Annals of Discrete Mathematics 6 (1980) 125–135

A COMBINATORIAL PROOF OF JACOBI'S IDENTITY*

Dominique FOATA

Département de Mathématique, Université de Strasbourg, 7, rue René-Descartes, 67084 Strasbourg, France

The Jacobi identity $\det\exp A = \exp \operatorname{Tr} A$ is proved by using combinatorial methods: partitional complex and free monoid factorizations.

1. Introduction.

Jacobi's identity

$$\det\exp A = \exp \operatorname{Tr} A \tag{1.1}$$

holds for every square matrix A with complex entries with $\operatorname{Tr} A$ standing for the trace of A. (See [8, pp. 152–156] for a classical Linear Algebra proof.) As the identity has been applied to several problems in Enumeration [4, 6], the question of doing the converse has been raised: use combinatorial techniques to derive Jacobi's identity. This is the purpose of this paper.

Let n be a fixed positive integer and $B = (b(i, j))$ $(1 \leqslant i, j \leqslant n)$ be a square matrix with n^2 commuting variables $b(i, j)$ $(1 \leqslant i, j \leqslant n)$. Using combinatorial techniques (essentially, partitional complex and free monoid factorizations) we show that the identity

$$\frac{1}{\det(I-B)} = \exp \operatorname{Tr} \log \frac{1}{I-B} \tag{1.2}$$

holds in the $\mathbb{C}$-algebra of formal power series with the n^2 variables $b(i, j)$ $(1 \leqslant i, j \leqslant n)$. From (1.2) it is straightforward to derive (1.1) with A a matrix whose entries are formal power series with constant terms equal to zero.

There are three steps in the proof of (1.2). First, the ratio $1/\det(I-B)$ is shown (see Section 2) to be the generating function for the so-called *color words* by a certain monomial-valued function β_{dec}, i.e.

$$\frac{1}{\det(I-B)} = 1 + \sum_{m \geqslant 1} \sum \{\beta_{\text{dec}}(w) : w \in X^m\}. \tag{1.3}$$

* Supported in part by N.S.F. Grant MCS 77–02113 (Department of Mathematics, University of California, San Diego, La Jolla, CA 92093, U.S.A.).

Second, $\exp \operatorname{Tr} \log(1/I-B)$ is proved (see Section 3) to be the exponential generating function for the so-called *colored permutations* by another monomial-valued function β, i.e.

$$\exp \operatorname{Tr} \log \frac{1}{I-B} = 1 + \sum_{m \geq 1} \frac{1}{m!} \sum \{\beta(p) : p \in P_m\}. \tag{1.4}$$

The third (and crucial) step consists of constructing a bijection $(w, \sigma) \to p$ of $X^m \times \mathfrak{S}_m$ (with $\mathfrak{S}_m$ the symmetric group of degree m) onto P_m with the property that

$$\beta_{\text{dec}}(w) = \beta(p). \tag{1.5}$$

This is done in Section 4. Clearly, (1.3), (1.4) and the existence of the above bijection imply (1.2).

The following notation will be used throughout. If $\alpha(y)$ is a monomial for every element y of a finite set Y, then $\alpha\{Y\}$, will stand for the polynomial

$$\alpha\{Y\} = \sum \{\alpha(y) : y \in Y\}.$$

2. Decreasing words

Let X designate the totally ordered set $[n] = \{1 < 2 < \cdots < n\}$. Its elements are called *colors*, and X itself is referred to as the *color set*. For each $m \geq 1$ any word $w = x_1 x_2 \cdots x_m$ of length m with letters taken from X will be called a *color word* and X^m will denote the set of all color words of length m. When the m letters of the word $w = x_1 x_2 \cdots x_m$ are rearranged in non-decreasing order, we obtain a word $\bar{w} = \bar{x}_1 \bar{x}_2 \cdots \bar{x}_m$ called the *non-decreasing rearrangement* of w. Define $\alpha(w)$ to be the monomial

$$\alpha(w) = b(\bar{x}_1, x_1) b(\bar{x}_2, x_2) \cdots b(\bar{x}_m, x_m)$$

in the variables $b(i, j)$. Then the identity

$$\frac{1}{\det(I-B)} = 1 + \sum_{m \geq 1} \alpha\{X^m\} \tag{2.1}$$

is essentially the MacMahon's Master theorem identity ([7, pp. 93–98]; see also the nine proofs given by Cartier [1], or [2, (Chapter 5]). To transform (2.1) into (1.3) that with the above notations may be rewritten

$$\frac{1}{\det(I-B)} = 1 + \sum_{m \geq 1} \beta_{\text{dec}}\{X^m\},$$

it remains to define β_{dec} and show that

$$\alpha\{X^m\} = \beta_{\text{dec}}\{X^m\}.$$

A color word $w = x_1 x_2 \cdots x_m$ is said to be (initially) *minorized* if its first letter is strictly less than all its other letters. The *decreasing factorization* of a word w is a sequence $(d_1, d_2, \ldots, d_s)$ of minorized words the first letters of which are in non-increasing order ($Fd_1 \geqslant Fd_2 \cdots \geqslant Fd_s$; the symbol Fd_i standing for the first letter of d_i) and $w = d_1 d_2 \cdots d_s$. Clearly, every word has one and only one decreasing factorization.

For each color word $w = x_1 x_2 \cdots x_m$ of positive length let

$$\beta(w) = b(x_1, x_2) b(x_2, x_3) \cdots b(x_{m-1}, x_m) b(x_m, x_1), \tag{2.2}$$

Furthermore, if $(d_1, d_2, \ldots, d_s)$ is the decreasing factorization of w, let

$$\beta_{\text{dec}}(w) = \beta(d_1)\beta(d_2) \cdots \beta(d_s). \tag{2.3}$$

When w is the empty word, let $\beta(w) = \beta_{\text{dec}}(w) = 1$.

For instance, the decreasing factorization of the following word w of length $m = 20$ is shown with vertical bars drawn after every minorized factor

$$w = 6 \mid 23 \mid 243 \mid 13 \mid 1 \mid 13 \mid 1 \mid 13 \mid 1 \mid 13 \mid 1 \mid 13 \mid. \tag{2.4}$$

Accordingly

$$\beta_{\text{dec}}(w) = b(6, 6) \cdot b(2, 3) b(3, 2) \cdot b(2, 4) b(4, 3) b(3, 2) \cdot b(1, 3)^5 \cdot b(3, 1)^5 \cdot b(1, 1)^4.$$

Finally, the identity

$$\alpha\{X^m\} = \beta_{\text{dec}}\{X^m\} \quad (m \geqslant 1)$$

is a consequence of the next theorem that can be found in [2, p. 51].

Theorem 2.1. *For each $m \geqslant 1$ there exists a bijection Φ of X^m onto itself with the following property: for each color word $w = x_1 x_2 \cdots x_m$ the color word $\Phi(w) = w' = x'_1 x'_2 \cdots x'_m$ is a rearrangement of w and*

$$\alpha(w') = \beta_{\text{dec}}(w).$$

This completes the proof of (1.3).

3. Colored Permutations

To establish (1.4) first expand $\operatorname{Tr} \log(1/I - B)$ to obtain

$$\operatorname{Tr} \log \frac{1}{I - B} = \operatorname{Tr} \sum_{m \geqslant 1} \frac{B^m}{m}.$$

As the (i, j)-entry in the matrix B^m is

$$\sum_{x_2, \ldots, x_m} b(i, x_2) b(x_2, x_3) \cdots b(x_m, j),$$

the trace of B^m is also equal to

$$\operatorname{Tr} B^m = \sum_i \sum_{x_2,\dots,x_m} b(i, x_2)b(x_2, x_3)\cdots b(x_m, i)$$
$$= \sum \{\beta(w) : w \in X^m\}$$

(see the definition of β in (2.2)). Thus

$$\operatorname{Tr}\log\frac{1}{I-B} = \sum_{m\geq 1} \frac{1}{m}\beta\{X^m\}. \tag{3.1}$$

The next step is to transform the right-side of (3.1) into an *exponential* generating function. This is achieved with the introduction of colored permutations and cycles. A *colored m-permutation* is nothing but a permutation graph with m vertices $1, 2, \dots, m$ that has the further property that each arc or loop is colored. This means that an element of X is assigned to each arc or loop. For instance, the graph in Fig. 1 is a colored m-permutation with $m = 20$ and $X = \{1, 2, \dots, 6\}$. In (1.4) the symbol P_m stands for the set of colored m-permutations and β enumerates the colors *by adjacency*. More precisely, let n_{ij} be the number of vertices of a colored m-permutation p that are ends of an arc colored i and beginnings of an arc colored j. Then

$$\beta(p) = \prod_{i,j} b(i, j)^{n_{ij}}. \tag{3.2}$$

For instance, with the above example

$$\beta(p) = b(6, 6)b(2, 3)b(3, 2)^2 b(2, 4)b(4, 3)b(1, 3)^5 b(3, 1)^5 b(1, 1)^4. \tag{3.3}$$

Next a *colored m-cycle* is simply defined as a *connected* colored m-permutation. The set of all colored m-cycles will be denoted by C_m. An alternate definition can be given as follows. A two-row matrix

$$\binom{w}{\sigma} = \begin{pmatrix} x_1 x_2 \cdots x_m \\ v_1 v_2 \cdots v_m \end{pmatrix} \tag{3.4}$$

with $w = x_1 x_2 \cdots x_m$ a color word of length m and $\sigma = v_1 v_2 \cdots v_m$ a permutation of $1\ 2 \cdots m$ is called a *colored m-biword*. Two colored m-biwords $\binom{w}{\sigma}$ and $\binom{w'}{\sigma'}$ are (cyclically) *equivalent* if they only differ by a cyclic rearrangement of their columns. Each equivalence class is then a *colored m-cycle*. Each colored m-cycle will be represented as a circular biword

$$c = \begin{bmatrix} x_1 x_2 \cdots x_m \\ v_1 v_2 \cdots v_m \end{bmatrix}. \tag{3.5}$$

The graphical representation of c can be obtained by drawing a graph with m vertices labeled $1, 2, \dots, m$ and joining the vertex labeled v_i (resp. v_m) with the vertex labeled v_{i+1} (resp. v_1) by an arc colored x_i (resp. x_m) for every $i = 1, 2, \dots, m-1$. Furthermore, when applied to the cycle c, the function $\beta(c)$

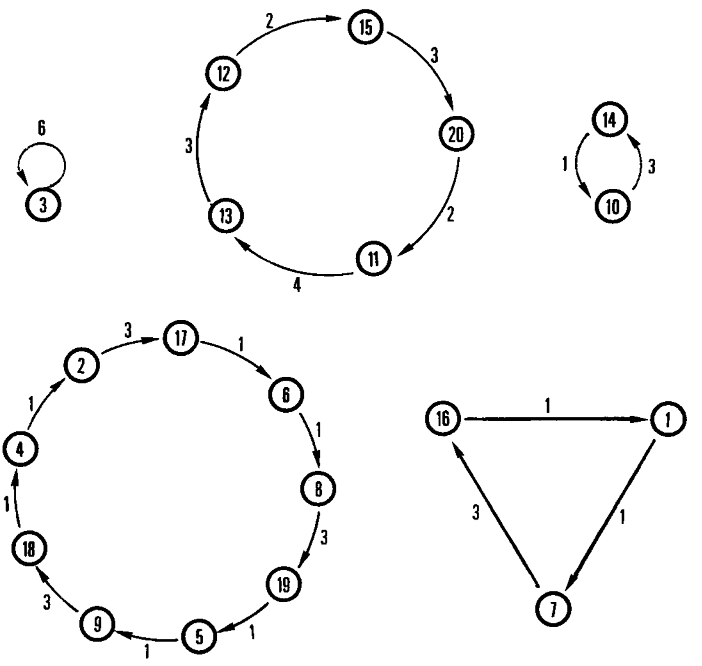

Fig. 1.

(given in (3.2)) simply reads

$$\begin{aligned}\beta(c) &= b(x_1, x_2)b(x_2, x_3)\cdots b(x_{m-1}, x_m)b(x_m, x_1) \\ &= \beta(w)\end{aligned} \tag{3.6}$$

with $w = x_1x_2\cdots x_m$.

Lemma 3.1. *For each* $m \geqslant 1$

$$\beta\{C_m\} = (m-1)!\,\beta\{X^m\}. \tag{3.7}$$

Proof. Let $\binom{w}{\sigma}$ be a colored m-biword. As σ is a permutation, the m cyclic rearrangements of $\binom{w}{\sigma}$ (including itself) are all distinct. Thus each colored m-cycle has exactly m representatives. Hence

$$\begin{aligned} m\beta\{C_m\} &= \sum \{\beta(w) : w \in X^m, \sigma \in \mathfrak{S}_m\} \quad \text{(from (3.6))} \\ &= m!\sum\{\beta(w) : w \in X^m\} \\ &= m!\,\beta\{X^m\}. \end{aligned}$$

Thus (3.1) and (3.7) imply

$$\operatorname{Tr}\log\frac{1}{I-B} = \sum_{m\geqslant 1}\frac{1}{m!}\beta\{C_m\}. \tag{3.8}$$

The proof of (1.4) will be completed, if the following identity

$$1+\sum_{m\geqslant 1}\frac{1}{m!}\beta\{P_m\}=\exp\sum_{m\geqslant 1}\frac{1}{m!}\beta\{C_m\} \tag{3.9}$$

can be established. But this can be achieved by using partitional complex techniques as follows. Let $I=\{i_1<i_2<\cdots<i_m\}$ be a finite subset of positive integers. When in (3.4) (resp. in (3.5)) $v_1v_2\cdots v_m$ stands for a permutation of $i_1i_2\cdots i_m$, the two-row matrix $\binom{w}{\sigma}$ (resp. $\left[{w \atop \sigma}\right]$) will be called a *colored I-biword* (resp. a *colored I-cycle*).

Let $c'=\left[{w \atop \sigma}\right]$ be a colored I-cycle and let φ be the unique increasing bijection of I onto $[m]$. Then, c' may be expressed as a pair (c, I) with c the colored m-cycle obtained from c' by replacing each label vertex v by $\varphi(v)$. Let p be a colored m-permutation. It is an unordered collection of its connected components $c'_1, c'_2, \ldots, c'_r$, namely

$$\{(c_1, I_1), (c_2, I_2), \ldots, (c_r, I_r)\} \tag{3.10}$$

with $\{I_1, I_2, \ldots, I_m\}$ a partition of $[m]$ and for $1\leqslant i\leqslant r$ the component c_i a colored $|I_i|$-cycle. Moreover, from the very definition of β ((3.2) and (3.6)) the following *multiplicative* property holds

$$\beta(p)=\beta(c_1)\beta(c_2)\cdots\beta(c_r). \tag{3.11}$$

The exponential formula (3.9) is a consequence of (3.10) and (3.11). More explicitly, (3.10) says that P_m is the so-called *partition complex* of degree m of the set $\bigcup_{m\geqslant 1} C_m$ (see e.g. [5, p. 51]). For each $p=\{(c_1, I_1), (c_2, I_2), \ldots, (c_r, I_r)\}$ let $\psi(p)=c_1c_2\cdots c_r$ be regarded as a commutative monomial in the variables c_i's. The abstract exponential formula (see e.g. [5, p. 51]) reads

$$1+\sum_{m\geqslant 1}\frac{1}{m!}\sum\{\psi(p):p\in P_m\}=\exp\sum_{m\geqslant 1}\frac{1}{m!}\sum\{c:c\in C_m\}. \tag{3.12}$$

As β is multiplicative (property (3.11)) and the degree of the monomial $\beta(p)$ is m whenever p belongs to P_m, the map β can be extended to a continuous homomorphism. Therefore, the image under β of (3.12) yields (3.9).

This completes the proof of (1.4).

4. The standard factorization

Now comes the crucial step: combine identities (1.3) and (1.4), that is, prove the combinatorial theorem described as step (iii) in the introduction.

First, a few basic notions are recalled. A color word is *primitive* if it is not equal to the power w_1^k of another word w_1 with $k\geqslant 2$. It is *standard* if it is less (with respect to the lexicographic order) than each of its cyclic rearrangements.

For instance, $w=23243$ is less than each of its cyclic rearrangements 32432, 24323, 43232 and 32324. It is then standard.

A standard word is always primitive. Each word w is the power of a unique primitive word denoted by $\sqrt{w}$ and referred to as the *root* of w. As before, Fw will designate the first letter of w.

The notion of standard word was introduced by Chen et al. [3] in the study of free differential calculus. Then, Schützenberger [9] developed a theory of free monoid factorizations, an example of which is the standard factorization described in Proposition 4.1 below. Finally, Viennot [10] made a systematic study of these factorizations and built up a unified theory of the basic commutator calculus. The proof of the next proposition can be found in Viennot ([10, p. 40, Lemma 1.11]).

Proposition 4.1. *Each non-empty color word w admits a unique factorization $(u_1, u_2, \ldots, u_t)$, called its lexicographic standard factorization, having the following three properties*

(i) $u_1, u_2, \ldots, u_t$ *are standard*;
(ii) $u_1 \geq u_2 \geq \cdots \geq u_t$ (*with respect to the lexicographic order*);
(iii) $w = u_1 u_2 \cdots u_t$.

For instance, the standard factorization of 623243131131131131113 reads

$$(6, 23243, 13, 113, 113, 113, 113). \tag{4.1}$$

The notion of standard factorization is now extended to biwords. A colored I-biword

$$\binom{w}{\sigma} = \binom{x_1 x_2 \cdots x_m}{v_1 v_2 \cdots v_m}$$

is *bistandard*, if

(i) $w = u^k$ $(k \geq 1)$ and u is standard:
(ii) if $k \geq 2$ and $m = kl$, then

$$v_1 < \min\{v_{l+1}, v_{2l+1}, \ldots, v_{(k-1)l+1}\}.$$

For instance, the biword

$$\binom{w}{\sigma} = \begin{pmatrix} 1 & 1 & 3 & 1 & 1 & 3 & 1 & 1 & 3 \\ 16 & 6 & 8 & 19 & 5 & 9 & 17 & 4 & 2 \end{pmatrix}$$

is bistandard. With the above notations $u = 113$ is standard. Further $m = 9$, $k = 3$, $l = 3$ and $v_1 = 16 < \min\{v_4 = 19, v_7 = 17\}$.

Now a sequence

$$\binom{w_1}{\sigma_1}, \binom{w_2}{\sigma_2}, \ldots, \binom{w_q}{\sigma_q}$$

of colored biwords is said to be a *bistandard factorization* of a colored m-biword

$\binom{w}{\sigma}$ if the following properties hold

(i) each biword $\binom{w_i}{\sigma_i}$ is bistandard;

(ii) for $1 \leqslant i \leqslant q-1$ either $\sqrt{w_i} > \sqrt{w_{i+1}}$, or $\sqrt{w_i} = \sqrt{w_{i+1}}$ and $F\sigma_i > F\sigma_{i+1}$;

(iii) $\binom{w}{\sigma} = \binom{w_1}{\sigma_1}\binom{w_2}{\sigma_2} \cdots \binom{w_q}{\sigma_q}$.

In the next example it can be verified that the factorization of $\binom{w}{\sigma}$ determined by the vertical bars is a bistandard factorization.

$$\binom{w}{\sigma} = \left(\begin{array}{c|ccccc|cc|ccccccccc|ccc} 6 & 1 & 3 & 2 & 4 & 3 & 1 & 3 & 1 & 1 & 3 & 1 & 1 & 3 & 1 & 1 & 3 & 1 & 1 & 3 \\ 3 & 12 & 15 & 20 & 11 & 13 & 14 & 10 & 17 & 6 & 8 & 19 & 5 & 9 & 18 & 4 & 2 & 10 & 1 & 7 \end{array}\right). \tag{4.2}$$

The procedure described next will yield a bistandard factorization of each colored biword.

Procedure. (i) let $\binom{w}{\sigma}$ be a colored m-biword and $(u_1, u_2, \ldots, u_t)$ be the standard factorization of w;

(ii) cut the biword $\binom{w}{\sigma}$ at the end of each factor $u_1, u_2, \ldots, u_t$; this determines a factorization $(\tau_1, \tau_2, \ldots, \tau_t)$ of the second row σ of the biword, namely

$$\begin{array}{c|c|c|c|} u_1 & u_2 & \cdots & u_t \\ \tau_1 & \tau_2 & \cdots & \tau_t \end{array}. \tag{6.3}$$

(iii) For each $j = 1, 2, \ldots, t-1$ remove the jth vertical bar (standing between u_j and u_{j+1}) whenever the following two conditions hold:

(a) $u_{i-1} > u_i = u_{i+1} = \cdots = u_{j-1} = u_j$ for some i with $1 \leqslant i \leqslant j$ (the inequality $u_{i-1} > u_i$ being discarded when $i = 1$);

(b) $F\tau_{j+1}$ is *not* the minimum of the set $\{F\tau_i, F\tau_{i+1}, \ldots, F\tau_j, F\tau_{j+1}\}$.

(iv) The remaining vertical bars determine a bistandard factorization of $\binom{w}{\sigma}$.

Take again example (4.2) and disregard the vertical bars. By (4.1) its (4.3)-factorization reads

u_1	u_2	u_3	u_4	u_5	u_6	u_7
6	2 3 2 4 3	1 3	1 1 3	1 1 3	1 1 3	1 1 3
3	12 15 20 11 13	14 10	17 6 8	19 5 9	18 4 2	16 1 7
τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7

Then apply step (iii) of the above procedure. The first bar remains for $u_1 = 6 > 23243 = u_2$. Also the second and the third ones. Next $u_3 > u_4 = u_5$ and $17 = F\tau_4 < F\tau_5 = 19$. Thus the 4th bar is removed. Also, as $u_3 > u_4 = u_5 = u_6$ and $17 = F\tau_4 < F\tau_6 = 18$, the 5th bar is removed. But as $16 = F\tau_7$ is minimum among $F\tau_4$, $F\tau_5$, $F\tau_6$, $F\tau_7$, the 6th bar is not removed. This yields the factorization shown in (4.2).

Proposition 4.2. *Each colored m-biword* $\binom{w}{\sigma}$ *has a unique bistandard factorization, that can be obtained by applying to* $\binom{w}{\sigma}$ *the above procedure.*

Proof. Let $(u_1, u_2, \ldots, u_t)$ be the standard factorization of w. Each bistandard factorization

$$\binom{w_1}{\sigma_1}, \binom{w_2}{\sigma_2}, \ldots, \binom{w_q}{\sigma_q}$$

of $\binom{w}{\sigma}$ is such that $w_1 = u_1'^{k_1}$, $w_2 = u_2'^{k_2}, \ldots, w_q = u_q'^{k_q}$ with $u_1', u_2', \ldots, u_q'$ standard and $u_1' \geqslant u_2' \geqslant \cdots \geqslant u_q'$. As the standard factorization of w is unique, the sequence $(u_1', \ldots, u_1', u_2', \ldots, u_2', \ldots, u_q', \ldots, u_q')$ where each word u_i' is repeated k_i times, must be equal to $(u_1, u_2, \ldots, u_t)$. thus each w_i is a product of *equal* successive factors u_j.

Suppose now that $\binom{w}{\sigma}$ has two different bistandard factorizations

$$\binom{w_1}{\sigma_1}, \binom{w_2}{\sigma_2}, \ldots, \binom{w_q}{\sigma_q} \quad \text{and} \quad \binom{w_q'}{\sigma_q'}, \binom{w_2'}{\sigma_q'}, \ldots, \binom{w_{q'}'}{\sigma_{\sigma'}}.$$

Let i be the smallest integer with $\binom{w_i}{\sigma_i} \neq \binom{w_i'}{\sigma_i'}$. Assume that w_i is shorter than w_i'. Then $w_i = u^{p_i}$ and $w_i' = u^{p_i'}$ with u standard and $p_i < p_i'$. Also $w_{i+1} = u^{p_{i+1}}$ for some $p_{i+1} \geqslant 1$ and the same u because $p_i < p_i'$. Therefore, $F\sigma_i > F\sigma_{i+1}$. Also $F\sigma_i = F\sigma_i' < F\sigma_{i+1}$, because $\binom{w_i'}{\sigma_i'}$ is bistandard. This is a contradiction. Thus there exists at most one bistandard factorization.

Finally, it is straightforward to verify that the above procedure does yield a bistandard factorization.

Corollary 4.3. *Each colored I-cycle contains one and only one bistandard word.*

Proof. Let

$$\binom{w}{\sigma} = \binom{x_1 x_2 \cdots x_m}{v_1 v_2 \cdots v_m}$$

be a colored I-biword and u' be the root of w. Then there exist u' primitive and positive integers, k, l with $w = u'^k$ and $m = kl$. The rearrangement class of u' contains, by definition, a unique standard word u. Thus $\binom{w}{\sigma}$ is equal to the juxtaposition product

$$\binom{u_0'}{\sigma_0'}\binom{u_1}{\sigma_1}\binom{u_2}{\sigma_2}\cdots\binom{u_{k-1}}{\sigma_{k-1}}\binom{u_k'}{\sigma_k'}$$

where $u_1 = u_2 = \cdots = u_{k-1} = u_k' u_0' = u$. Let h be the integer defined by

$$F\sigma_h = \min\{F\sigma_1, F\sigma_2, \ldots, F\sigma_{k-1}, F\sigma_k'\}. \tag{4.4}$$

Then the product

$$\binom{u_h}{\sigma_h}\cdots\binom{u_{k-1}}{\sigma_{k-1}}\binom{u_k' u_0'}{\sigma_k' \sigma_0'}\binom{u_1}{\sigma_1}\cdots\binom{u_{h-1}}{\sigma_{h-1}}$$

is bistandard. Because of property (4.4) it is the only one in the cyclic rearrangement class.

Let $\binom{w}{\sigma}$ be a bistandard biword. Then $\binom{w}{\sigma} \to \left[{w \atop \sigma}\right]$ is a bijection onto the set of colored cycles. The inverse bijection will be denoted by

$$c \to \mathrm{BIS}(c).$$

The main theorem of this paper can now be stated.

Theorem 4.4. *Let $\binom{w}{\sigma}$ be a colored m-biword with bistandard factorization*

$$\binom{w_1}{\sigma_1}, \binom{w_2}{\sigma_2}, \ldots, \binom{w_q}{\sigma_q}.$$

Then, the mapping that sends $\binom{w}{\sigma}$ to the unordered collection of colored cycles

$$p = \left\{ \begin{bmatrix} w_1 \\ \sigma_1 \end{bmatrix}, \begin{bmatrix} w_2 \\ \sigma_2 \end{bmatrix}, \ldots, \begin{bmatrix} w_q \\ \sigma_q \end{bmatrix} \right\}.$$

is a bijection of $X^m \times \mathfrak{S}_m$ onto P_m with the property that

$$\beta_{\mathrm{dec}}(w) = \beta(p).$$

Proof. The bijectivity property follows from Proposition 4.2 and its corollary. Only the last property is to be verified. The function β_{dec} was defined in (2.3). Let $(d_1, d_2, \ldots, d_s)$ be the decreasing factorization of w. then

$$\beta_{\mathrm{dec}}(w) = \beta(d_1)\beta(d_2) \cdots \beta(d_s).$$

Also, by (2.2) and (3.11)

$$\beta(p) = \beta(w_1)\beta(w_2) \cdots \beta(w_q).$$

Accordingly, it suffices to verify that each factor w_i of the bistandard factorization is a product $d_j d_{j+1} \cdots d_{j+r}$ $(1 \leqslant j \leqslant j + r \leqslant s)$ and

$$\beta(w_i) = \beta(d_j)\beta(d_{j+1}) \cdots \beta(d_{j+r}).$$

But each w_i is a power of a standard word u, say, $w_i = u^k$, and $\beta(w_i) = \beta(u)^k$. Let $(u_1, u_2, \ldots, u_t)$ be the standard factorization of w. It remains to verify that

$$\beta(d_1)\beta(d_2) \cdots \beta(d_s) = \beta(u_1)\beta(u_2) \cdots \beta(u_t).$$

As $u_1 \geqslant u_2 \geqslant \cdots \geqslant u_t$, then $Fu_1 \geqslant Fu_2 \geqslant \cdots \geqslant Fu_t$. Hence, each u_i is a product of factors of the decreasing factorization. Write

$$u_i = d_j d_{j+1} \cdots d_{j+r}.$$

Then $Fu_i = Fd_j \geqslant Fd_{j+1} \geqslant \cdots \geqslant Fd_{j+r}$.

As u_i is standard, none of the letters $Fd_{j+1}, \ldots, Fd_{j+r}$ can be smaller than Fu_i.

Hence, $Fd_j = Fd_{j+1} = \cdots = Fd_{j+r}$, and, accordingly,

$$\beta(u_i) = \beta(d_j d_{j+1} \cdots d_{j+r}) = \beta(d_j)\beta(d_{j+1}) \cdots \beta(d_{j+r}).$$

Remark 4.5. The inverse bijection $p \to \binom{w}{\sigma}$ is obtained

(i) by decomposing the colored m-permutation p into its colored cycles

$$\{c_1, c_2, \ldots, c_r\};$$

(ii) taking for each c_i the unique bistandard biword BIS $c_i = \binom{w_i}{\sigma_i}$ contained in c_i (Corollary 4.3);

(iii) then $\binom{w}{\sigma}$ is the juxtaposition product of the $\binom{w_i}{\sigma_i}$'s written in *decreasing order* ($\binom{w_i}{\sigma_i} > \binom{w_j}{\sigma_j}$ if and only if either $w_i > w_j$, or $w_i = w_j$ and $F\sigma_i > F\sigma_j$).

For instance the colored m-biword (4.2) corresponds to the colored m-permutation of Fig. 1. Furthermore, the monomial $\beta(p)$ that counted the color adjacencies in p (see (3.3)) is precisely equal to $\beta_{\text{dec}}(w)$ determined in (2.4).

Acknowledgements

The author is grateful to Professor Garsia for the enthusiasm he showed during the preparation of [4], the paper that originated the present one, and especially for the content of Lemma 3.1. The author is also thankful to Professor Srivastava for his masterly editorial work.

References

[1] P. Cartier, La série génératrice exponentielle, Applications probabilistes et algébriques, Publ. I.R.M.A. Univ. Strasbourg (1971–1972) 68 pp.

[2] P. Cartier and D. Foata, Problèmes combinatoires de commutation et réarrangements, Lecture Notes in Math., 85 (Springer-Verlag, Berlin, Heidelberg, New York, 1969).

[3] K.T. Chen, R.H. Fox and R.C. Lyndon, Free differential calculus, IV. The quotient groups of the lower central series, Ann. of Math. 68 (1958) 81–95.

[4] D. Foata and A.M. Garsia, A combinatorial approach to the Mehler formulas for Hermite polynomials, in: Proceedings of Symposia in Pure Mathematics, Vol. 34 (Relations between Combinatorics and Other Parts of Mathematics) (Am. Math. Soc., Providence, RI, 1979) 163–179.

[5] D. Foata and M.P. Schützenberger, Théorie géométrique des polynômes eulériens, Lecture Notes in Math. 138 (Springer-Verlag, Berlin, Heidelberg, New York, 1970).

[6] D.M. Jackson, The combinatorial interpretation of the Jacobi identity from Lie algebras, J. Combin. Theory 23 (A) (1977) 233–256.

[7] P.A. MacMahon, Combinatory Analysis, Vol. 1 (Cambridge Univ. Press, Cambridge, 1915) (reprinted: Chelsea Publ. Co., New York, 1960).

[8] W. Miller Jr., Symmetry Groups and Their Applications (Academic Press, New York, London, 1972).

[9] M.P. Schützenberger, On a factorization of free monoids, Proc. Am. Math. Soc. 16 (1965) 21–24.

[10] G. Viennot, Algébres de Lie libres et monoïdes libres, Lecture Notes in Math., 691 (Springer-Verlag, Berlin, Heidelberg, New York, 1978).