Bull. London. Math. Soc. **11** (1979), p. 208-214

HERBERT OWEN FOULKES

D. FOATA, A. O. MORRIS and H. PERFECT

(a) Personal history and career (H.P.)

Herbert Owen Foulkes, the youngest of four children of John and Margaret Foulkes, was born on 18 April 1907 in the small country town of Mold in Flintshire, North Wales, and received his secondary education at the Alun Grammar School there. In those early years he had already distinguished himself academically by reaching the sixth form at the age of 13. From the Grammar School, H. O. Foulkes proceeded, on an Eyton Williams Entrance Scholarship, to the University College of North Wales, Bangor, in 1924. He was the top scholar of his year. As an underegraduate he remained outstanding and was awarded the R. A. Jones Mathematical Prize in 1926. The University of Wales was sufficiently enlightened to award degrees in the separate disciplines of Pure Mathematics and Applied Mathematics as early as about 1910, and H. O. Foulkes graduated in Pure Mathematics with first class honours in 1927. He was offered a research scholarship, tenable at Jesus College, Oxford, but the difficulty of finding employment for graduates was present then as today and this, together with family duties, persuaded him instead to take up a teaching appointment at the County School, Port Talbot; and the next 19 years from 1928 to 1947 were spent on the mathematics staff there. Although his teaching commitments in both mathematics and physics were very considerable, and H. O. Foulkes was a very gifted as well as a patient and understanding teacher, he was from the start devoted mathematical research. He proceeded to work for his M.Sc. degree, under the direction of the late Professor W. E. H. Berwick of Bangor, on the subject of the Galois theory of equations. The degree was duly awarded in 1931, with high commendations from the examiners, and two papers were published in its wake. Encouraged by the late Professor A. R. Richardson of the University College of Swansea, H. O. Foulkes continued with his mathematical researches, veering to some extent from the theory of equations to matrix algebra, and some years later his next few papers were published in this field. His interest in group representations was also beginning to develop and this was reinforced by his introduction to D. E. Littlewood, who was then in Swansea and was later to become Professor in Bangor.

While still teaching in Port Talbot, H. O. Foulkes had assisted for a time in the post-war training of teachers at the University College of Swansea.

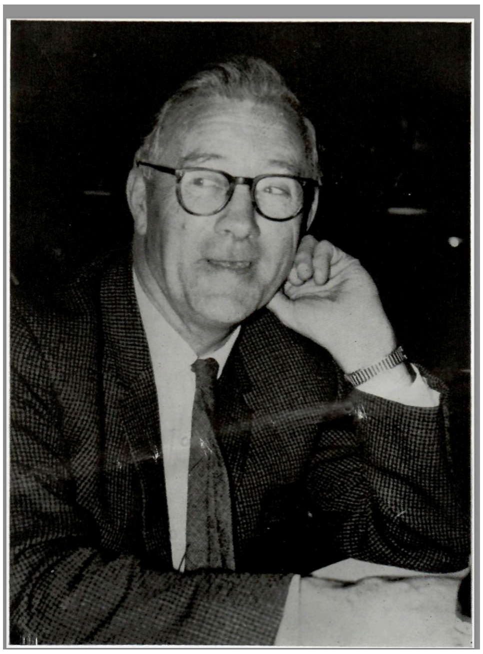
It was in 1947, at the age of 40 and upon his appointment as Assistant Lecturer in Mathematics, that he entered the University sphere as a full-time teacher and research worker. A year later he became a Lecturer and three years later a Senior Lecturer. Although his University career started late, it was remarkable for the extent of his achievement. On the one hand, he was a dedicated and very successful teacher both of undergraduates and postgraduates and gave a great deal of time to this, for he willingly and generously devoted himself to his students; four of his postgraduate students received Ph.D. degrees and six of them M.Sc. degrees. On the other hand, his own research work, now on the subject of Schur functions, was steadily progressing. This was marked in 1950 by the award of the degree of D.Sc. by the University of Wales. Professor Weston has written to me in the following terms of his teaching and research : "There is no doubt that he was a very fine teacher, at all levels. His lectures were meticulously prepared, and were delivered with clarity and economy but without austerity; and in his personal work with students he was patient and painstaking, and very approachable. Although he gave so much time and energy to teaching, he was steadily active in research (and the supervision of research) with quite impressive results." In 1961 H. O. Foulkes became a Reader in Pure Mathematics and in 1970 was appointed to a Personal Chair in Pure Mathematics. Apart from the academic year 1955-56, which was spent in Cambridge where he held a Leverhulme Research Fellowship, Professor Foulkes's University work was all based in Swansea. His contributions to Algebra and Combinatorics are discussed in some detail below. Here we mention in passing his continuing and deepening interest in Schur functions and their applicability in Enumerative Combinatorics; also the great debt owed to him by the mathematical community for his elucidation and publicising of the much-neglected work of J. H. Redfield. Particularly in the years around the time of his retirement, Professor Foulkes was much in demand as a lecturer in this country and abroad; he visited the Universities of Calgary, René-Descartes in Paris, and Strasbourg and, in 1973, participated in the Combinatorics Colloquium in Oberwolfach. Professor Foulkes retired officially in 1974 and was given the title of Emeritus Professor. However, he remained firmly attached to the Pure Mathematics Department in Swansea by becoming an Honorary Research Fellow. For some years he had been in poor health and his illness became serious at about this time. Despite considerable physical handicaps, involving the loss of the sight of one eye, he continued to pursue his research work and his last years of research were a particularly fruitful period for him. He died in July 1977.

The local mathematical society, not less than the University College, have cause for gratitude for Professor Foulkes's work. With his eldest brother the late T. G. Foulkes of Swansea Grammar School he was a leading member and one-time president of the South-West Wales Branch of the Mathematical Association. Professor Foulkes and his future wife Beryl were fellow mathematics students in Bangor. They married in 1933 and thus began a peculiarly close and understanding partnership lasting well over 40 years. Those of us who knew them well and visited their home were aware of the deep indebtedness which Professor Foulkes felt to his wife for her support and encouragement, and the intimate and happy home background which she provided, in which Professor Foulkes was able to pursue his mathematical research literally until the day of his death. Both were keen gardeners and Professor Foulkes, who came of a musical family, was no mean performer on the flute and obviously greatly enjoyed this recreation. The University College of Swansea has inaugurated a Prize Fund to the memory of Professor Foulkes, and a special volume of "Discrete Mathematics" in tribute to him will shortly be published.

Acknowledgments

Mrs. Foulkes has been kind enough to share personal memories with me on many occasions during the last few months while I have been writing these notes. I am also indebted to Professor Jeffrey Weston of the University College of Swansea and to Emeritus Professor R. Wilson, late of Swansea, for their reminiscences of Professor Foulkes; also to Dr. Carl Linden who, in particular, provided me with the list of Professor Foulkes's publications.

The accompanying photograph was taken by D. Foata at Oberwolfach.



(b) Mathematical work (A.O.M. and D.F.)

Foulkes's first research publication [2], which was based on his M.Sc. thesis [1] written under Professor W. E. H. Berwick at Bangor, involved classical Galois theory of equations. Immediately on graduation he was appointed to a teaching post at Port Talbot near Swansea. Early in this period he came into contact with A. R. Richardson who was then Professor of Mathematics at the University College of Swansea and also D. E. Littlewood who had recently joined the lecturing staff there. Both of them influenced him greatly in the choice of research problems both then and in his later research work. Apart from the expository paper [3], his next paper [4] was submitted for publication in early 1939, nine years after his earlier paper. In it, he obtains more explicit results about matrices which satisfy a given algebraic equation through using the representations of the Galois group; he restricts his attention to equations of degree ≤ 4. By considering some elementary results on associative algebras he shows [5] how matrices X may be obtained such that $YA = A^t X$, where A is a $n \times n$ matrix, and gives explicit solutions for the cases $n = 2, 3$. The more general problem of obtaining rational solutions of the matrix equation $XA = BX$, where A and B are $n \times n$ matrices, is considered in [6]. In [7] a complete set of irreducible monomial representations of all groups of orders pq, pqr, p^3 and $p^2 q$, where p, q and r are distinct primes, is determined explicitly in terms of the generating relations of these groups.

All these papers ([4], [5], [6], [7]) involve quite elementary, but sometimes involved matrix manipulations. These earlier papers bear little relationship to his later work, but through them we see how his attitude towards mathematics had developed and become established. Throughout his research life, he was predominantly concerned with obtaining explicit solutions to concrete and specific problems which required powerful and elaborate apparatus of algebraic identities rather than abstract ideas. He could in later years best be described as a combinationalist who specialized in the application of group theory, especially the representation theory of symmetric groups, to combinatorial problems. He had a remarkable skill in handling and improving these methods, a mastery of technical details and a will to persevere with highly complex and lengthy calculations.

In the post-war period, after his appointment in April 1947 to the lecturing staff at the University College of Swansea at the age of 40, his research interests changed considerably. Since the early 1930's, D. E. Littlewood, initially in collaboration with A. R. Richardson, had developed a close interest in a class of symmetric functions which had been introduced by I. Schur 1n his dissertation at the turn of the century. Littlewood called these functions Schur functions, or S-functions. Most of Foulkes's subsequent research was involved, directly or indirectly, with S-functions and their applications, no doubt inspired by Littlewood.

Schur functions may be defined in various ways; for example, if $x_1, x_2, \ldots$ form a countably infinite set of indeterminates and $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a partition of a positive integer n (written $\lambda \vdash n$), then the S-function $\{\lambda\}$ is defined by

$$\{\lambda\} = \frac{1}{n!} \sum_{\rho \vdash n} h_\rho \chi_\rho^\lambda s_1^{\rho_1} s_2^{\rho_2} \cdots, \tag{1}$$

where h_ρ is the order of the conjugacy class of the symmetric group S_n, corresponding to the partition $\rho = (1^{\rho_1} 2^{\rho_2} \ldots)$ of n, χ_ρ^λ is the value of the irreducible character of S_n, corresponding to λ at the class ρ, and s_i $(i = 1, 2, \ldots)$ is the i-th power sum of x_1, $x_2, \ldots$.

Many of Foulkes's papers in this area were concerned with a "new" multiplication of S-functions introduced by Littlewood in 1936 called by him the plethysm of S-functions, when he applied group characters and S-functions to classical invariant theory, an application which again was originally due to Schur. According to Schur's definition of an invariant matrix, if A is any matrix, then corresponding to each partition $\lambda \vdash n$ there is an irreducible invariant matrix $A^{\{\lambda\}}$ and, furthermore, an irreducible invariant matrix of an irreducible invariant matrix is equivalent to a direct sum of irreducible invariant matrices. Thus, if λ, μ, ν are partitions of appropriate positive integers, then

$$[A^{\{\lambda\}}]^{\{\mu\}} = \sum A^{\{\mu\}};$$

thus the plethysm $\otimes$ of S-functions was defined by

$$\{\lambda\} \otimes \{\mu\} = \sum \{\nu\}.$$

(Alternative definitions are now available, e.g. A. Kerber [Representations of Permutalion Groups II, Lecture Notes in Mathematics, Vol. 495 (Springer-Verlag) 1975].) Littlewood and others had been concerned with explicitly evaluating these plethysms, but their methods were only successful in the easiest cases and entailed substantial computation. Foulkes's earliest papers on S-functions were concerned with attempts to extend these computations. His method was based on a differential operator for S-functions introduced in [9] which was to play a significant part in his later research. For each partition $\lambda \vdash n$, he defined an operator D_λ on the algebra of symmetric functions (c.f. (1))

$$D_\lambda = \frac{1}{n!} \sum_{\rho \vdash n} h_\rho \chi_\rho^\lambda \, 1^{\rho_1} 2^{\rho_2} \cdots \frac{\partial^{\rho_1 + \rho_2 \cdots}}{\partial s_1^{\rho_1} \partial s_2^{\rho_2} \cdots}.$$

After determining the effect of these operators on S-functions and a product of S-functions, he proceeded to show how they may be used in evaluating plethysms : indeed, theoretically to give the coefficient of any S-function in $\{\lambda\} \otimes \{\mu\}$. He applied this method in [10] to evaluate $\{m\} \otimes \{n\}$, where $m = 5, 6$; $n = 2, 3, 4$, in [15] to extend H. W. Turnbull's work on the differentiation of a matrix and in [12] to give simple formulae for $D_m\{\lambda\}$, $D_{1^n}\{\lambda\}$ and $D_{m1^n}\{\lambda\}$ in determinantal form of S-functions. In [13], he was concerned with the special case $\{\lambda\} \otimes s_r$ and gave a new procedure for the evaluation of $\{1^m\} \otimes s_r$, and $\{m\} \otimes s_r$. His work in this area culminated in the impressive memoir [16]. Earlier work had been mainly concerned with methods which could be applied to give a full evaluation of $\{\lambda\} \otimes \{\mu\}$; Foulkes's aim was to prove more general theorems which would give the coefficients of particular partitions, e.g. $\{4m-k, k\}$, $\{m+k, m+k, m-k, m-k\}$, $\{m+k, m, m, m-k\}$, $\{4m-2k, k, k\}$ where $m \geq k \geq 0$ in $\{m\} \otimes \{\mu\}$, where m is any positive integer and $\mu \vdash 4$. The main tool was again Foulkes's own differential operator. He was able to prove some of these results for arbitrary μ. In all these papers, his objective was to produce methods which were not dependent on knowledge of character tables of S_n.

There were other less major contributions to pure S-function theory in [8], [11], [14], In [11] he gave an alternative proof of a result on bialternants due to K. A. Hirsch, at the same time extending this to modified bialternants; [14] is concerned with the determination of an explicit expression of a monomial symmetric function in terms of S-functions, and also an application of the work to the determination of the characters of the symmetric group.

Foulkes was always perceptive in finding new applications for S-functions. S. Kakeya had proved in 1925 that any symmetric function on n indeterminates $x_1, \ldots, x_n$ could be expressed as a rational function of a fundamental set $s_{\nu_1}, \ldots, s_{\nu_n}$, where $\nu_1, \ldots, \nu_n$ are the first of n positive integers which do not belong to an additive sequence of positive integers. Foulkes [17] used S-functions to give a short, elegant proof of Kakeya's theorem and, typically, his proof had the advantage of leading to explicit results in special cases. Simultaneously, he dealt similarly with a result due to Pólya. He showed in [18] how S-functions may be used in the evaluation of the characters of the Lie representations of the general linear group. In these later papers, he continued to use extensively his differential operator.

The papers [19] and [21] may in many respects be regarded as Foulkes's most important contributions. In them he considered a long neglected, little known and remarkable pioneering paper by J. H. Redfield ("The theory of group reduced distributions," *American J. Math.* 49 (1927), 433-455) which seemed to contain or anticipate many of the enumeration results for graphs which had been discovered in the intervening years. Redfield had associated every permutation group with a type of symmetric function which he called a group reduction function. What Foulkes does in [19] is to interpret these symmetric functions in terms of group representation theory; and then certain operations on group reduction functions can be interpreted in terms of the well-known inner product and direct product of group characters. Special cases of these functions had been considered by others (Pólya, Riordan, Read) independently of Redfield's work and had been called the cycle index. Redfield had also been concerned with the resolution of a permutation character of the symmetric group into transitive characters; Foulkes was able to resolve Redfield's difficulty concerning the lack of uniqueness by considering the marks of the distinct subgroups rather than the irreducible characters. The second paper [21] is involved with the application of this work via Redfield's range correspondence to graph theory, in particular to the enumeration of linear graphs. He does this by expanding in terms of S-functions a certain symmetric function; the number of inequivalent linear graphs of k-edges on n vertices is then the sum of the multiplicities of the S-functions of less than three parts in this expansion. He then proceeds to consider various generalisations and shows how character theory may be used in calculations of this type. The paper [23] is an elementary exposition of this work. He was also able to apply this kind of analysis in an interesting paper [20] to some enumerations of importance in genetics which had been considered by R. A. Fisher.

The subject matter of [22], [25], [27], of lesser importance, was of a combinatorial nature. In [22] Foulkes showed that a numerical function introduced by T. Tsuzuku in connection with group transitivity is related to exponential polynomials and, when considered as a function of a partition, has multiplicative properties : the proof again involved his useful differential operator on S-functions. In [25] explicit expressions are given for characters of S_n, induced from irreducible characters of cyclic subgroups and vice-versa. This involves interesting generalizations of symmetric functions introduced via Euler and Möbius functions, the motivation for this work again being the Redfield-Pólya enumeration problems mentioned earlier. He indicates in [27], via a simple argument, that a combinatorial lemma relating standard Young tableaux due to B. Gordon was essentially well known.

Foulkes's contributions to conferences on combinatorial theory held at Oxford in 1969 and 1972 were published in [23] and [25]. Although Foulkes's earlier work on plethysms of S-functions had widespread recognition, especially among physicists, his research had proceeded fairly independently at Swansea with very little direct contact with others either nationally or internationally. The response to this work in these conferences and the greater prominence given to, and the international recognition of his work, especially after his retirement, was a source of great pleasure to him.

Foulkes took his first trip to the European continent in July 1972, when he attended the meeting on permutations organized by the Universit René-Descartes in Paris. He presented there in [26] a very elegant and concise review of the theory of symmetric functions. In particular, he suggested that the coefficients of the Hall-Littlewood functions should receive a combinatorial interpretation that would prove a natural q-extension for the Kostka numbers, a question that has just been settled by Lascoux and Schützenberger.

In March 1973 he participated in the Combinatorics Colloquium at Oberwolfach, Germany. It was the occasion for him to discuss his results on paths in ordered structures of partitions. He showed [28] how several path countings in partition structures could be expressed in terms of characters of the symmetric group. The Oberwolfach meeting was in fact his first contact with enumerative combinatorics proper, and there he heard of the various combinatorial interpretations for the classical numbers of analysis, such as tangent, secant, Eulerian, Genocchi numbers. It was, at once, clear to him that the numerous ingenious counting devices which he had developed in the study of symmetric functions should also be applied to the combinatorial theory of numbers. Actually, the material of the paper he had just discussed in Oberwolfach was the starting point of the four next papers he was to write.

He spent the months of April and May 1974 at the University of Strasbourg, as a visiting professor. There he expounded his latest results, essentially the contents of [29] and [30]. He had obtained a beautiful identity for the number of permutations with a prescribed up-down sequence, which involved the Littlewood-Richardson coefficients $g_{\lambda\mu\nu}$ and the dimensions of the irreducible representations of the symmetric group $f(\mu)$. Each up-down sequence was associated with a Ferrers diagram. Consequently the alternating permutations corresponded to diagrams of the form $(n, n-1, \ldots, 2, 1)$. As the number of alternating permutations of length n is equal to the tangent or secant number D_n (depending on whether n is even or odd), each number D_n could then be expressed as a function of the coefficients $g_{\lambda\mu\nu}$, and $f(\mu)$. He then obtained several remarkable and unexpected formulae on tangent and secant numbers.

In the same vein, he was able in [31] to obtain some well-known recurrence relations concerning Kostka numbers and irreducible characters of symmetric groups by using similar purely combinatorial arguments. In [32] an interesting simple nonrecursive combinatorial rule for Eulerian numbers was obtained.

He enjoyed his visit to Strasbourg, and was always eager to taste the recipes of the Alsatian cuisine, to go hiking in the Vosges mountains, or just to walk around the ancient streets of the city, especially after his wife had joined him during the second half of his stay. Unfortunately his health failed him on his return to Swansea.

1975 was a critical year for him. He could not make the trip to Oberwolfach in February, where a Colloquium on Combinatorics was held. In some measure he recovered his health in 1976 and came back to Strasbourg to participate in the Table Ronde on Combinatorics and Representation of the Symmetric Group. His talk on parts of the material that made up his very last paper was one of the highlights of the meeting. His main desire was to write up all his results on the relations between Enumeration and Symmetric Group Representation in a single paper, but as the Proceedings of the Table Ronde had to be published quickly, he could not submit his paper in time. This however enabled him to complete a major piece of work [33] which will appear in a special issue of Discrete Mathematics dedicated to his memory.

He died in July 1977 a few days after completing this work.

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