d'_{ij})_{(1 \leqslant i, j \leqslant n)}$ be two distinct Viennot matrices with ERASE (D) and ERASE (D') both equal to the same permutation matrix C. Let (k, l) be the smallest pair (with respect to the lexicographic order) such that $d_{kl} \neq d'_{kl}$. In other words, $d_{ij} = d'_{ij}$ whenever $1 \leqslant i \leqslant k-1$ and $1 \leqslant j \leqslant n$, or $i = k$ and $1 \leqslant j \leqslant l-1$, but $d_{kl} \neq d'_{kl}$. As ERASE (D) = ERASE (D'), both entries d_{kl} and d'_{kl} are different from 1. Let $d_{kl} = y$, $d'_{kl} = y'$ and without loss of generality assume that $y < y'$. Two cases are to be considered: (i) $0 = y < 2 \leqslant y'$; (ii) $2 \leqslant y < y'$.

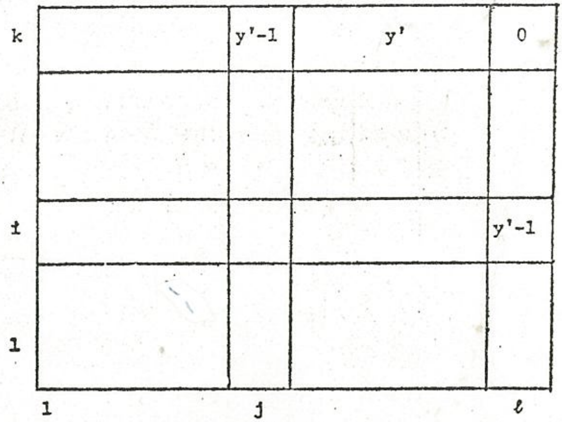

FIGURE 4

In case (i) consider the hook H with ends $(k, 1)$ and $(1, l)$. In the matrix D′ the hook H carries an entry equal to $(y'-1)$ in its arm, and its leg, in cells (say) (k,j) and (i, l) with $1 \leqslant i \leqslant k-1$ and $1 \leqslant j \leqslant l-1$. But D and D' have the same entries in these two cells. Hence, $d_{kj} = d'_{kj} = d_{il} = d'_{il} = y'-1$. Now consider the hook with ends (k,j) and (i, l). In D it carries an entry equal to y', and that entry is not in cell (k, l) since $d_{kl} = 0$. But that entry occurs in the same cell in D'. There are then two entries y' either in the same row, or in the same column in D'. But this contradicts axiom (ii).

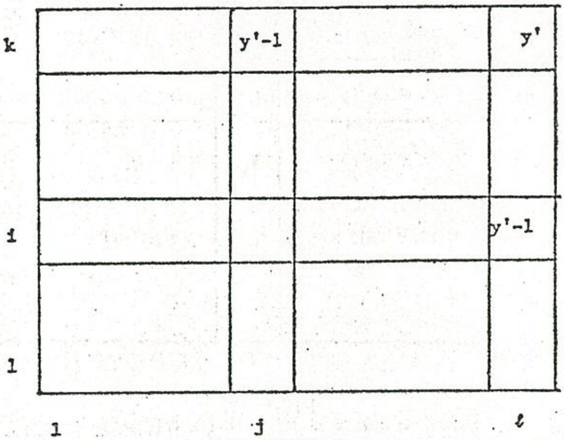

FIGURE 5

Finally, examine case (ii) $2 \leqslant y < y'$. In the matrix D' the arm and the leg of the hook with ends $(k, 1), (1, l)$ contains each an entry y. Again, by definition of (k, l) the same hook in D will contain two entries equal to y. This contradicts axiom (ii) applied to D.

Q.E.D.

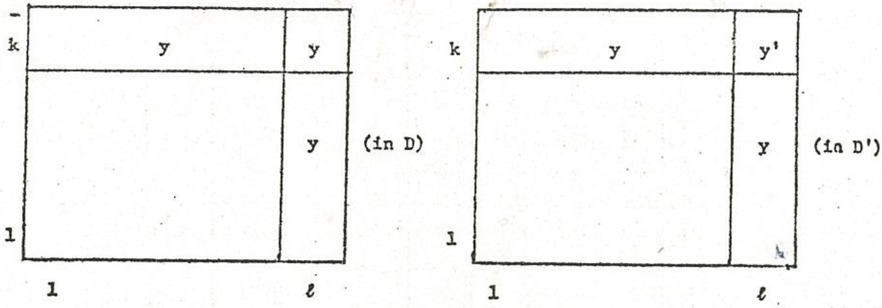

FIGURE 6

4. THE FILLING TRANSFORMATION

The erasing transformation will be proved to be *bijective* if it is shown that every permutation matrix can be embedded into a Viennot matrix. In other words, a transformation defined on S_n is to be found that preserves the entries 1, but replaces the entries zero by integers strictly greater than 1, in such a way that the matrix obtained is Viennot. This will be achieved by the filling transformation described next.

Let $C=(c_{ij})_{(1\leqslant i,\, j\leqslant n)}$ be a permutation matrix. Construct a matrix $D=(d_{ij})_{(1\leqslant i,\, j\leqslant n)}$ as follows. First, let $d_{ij}=c_{ij}$ for each entry in the bottom row or in the leftmost column. Next let $k\geqslant 2$, $l\geqslant 2$ and suppose that d_{ij} has been defined for every (i,j) less than (k,l) (with respect to the lexicographic order).

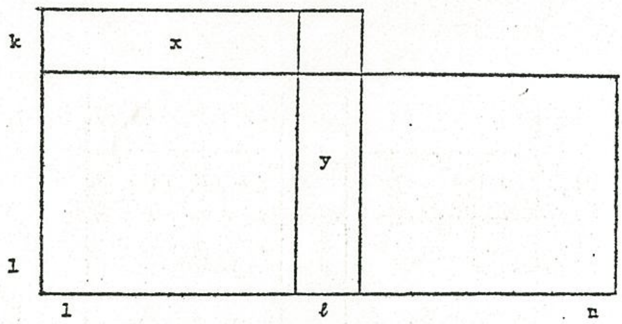

FIGURE 7

Let $x=\max_{1\leqslant j\leqslant l-1} d_{kj}$ (resp. $y=\max_{1\leqslant i\leqslant k-1} d_{il}$) be the maximum entry among the elements of D in the kth row (resp. lth column) and to the left of (resp. and under) d_{kl}. Define

$$\begin{aligned} d_{kl} &= x+1 \quad \text{if} \quad x=y\geqslant 1 \\ &= c_{kl} \qquad \text{otherwise.} \end{aligned} \tag{1}$$

PROPOSITION 2 *The matrix D is Viennot.*

Proof In the above construction the entries 1 of C are preserved. Also, in every row and column the nonzero entries occur in natural order. To verify axiom (iii) let H be a hook with ends (i_2,j_1) and (i_1,j_2) and assume that $d_{i_2 j_1} = d_{i_1 j_2} = z$.

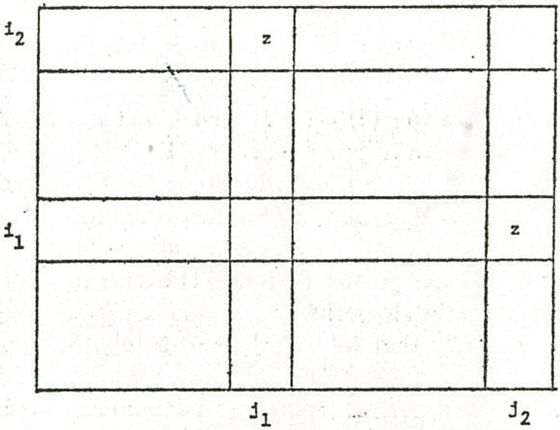

FIGURE 8

If $H\backslash(i_2,j_2)$ contains no entry equal to $(z+1)$ then

$$x = \max_{1 \leqslant j \leqslant j_2 - 1} d_{i_2 j} = z \quad \text{and} \quad y = \max_{1 \leqslant i \leqslant i_2 - 1} d_{i j_2} = z.$$

Accordingly, when the filling algorithm is applied to the cell (i_2,j_2), the entry $d_{i_2 j_2}$ is assigned to be $(z+1)$.

Q.E.D.

It follows from propositions 1 and 2 that the transformation ERASE is a *bijection of* V_n *onto* S_n. Before actually using the Viennot matrix set-up a few basic properties on Yamanouchi words are recalled in the next two sections.

5. YAMANOUCHI WORDS

Let $a = a_1 a_2 \dots a_n$ be a word of length n with n letters $a_1, a_2, \dots, a_n$ taken from $\mathbb{N}^*$. For each integer i denote by $|a|_i$ the *number of occurrences* of i in a.

DEFINITION *A word* $a = a_1 a_2 \dots a_n$ *is* Yamanouchi *if for each* $i = 1, 2, \dots, n$ *either* $a_i = 1$, *or* $a_i \geqslant 2$ *and* $|a_1 a_2 \dots a_i|_{a_i - 1} \geqslant |a_1 a_2 \dots a_i|_{a_i}$.

By convention the empty word is Yamanouchi. Note that 1 is the only one-letter Yamanouchi word. Furthermore, each left factor $a_1 a_2 \dots a_i$ $(1 \leqslant i \leqslant n)$

of a Yamanouchi word $a_1 a_2 \dots a_n$ is Yamanouchi. In particular, the first letter a_1 is always equal to 1. Also

$$|a|_{a_n} \geqslant |a|_{a_n+1} + 1. \qquad (2)$$

PROPOSITION 3 *Let $a = a_1 a_2 \dots a_n$ be a word. Then the following two statements are equivalent:*

1) *a is Yamanouchi;*

2) *$a_1 a_2 \dots a_{n-1}$ is Yamanouchi, $|a|_1 \geqslant |a|_2 \geqslant \cdots \geqslant |a|_n$, and no letter greater than n occurs in a.*

Proof The implication (2) ⇒ (1) is trivial. Let $a = a_1 a_2 \dots a_n$ be Yamanouchi. There is nothing to prove when $n = 1$.
Assume then $n \geqslant 2$. If $|a|_j \geqslant 1$ holds for some $j \geqslant n+1$, then $a_k = j$ for some k with $2 \leqslant k \leqslant n$. As $a_1 a_2 \dots a_k$ is Yamanouchi, then $|a_1 a_2 \dots a_{k-1}|_{j-1} = |a_1 a_2 \dots a_k|_{j-1} = |a_1 a_2 \dots a_k|_{a_k-1} \geqslant |a_1 a_2 \dots a_k|_{a_k} = |a_1 a_2 \dots a_k|_j \geqslant 1$. As $j-1 \geqslant n > k-1$, a letter greater than $(k-1)$ occurs in $a_1 a_2 \dots a_{k-1}$ and this contradicts the induction hypothesis.

Let us prove finally that $|a|_1 \geqslant |a|_2 \geqslant \cdots \geqslant |a|_n$. Suppose $a_n = j$. By induction $|a|_1 \geqslant \cdots \geqslant |a|_{j-1}$ and $|a|_{j+1} \geqslant \cdots \geqslant |a|_n$. Also by induction $|a_1 \dots a_{n-1}|_j \geqslant |a_1 \dots a_{n-1}|_{j+1}$. As a is Yamanouchi, then $|a|_{j-1} \geqslant |a|_j$. Finally $|a|_j = |a_1 \dots a_{n-1}|_j + 1 \geqslant |a_1 \dots a_{n-1}|_{j+1} + 1 = |a|_{j+1} + 1$. Altogether $|a|_1 \geqslant |a|_2 \geqslant \cdots \geqslant |a|_n$.

Q.E.D.

The Yamanouchi words of length n are in a one-to-one correspondence with the standard tableaux of size n. The latter objects are the pairs (A, ϕ) with A a Ferrers diagram of cardinality n (that is, an interval of $\mathbb{N} \times \mathbb{N}$ having a minimum element) and ϕ an increasing bijection of A onto $[n]$. Ferrers diagrams are usually represented as sets of squares with $\phi(i,j)$ written on the square of coordinates (i,j).

For instance,

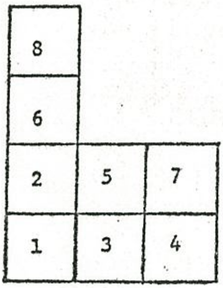

FIGURE 9

is a standard tableau of order 8.

A correspondence between standard tableaux and Yamanouchi words can be achieved by associating with each standard tableau (A, ϕ) the word $a_1 a_2 \dots a_n$ with a_i the label of the row of A that carries integer i.

For instance, to the above tableau there corresponds the Yamanouchi word

$$a = 1\,2\,1\,1\,2\,3\,2\,4.$$

6. ADJUNCT AND SUBTRACT MAPPINGS

Let $n \geqslant 2$ and $b' = b'_1 b'_2 \dots b'_{n-1}$ be a word with positive integral letters. To each integer l with $1 \leqslant l \leqslant n$ there corresponds a unique factorization

$$(w_0, w_1, 1, w_2, 2, \dots, (r-2), w_{r-1}, (r-1), w_r) \tag{3}$$

of b having the following properties

i) the length of w_0 is $(l-1)$;

ii) for each m $= 1, 2, \dots, r$ the word w_m has no occurrence of m ($|w_m|_m = 0$).
Such a factorization is called the *l-position factorization* of b.

For instance, with $b' = 1\,2\,1\,1\,2\,2\,3$ and $l = 4$, the l-position factorization of b' reads

$$\begin{array}{c} (1\,2\,1,\ e,\ 1,\ e,\ 2,\ 2,\ 3,\ e) \\ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\ w_0 \quad w_1 \quad w_2 \quad w_3 \quad w_4 \end{array}$$

(e the empty word). Also $r = 4$.

Note that the integer r is completely determined by (b', l), and b' is the juxtaposition product

$$b' = w_0 w_1 1 w_2 2 \dots (\mathrm{r}-2) w_{r-1} (r-1) w_r. \tag{4}$$

Define

$$b = w_0 1 w_1 2 w_2 3 \dots (r-1) w_{r-1} r w_r \tag{5}$$

and

$$\tau(b', l) = (b, r). \tag{6}$$

Conversely, let b be a word of length n containing the subword $1, 2, \dots, r$. Then b admits a unique factorization

$$(w_0, 1, w_1, 2, w_2, 3, \dots, (r-1), w_{r-1}, r, w_r) \tag{7}$$

with the property that

$$|w_m|_m = 0 \quad \text{for each} \quad \mathrm{m} = 1, 2, \dots, Y.$$

Such a factorization is called the *r-value factorization* of b. Also let $(l-1)$ be the length of the factor w_0, and define

$$b' = w_0 w_1 1 w_2 2 \dots (r-2) w_{r-1} (r-1) w_r$$

and

$$\tau'(b,r) = (b',l). \tag{8}$$

Clearly

$$\tau'\tau(b',l) = (b',l) \tag{9}$$

for every word b' of length $(n-1)$ and l in $[n]$. Also

$$\tau\tau'(b,r) = (b,r) \tag{10}$$

if b has length n and contains the subword $1, 2, \dots, r$.

The mappings τ and τ' are called *adjunct* and *subtract* mappings. They are now applied to Yamanouchi words. The next proposition is the transcription to Yamanouchi words of the result established by Schensted (1961) for standard tableaux.

PROPOSITION 4 *Let $1 \leqslant l \leqslant n$ and $\tau(b',l) = (b,r)$. Then, b' is Yamanouchi if and only if b is Yamanouchi and $|b|_r \geqslant |b|_{r+1} + 1$.*

Proof From the definition of τ it follows that $|b|_r = |b'|_r + 1$ and $|b|_{r+1} = |b'|_{r+1}$. Hence, if b' is Yamanouchi, the inequality $|b|_r \geqslant |b|_{r+1} + 1$ holds.

Now when 1 is juxtaposed at the end of a Yamanouchi word or when the word's last letter is deleted, it remains Yamanouchi. Thus, when $l = n$, or $r = 1$, then $b = b'1$ and the proposition holds. Let $1 \leqslant l \leqslant n-1$. For the same reason as above the left factor $w_0 1$ of b is Yamanouchi if and only if the factor w_0 of b' is Yamanouchi. Let $1 \leqslant m \leqslant r$ and x be a letter of w_m so that $w_m = uxv$ for some words u, v. Remember that w_m has no occurrence of m. Compare the two left factors

$$c = w_0 w_1 1 w_2 2 \dots w_{m-1} (m-1) u x$$

$$d = w_0 1 w_1 2 w_2 3 \dots w_{m-1} m u x$$

of b' and b, respectively. (By convention, the letter $(m-1)$ vanishes in c when r = 1.) For every y the occurrences $|c|_y$ and $|d|_y$ are the same, except for $y = m$ where $|d|_m = |c|_m + 1$.

Assume b' to be Yamanouchi. When $x \neq 1$, $m+1$, we have $|d|_{x-1} = |c|_{x-1} \geqslant |c|_x = |d|_x$. If $x = m+1$, then $|d|_{x-1} = |c|_{x-1} + 1 \geqslant |c|_x + 1 = |d|_x + 1$.

If b is Yamanouchi and $x \neq 1$, m, $m+1$, then

$$|c|_{x-1} = |d|_{x-1} \geqslant |d|_x = |c|_x.$$

When $x = m+1$, then $|c|_{x-1} = |d|_{x-1} - 1$ and $|c|_x = |d|_x$. There remains to prove $|d|_m \geqslant |d|_{m+1} + 1$ for $m = 1, 2, \ldots, r$. By assumption $|b|_r \geqslant |b|_{r+1} + 1$. If x is an occurrence of $(r+1)$ in w_r, we may write

$$w_r = uxv, \quad \text{so that} \quad b = dv.$$

But $|d|_r + |v|_r = |b|_r \geqslant |b|_{r+1} + 1 = |d|_{r+1} + |v|_{r+1} + 1$. As v is a factor of w_r, there is no occurrence of r in v. Hence

$$|d|_r \geqslant |d|_{r+1} + 1.$$

Assume now that x is an occurrence of $(m+1)$ in w_m with $1 \leqslant m \leqslant r-1$. With the above notations $w_m = uxv$ and $dv = w_0 1 w_1 2 w_2 \ldots m w_m$ the following relations hold

$$\begin{aligned} |d|_m = |dv|_m = |dv(m+1)|_m &\geqslant |dv(m+1)|_{m+1} \\ &= |dv|_{m+1} + 1 = |d|_{m+1} + |v|_{m+1} + 1. \end{aligned}$$

Thus

$$|d|_m \geqslant |d|_{m+1} + 1.$$

It remains to compare the two factors $c = w_0 w_1 1 w_2 2 \ldots w_m m$ and $d = w_0 1 w_1 2 w_2 \ldots w_m (m+1)$ of b' and b, respectively, when $1 \leqslant m \leqslant r-1$. When c is Yamanouchi, it follows from (2) that $|c|_m \geqslant |c|_{m+1} + 1$. Hence $|d|_m = |c|_m \geqslant |c|_{m+1} + 1 = |d|_{m+1}$. When d is Yamanouchi, then

$$\begin{aligned} |c|_m = |d|_m = |w_0 1 w_1 2 w_2 \ldots w_m|_m &\geqslant |w_0 1 w_1 2 w_2 \ldots w_m|_{m+1} \\ = |w_0 w_1 1 w_2 \ldots w_m m|_{m+1} &= |c|_{m+1}. \end{aligned}$$

Q.E.D.

Each ordered pair (a, b) with a and b both Yamanouchi words and rearrangements of each other is called a *Yamanouchi rearrangement pair*. The set of those pairs with length n is denoted by $Y_n^{(2)}$, while Y_n will designate the set of Yamanouchi words themselves of length n. Now associate with each (a', b', l) in $Y_{n-1}^{(2)} \times [n]$ the pair (a, b) defined by

$$(b, r) = \tau(b', l) \tag{11}$$
$$a = a'r.$$

By proposition 4 the word b is Yamanouchi. Furthermore, by definition of τ (in (6)), the words b and b' have the same occurrences, except for the letter r, that is,

$$|b|_m = |b'|_m \quad \text{if} \quad m \neq r$$

and

$$|b|_r = |b'|_r + 1.$$

As a' is a rearrangement of b', the word $a = a'r$ will be a rearrangement of b. Therefore $|a|_{r-1} = |b|_{r-1} \geqslant |b|_r = |a|_r$, and (a, b) belongs to $Y_n^{(2)}$.

Conversely, let (a, b) be in $Y_n^{(2)}$, and associate with it the triple (a', b', l) defined as follows. Let r be the last letter of a and write $a = a'r$. Then, define (b', l) by

$$(b', l) = \tau'(b, r). \tag{12}$$

As b is a rearrangement of a, the integer r occurs in b. Furthermore, as b is Yamanouchi, it contains the subword $1, 2, \ldots, r$. Thus (12) makes sense by definition of τ. Finally, as r is the last letter of a, the inequality

$$|a|_r \geqslant |a|_{r+1} + 1$$

follows from (2). Hence

$$|b|_r \geqslant |b|_{r+1} + 1,$$

and, by proposition 4, the word b' is Yamanouchi. The same reasoning as before shows that a' is a rearrangement of b'. Thus, (a', b') is in $Y_{n-1}^{(2)}$.

The next proposition then follows from (9) and (10).

PROPOSITION 5 *The mapping*

$$(a', b' l) \rightarrow (a, b)$$

defined by (11) *is a bijection of* $Y_{n-1}^{(2)} \times [n]$ *onto* $Y_n^{(2)}$. *The inverse mapping is given by* (12).

7. THE ROBINSON CORRESPONDENCE

If $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$ and $\sigma(l) = n$, let $\sigma' = \sigma(1)\ldots\sigma(l-1)\sigma(l+1)\ldots\sigma(n)$. Then $\sigma \rightarrow (\sigma', l)$ is a bijection of $\mathfrak{S}_n$ onto $\mathfrak{S}_{n-1} \times [n]$. Therefore, the sequence

$$\begin{array}{cccc} \sigma \rightarrow & (\sigma', l) \rightarrow & ((a', b'), l) \rightarrow & (a, b) \\ \mathfrak{S}_n & \mathfrak{S}_{n-1} \times [n] & Y_{n-1}^{(2)} \times [n] & Y_n^{(2)} \end{array} \tag{13}$$

defines, by induction on n, a bijection of $\mathfrak{S}_n$ onto $Y_n^{(2)}$. That bijection is in fact the Robinson correspondence, as described by Schensted (1961), for the pair (a, b) is obtained from (a', b') by using the adjunct mapping τ, which is nothing but the Yamanouchi version of Schensted's bumping algorithm. The description of the correspondence by means of Viennot matrices is stated next.

PROPOSITION 6 *Let D be a Viennot matrix of order n, let a and b be its maximum column and row words, respectively. Then,* $D \rightarrow (a, b)$ *is a bijection of* V_n *onto* $Y_n^{(2)}$. *Let* σ *be the permutation associated with the permutation matrix* $C =$ ERASE (D). *Then,* $\sigma \rightarrow (a, b)$ *is the Robinson correspondence.*

Proof For n = 1 the proposition is trivial. Let $n \geqslant 2$ and $D = (d_{ij})$ be a Viennot matrix of order n with maximum row and column words $a = a_1 a_2 \dots a_n$ and $b = b_1 b_2 \dots b_n$, respectively. Let (n, l) be the cell that carries the entry 1 in the top row of D and denote by D' the matrix obtained from D by deleting its nth row and lth column. Clearly, D' is a Viennot matrix of order $(n-1)$. Let $a' = a'_1 a'_2 \dots a'_{n-1}$ and $b' = b'_1 b'_2 \dots b'_{n-1}$ be the maximum row and column words of D'. Assume further that the maximum entry in the top row of D is r. Thus, $a_n = r$ and $a = a_1 a_2 \dots a_{n-1} a_n = a'_1 a'_2 \dots a'_{n-1} r$. Furthermore, the top row of D contains the entries $1, 2, \dots, r$ in cells, say, $(n, l_1) (= (n, l)), (n, l_2), \dots, (n, l_r)$, respectively. The other entries of the top row of D are zero. For convenience, write

$$b'' = b''_1 b''_2 \dots b''_n = b'_1 \dots b'_{l-1} 0 b'_l \dots b'_{n-1},$$

and let d_n be the top row of D. As indicated in the following figure the words b and b'' coincide except for the letters with positions $l_1, l_2, \dots, l_r$.

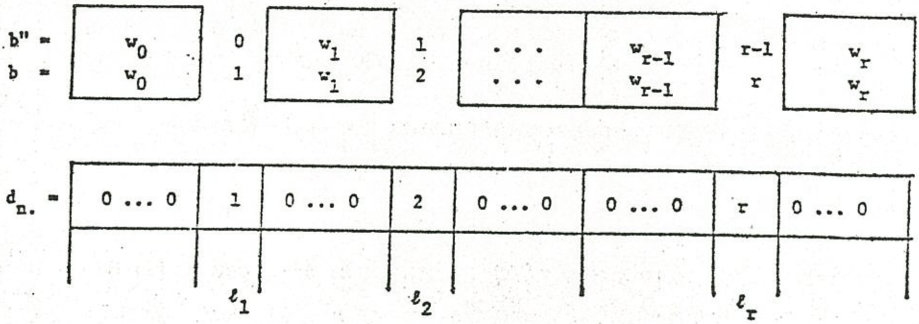

FIGURE 10

With the conventions $l_0 = 0$ and $l_r = r+1$ let

$$w_m = b_{l_m+1} b_{l_m+2} \dots b_{l_{m+1}-1} \qquad (0 \leqslant m \leqslant r).$$

Then

$$b' = w_0 w_1 1 w_2 2 \dots (r-2) w_{r-1} (r-1) w_r$$

$$b = w_0 1 w_1 2 w_2 3 \dots (r-1) w_{r-1} r w_r.$$

On the other hand, for every $m = 1, 2, \dots, r$ the word w_m has no occurrence of m. This follows immediately from the definitions of $l_1, l_2, \dots, l_r$ and the basic properties of Viennot matrices. Therefore, relations (11) and (12) hold, i.e.

$$(b, r) = \tau(b', l), \quad a = a'r,$$

$$(b', l) = \tau'(b, r).$$

Assume by induction on n that $D' \to (a', b')$ is a bijection of V_{n-1} onto $Y_{n-1}^{(2)}$ and consider the sequence.

$$D \to (D', l) \to (a', b', l) \to (a, b). \tag{15}$$

Let $C = \text{ERASE }(D)$ and $C' = \text{ERASE }(D')$. Clearly, $C \to (C', l)$ is a bijection of S_n onto $V_{n-1} \times [n]$. As ERASE is bijective, the first mapping of the above sequence is a bijection of V_n onto $V_{n-1} \times [n]$. By induction the second mapping is a one-to-one correspondence between $V_{n-1} \times [n]$ and $Y_{n-1}^{(2)} \times [n]$. Finally, the third one is a bijection onto $Y_n^{(2)}$ by proposition 5.

The last statement of the proposition follows from (13), (15) and the remark following (13).

Q.E.D.

8. THE ALGORITHM

The pair (a, b) of Yamanouchi words associated with the permutation σ under the Robinson correspondence can be quickly obtained (even for large values of n) by making use of proposition 6.

First, construct the permutation matrix $C = (c_{ij})$ defined by

$$\begin{aligned} c_{ij} &= 1 \quad \text{if} \quad \sigma(j) = i \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Second, apply to C the *filling* algorithm as described in (1) to obtain a Robinson matrix D.

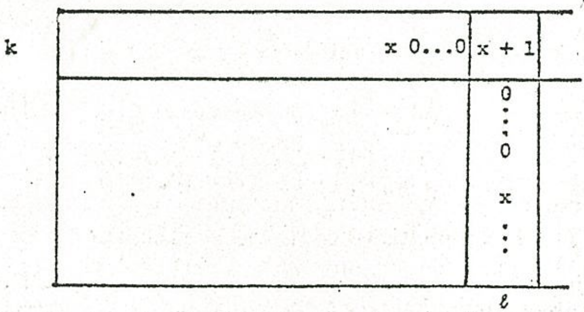

FIGURE 11

Third, set (a, b) to be the pair of the maximum row and column words of D.

The crucial part of this algorithm is part 2 (filling algorithm). The filling of C is actually a reading of the matrix from left to right and bottom to top in such a way that the entry $(x+1)$ is assigned to cell (k,l) every time the maximum among the entries already assigned that are located in the kth row and to the left of the cell (k,l), and also the maximum of the entries in the lth column and under (k,l) are both equal to x.

In the following example the starting permutation is $\sigma = 41687352$, and the filling algorithm applied to the corresponding matrix permutation C results in a Viennot matrix D whose maximum row and column words a and b read

$$a = 11231224 \qquad b = 12112324.$$

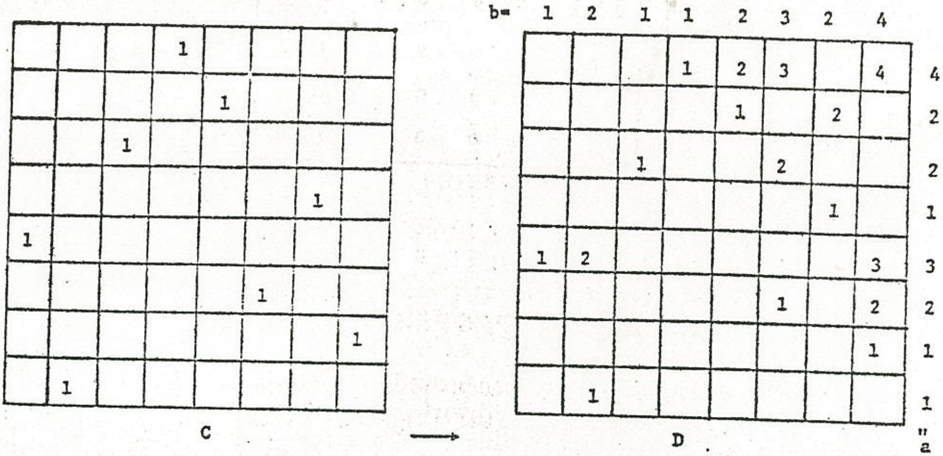

FIGURE 12

The *decoding* of (a,b), that is, the determination of C that corresponds to (a,b) is made row by row and downward using (15) and proposition 5. Practically, start with a square frame of $n \times n$ cells. Write $a = a_1 a_2 \dots a_n$ to its right side from bottom to top and $b = b_1 b_2 \dots b_n$ on its top from left to right. Repeat $b_1 b_2 \dots b_n$ in the top row of the frame and with $r = a_n$ underline the letters $1, 2, \dots, r$ of the r-value factorization of $b_1 b_2 \dots b_n$. Next write on the next to top row $b_1'' b_2'' \dots b_n''$ with

$$b_i'' = b_i \qquad \text{if } b_i \text{ is not underlined}$$

$$= b_i - 1 \quad \text{if } b_i \text{ is underlined.}$$

With $r = a_{n-1}$ underline the letters $1, 2, \dots, r$ of the r-value factorization of $b_1'' b_2'' \dots b_n'$ (disregarding the entries 0), and so on . . . until the bottom row is reached. The matrix with the entries 1 underlined is the matrix C corresponding to (a,b).

For instance, with $a = 11231224$ and $b = 12112324$, the following working matrix is obtained

$$
\begin{array}{r|cccccccc|c}
b = & 1 & 2 & 1 & 1 & 2 & 3 & 2 & 4 & \\
\hline
& 1 & 2 & 1 & \underline{1} & \underline{2} & \underline{3} & 2 & \underline{4} & 4 \\
& 1 & 2 & 1 & 0 & \underline{1} & 2 & \underline{2} & 3 & 2 \\
& 1 & 2 & \underline{1} & 0 & 0 & \underline{2} & 1 & 3 & 2 \\
& 1 & 2 & 0 & 0 & 0 & 1 & \underline{1} & 3 & 1 \\
& \underline{1} & \underline{2} & 0 & 0 & 0 & 1 & 0 & \underline{3} & 3 \\
& 0 & 1 & 0 & 0 & 0 & \underline{1} & 0 & \underline{2} & 2 \\
& 0 & 1 & 0 & 0 & 0 & 0 & 0 & \underline{1} & 1 \\
& 0 & \underline{1} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
& & & & & & & & & \uparrow \\
& & & & & & & & & a
\end{array}
$$

FIGURE 13

9. A SIMON-NEWCOMB PROPERTY

The following proposition has been proved by several authors (Schützenberger (1977), Foulkes (1976), Thomas (1977)).

PROPOSITION 7 *Let* $(a = a_1 a_2 \ldots a_r,\ b = b_1 b_2 \ldots b_r)$ *be the image of the permutation* $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$ *under the Robinson correspondence. Then, for each* $j = 1, 2, \ldots, n-1$ *the equivalences hold*

$$b_j < b_{j+1} \quad \text{iff} \quad \sigma(j) > \sigma(j+1)$$
$$b_j \geqslant b_{j+1} \quad \text{iff} \quad \sigma(j) < \sigma(j+1).$$

Proof Let $D = (d_{ij})$ be the Viennot matrix with maximum row and column words a, b. Let $b_j = x$, $b_{j+1} = y$ and $z = \min(x, y)$. For $1 \leqslant i \leqslant x$ (resp. $1 \leqslant i \leqslant y$) denote by $(\sigma_i(j), j)$ (resp. $(\sigma_i(j+1), j+1)$) the cell of the jth column (resp. $(j+1)$st) column that carries i. Assume that

$$\sigma(j) = \sigma_1(j) > \sigma_1(j+1) = \sigma(j+1),$$

and prove, by induction on i, that for every $i = 1, 2, \ldots, z$ the following inequality holds $\sigma_i(j) > \sigma_i(j+1)$.

The inequality holds for $i = 1$ by assumption. Now if $\sigma_i(j) > \sigma_i(j+1)$, the hook property applied to the above hook implies that

$$\sigma_i(j+1) < \sigma_{i+1}(j+1) \leqslant \sigma_i(j) < \sigma_{i+1}(j).$$

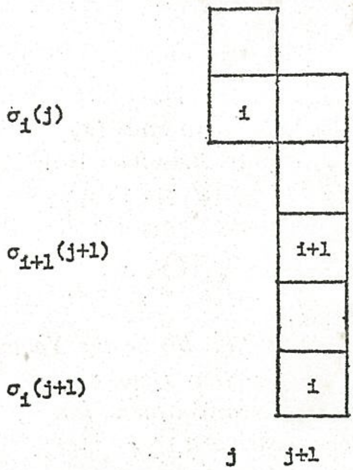

FIGURE 14

Therefore, the inequality also holds by replacing i by $(i+1)$. Hence $\sigma_z(j) < \sigma_z(j+1)$. Now if $x \geqslant y$, then $z = y$. But by considering the above hook for $i = z = y$ there would exist an entry $(y+1)$ in the $(j+1)$st column and this would contradict the definition of $b_{j+1} = z = y$. Therefore $\sigma(j) > \sigma(j+1)$ implies $b_j < b_{j+1}$.

Conversely, assume that $x = b_j < b_{j+1} = y$. The proof of the proposition will be completed if it is shown that for every $i = x, x-1, \ldots, 1$ the inequality

$$\sigma_i(j) > \sigma_i(j+1) \tag{16}$$

holds.

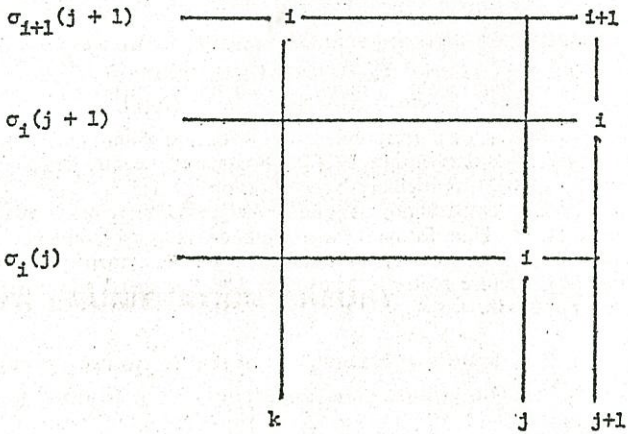

FIGURE 15

First the inequality is true for $i = x$. Otherwise, we would have

$$\sigma_x(j) < \sigma_x(j+1) < \sigma_{x+1}(j+1).$$

But then there is a cell $(\sigma_{x+1}(j+1), k)$ with $k < j$ containing x (see the above figure for $i = x$). The hook with ends $(\sigma_{x+1}(j+1), k)$ and $(\sigma_x(j), x)$ would contain $(x+1)$. This is a contradiction by definition of $b_j = x$. Thus inequality (16) holds for $i = x$. Finally, if it holds for $i+1$ with $i+1 \leqslant x$, the same reasoning applied to the above figure will show that the inequality also holds for i.

Q.E.D.

COROLLARY *Let (a, b) and (a', b') be the Yamanouchi rearrangement pairs corresponding to the permutation σ and σ', both of order n. If $b = b'$ then σ and σ' have the same up–down sequences. If $a = a'$, their inverses σ^{-1} and σ^{-1} have the same up–down sequences.*

Proof The corollary is a straightforward consequence of the previous proposition and the fact that the Yamanouchi rearrangement pair corresponding to σ^{-1} is (b, a).

Q.E.D.

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