

## FURTHER DIVISIBILITY PROPERTIES OF THE $q$ -TANGENT NUMBERS

DOMINIQUE FOATA

**ABSTRACT.** The  $q$ -tangent number  $T_{2n+1}(q)$  is shown to be divisible by  $(1+q)^{a(n,1)}(1+q^2)^{a(n,2)} \cdots (1+q^n)^{a(n,n)}$ , where the  $a(n,i)$ 's are positive integers having the maximal property that  $a(n,1) + a(n,2) + \cdots + a(n,n) = 2n$  whenever  $n$  is a power of 2.

**1. Introduction.** The  $q$ -tangent numbers are polynomials that may be defined by

$$\sum_{n>0} T_{2n+1}(q)x^{2n+1}/(q; q)_{2n+1} = \left( \sum_{n>0} (-1)^n x^{2n+1}/(q; q)_{2n+1} \right) / \left( \sum_{n>0} (-1)^n x^{2n}/(q; q)_{2n} \right), \quad (1.1)$$

where  $(a; q)_n = (1-aq) \cdots (1-aq^{n-1})$  for  $n \geq 1$  and  $(a; q)_0 = 1$ . When  $q$  equals 1, the  $q$ -tangent numbers become the ordinary tangent numbers  $T_{2n+1}$  ( $n > 0$ ) occurring in the Taylor expansion of  $\tan x$

$$\sum_{n>0} T_{2n+1}x^{2n+1}/(2n+1)! = \tan x. \quad (1.2)$$

Because of the relation

$$(n+1)T_{2n+1} = 2^{2n}G_{2n+2}, \quad (1.3)$$

where  $G_{2n+2}$  is an *odd* integer called the Genocchi number (see e.g. [3]), Schützenberger [6] raised the problem of finding a polynomial of the form

$$\prod_{i>1} (1+q^i)^{a(n,i)}$$

that divides  $T_{2n+1}(q)$ . Along these lines Andrews and Gessel [2] proved that  $T_{2n+1}(q)$  is divisible by

$$AG_n(q) = (1+q)^{\lfloor n/2 \rfloor + 1} (1+q^2) \cdots (1+q^n). \quad (1.4)$$

The purpose of this paper is to extend the result of the latter work as follows.

Every integer  $n$  may be written as  $n = m2^l$  with  $m$  odd and  $l > 0$ , so that the polynomial

$$Ev_n(q) = \prod_{0 < j < l} (1+q^{m2^j}) \quad (1.5)$$

Received by the editors October 25, 1979.

AMS (MOS) subject classifications (1970). Primary 05A15, 05A19; Secondary 33A30.

© 1981 American Mathematical Society  
 0002-9939/81/0000-0033/\$02.50

may be defined (and is always divisible by  $(1 + q^n)$ ). Let

$$\begin{aligned}
 D_n(q) &= \prod_{1 < i < n} Ev_i(q) \quad \text{for } n \text{ odd,} \\
 &= (1 + q^2) \prod_{1 < i < n} Ev_i(q) \quad \text{for } n \text{ even,}
 \end{aligned}
 \tag{1.6}$$

or, in an equivalent manner,

$$D_1(q) = 1 + q, \quad D_2(q) = (1 + q)^2(1 + q^2)^2
 \tag{1.7}$$

and for  $n \geq 3$

$$D_n(q) = D_{n-2}(q)Ev_{n-1}(q)Ev_n(q).
 \tag{1.8}$$

For convenience, let  $Ev_0(q) = D_0(q) = 1$ .

**THEOREM 1.** *For each  $n > 0$  the polynomial  $D_n(q)$  is a divisor of  $T_{2n+1}(q)$ .*

As  $(1 + q)(1 + q^{n-1})(1 + q^n)$  divides  $Ev_{n-1}(q)Ev_n(q)$  for every  $n \geq 2$ , formula (1.8) shows by induction that  $AG_n(q)$  (given in (1.4)) divides  $D_n(q)$  for all  $n$ . Thus Theorem 1 extends the result obtained by Andrews and Gessel. Now to compare the divisibility properties of  $AG_n(q)$  with  $D_n(q)$  write the latter polynomial in the form

$$D_n(q) = \prod_{1 < i < n} (1 + q^i)^{a(n,i)}.
 \tag{1.9}$$

The first values of the coefficients  $a(n, i)$  ( $1 < i < n$ ) are shown in Table 1.

$n \backslash i$	1	2	3	4	5	6	7	8
1	1							
2	2	2						
3	2	1	1					
4	3	3	1	1				
5	3	2	1	1	1			
6	3	3	2	1	1	1		
7	3	2	2	1	1	1	1	
8	4	4	2	2	1	1	1	1

TABLE 1

The number of factors in  $AG_n(q)$  (resp.  $D_n(q)$ ) is  $[n/2] + n$  (resp.  $a(n, 1) + \dots + a(n, n)$ ).

**PROPOSITION 2.** *If  $2^l < n < 2^{l+1}$  ( $l \geq 1$ ), then*

$$(a(n, 1) + \dots + a(n, n)) - ([n/2] + n) \geq 2^{l-1} - 1.
 \tag{1.10}$$

Finally, as  $D_n(1)$  divides the tangent number  $T_{2n+1}$ , it follows from (1.9) and (1.3), since  $G_{2n+2}$  is odd, that

$$a(n, 1) + \dots + a(n, n) \leq 2n.
 \tag{1.11}$$

Whenever equality holds, we may say that  $D_n(q)$  is maximal.

PROPOSITION 3. When  $n$  is a power of 2, then

$$a(n, 1) + \dots + a(n, n) = 2n, \tag{1.12}$$

i.e.  $D_n(q)$  is maximal.

The pattern of the paper by Andrews and Gessel is followed closely. In particular, the crucial part is played by a divisibility property of Gaussian polynomials (Lemma 2.2) that, roughly speaking, sorts the cyclotomic polynomials  $\phi_d$  according to the parity of  $d$ .

2. A divisibility property of Gaussian polynomials. The polynomials  $Ev_n(q)$  defined in (1.5) can be expressed in terms of cyclotomic polynomials  $\phi_d$  as follows.

LEMMA 2.1. For each  $n \geq 1$  we have

$$Ev_n(q) = \prod \{ \phi_d(q) : d|2n, d \text{ even} \}. \tag{2.1}$$

PROOF. Let  $n = m2^l$  with  $m$  odd and  $l \geq 0$ . For each  $j = 0, 1, \dots, l$  consider the set

$$A_j = \{ d : d|m2^{j+1}, d \nmid m2^j \}$$

and let

$$B = \{ d : d|m2^{l+1}, d \text{ even} \}.$$

By definition

$$1 - q^i = \prod_{d|i} \phi_d(q) \quad \text{for each } i \geq 1. \tag{2.2}$$

As  $(1 - q^{2^i}) = (1 - q^i)(1 + q^i)$ , we then derive

$$1 + q^i = \prod \{ \phi_d(q) : d|2i, d \nmid i \}. \tag{2.3}$$

In particular, if  $0 \leq j < l$ , then

$$1 + q^{m2^j} = \prod \{ \phi_d(q) : d|m2^{j+1}, d \nmid m2^j \} = \prod_{d \in A_j} \phi_d(q).$$

On the other hand, as the sets  $A_j$  are two by two disjoint, it suffices to show that  $B$  is the union of the  $A_j$ 's.

But if  $d|m2^{l+1}, d \nmid m2^l$  for some  $j$  with  $0 \leq j < l$ , then  $d|2n$  (equal to  $m2^{l+1}$ ) and  $d$  is even. Thus  $d$  belongs to  $B$ . Conversely, suppose  $d|2n$  and  $d$  even. Then  $d = m'2^{j+1}$  with  $m'$  odd,  $m'|m$  and  $0 \leq j < l$ . Consequently,  $d$  is an element of  $A_j$ . Q.E.D.

For each  $n \geq 1$  let

$$Od_n(q) = \prod \{ \phi_d(q) : d|2n, d \text{ odd} \}, \tag{2.4}$$

so that

$$1 - q^{2n} = Od_n(q)Ev_n(q). \tag{2.5}$$

On the other hand, let the *Gaussian polynomial* be defined by

$$\begin{aligned} \left[ \begin{matrix} N \\ M \end{matrix} \right] &= (q; q)_N / ((q; q)_M (q; q)_{N-M}) \quad \text{for } 0 < M < N, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

LEMMA 2.2. For nonnegative integers  $k$  and  $n$  the expression

$$\left[ \begin{matrix} 2n \\ 2k + 1 \end{matrix} \right] \frac{Ev_0(q)Ev_1(q) \dots Ev_k(q)}{Ev_{n-k}(q)Ev_{n-k+1}(q) \dots Ev_n(q)} \tag{2.6}$$

is a polynomial in  $q$ .

PROOF. The expression (2.6) is zero if  $k > n$ . Assume that  $0 < k < n - 1$ . Using (2.5) the Gaussian polynomial  $\left[ \begin{matrix} 2n \\ 2k + 1 \end{matrix} \right]$  may be factorized as a product of two factors,

$$\begin{aligned} \left[ \begin{matrix} 2n \\ 2k + 1 \end{matrix} \right] &= \frac{Od_n(q)(1 - q^{2n-1})Od_{n-1}(q) \dots (1 - q^{2n-2k+1})Od_{n-k}(q)}{(1 - q^{2k+1})Od_k(q)(1 - q^{2k-1}) \dots Od_1(q)(1 - q)} \\ &\cdot \frac{Ev_n(q)Ev_{n-1}(q) \dots Ev_{n-k}(q)}{Ev_k(q)Ev_{k-1}(q) \dots Ev_1(q)}. \end{aligned}$$

When numerators and denominators are expressed in terms of cyclotomic polynomials, the first factor, because of (2.2) and (2.4) (resp. the second factor, because of (2.1)) only involves cyclotomic polynomials  $\phi_d$  with  $d$  odd (resp.  $d$  even). As  $\left[ \begin{matrix} 2n \\ 2k + 1 \end{matrix} \right]$  is a polynomial and the cyclotomic polynomials are irreducible, each of these two factors is also a polynomial. But the first one is precisely equal to the expression given in (2.6). Q.E.D.

3. Proof of Theorem 1. Let  $T(x)$  be the generating function for the  $q$ -tangent numbers as written in (1.1). Andrews and Gessel [2, p. 282] found that

$$T(x) = (-i)((-ix; q)_\infty - (ix; q)_\infty) / ((-ix; q)_\infty + (ix; q)_\infty) \tag{3.1}$$

where  $(a; q)_\infty = \lim_n (a; q)_n$ . As  $(a; q)_\infty = (1 - a)(aq; q)_\infty$ , it is straightforward to obtain

$$T(x) - T(qx) = x + xT(qx)T(x); \tag{3.2}$$

that is,

$$\begin{aligned} \sum_{n > 0} T_{2n+1}(q)x^{2n} / (q; q)_{2n} &= 1 + \left( \sum_{k > 0} T_{2k+1}(q)q^{2k+1}x^{2k+1} / (q; q)_{2k+1} \right) \\ &\cdot \left( \sum_{j > 0} T_{2j+1}(q)x^{2j+1} / (q; q)_{2j+1} \right). \end{aligned}$$

Equating coefficients of  $x^{2n}$  in both members we find that

$$T_{2n+1}(q) = \sum_{0 < k < n-1} \left[ \begin{matrix} 2n \\ 2k + 1 \end{matrix} \right] q^{2k+1} T_{2k+1}(q) T_{2n-2k-1}(q) \quad (n > 1). \tag{3.3}$$

The proof of Theorem 1 is now completed as follows. First  $T_1(q) = T_{2 \times 0 + 1}(q) = 1$ . Proceed by induction on  $n > 1$ . For  $0 < k < n - 1$  the expression

$$\left[ \begin{matrix} 2n \\ 2k + 1 \end{matrix} \right] \frac{T_{2k+1}(q)T_{2n-2k-1}(q)}{Ev_1(q) \dots Ev_n(q)} \tag{3.4}$$

is a polynomial because it may be factorized as

$$\left[ \begin{matrix} 2n \\ 2k + 1 \end{matrix} \right] \frac{Ev_0(q) \dots Ev_k(q)}{Ev_{n-k}(q) \dots Ev_n(q)} \cdot \frac{T_{2k+1}(q)}{Ev_0(q) \dots Ev_k(q)} \cdot \frac{T_{2n-2k-1}(q)}{Ev_1(q) \dots Ev_{n-k-1}(q)},$$

the first factor being a polynomial by Lemma 2.2; the other two are also by the induction hypothesis. This shows, when  $n$  is odd, that each term in the sum on the right side of (3.3) is divisible by  $D_n(q) = Ev_1(q)Ev_2(q) \dots Ev_n(q)$ . Hence,  $D_n(q)$  divides  $T_{2n+1}(q)$ . When  $n$  is even, rewrite (3.3) by grouping the terms two by two to give

$$T_{2n+1}(q) = \sum_{0 < k < n/2-1} \left[ \begin{matrix} 2n \\ 2k + 1 \end{matrix} \right] q^{2k+1} (1 + q^{2(n-2k-1)}) T_{2k+1}(q) T_{2n-2k-1}(q). \tag{3.5}$$

As  $n$  is even,  $(1 + q^{2(n-2k-1)})$  is divisible by  $(1 + q^2)$ , and by (3.4) the expression  $\left[ \begin{matrix} 2n \\ 2k + 1 \end{matrix} \right] T_{2k+1}(q) T_{2n-2k-1}(q)$  is divisible by  $Ev_1(q)Ev_2(q) \dots Ev_n(q)$ . Hence, each term in the sum on the right side of (3.5) is divisible by  $D_n(q) = (1 + q^2)Ev_1(q) \dots Ev_n(q)$ . This completes the proof of Theorem 1.

**4. Proofs of Propositions 2 and 3.** Note that  $D_2(q) = (1 + q^2)Ev_1(q)Ev_2(q) = (1 + q)^2(1 + q^2)^2$ . Thus (1.12) holds for  $n = 2$ . Let  $n = 2^l$  ( $l > 2$ ) and proceed by induction on  $l$ . Clearly

$$Ev_{2i}(q) = Ev_i(q)(1 + q^{2i}) \quad (i > 1).$$

Hence,

$$\begin{aligned} D_n(q) &= (1 + q^2) \prod_{1 < i < n} Ev_i(q) \\ &= (1 + q^2) \prod_{\substack{1 < i < n \\ i \text{ odd}}} Ev_i(q) \cdot \prod_{\substack{1 < i < n \\ i \text{ even}}} Ev_i(q) \\ &= (1 + q^2) \prod_{1 < i < n/2} (1 + q^{2i-1}) \cdot \prod_{1 < i < n/2} Ev_{2i}(q) \\ &= (1 + q^2) \prod_{1 < i < n/2} (1 + q^{2i-1}) \cdot \prod_{1 < i < n/2} (1 + q^{2i}) \cdot \prod_{1 < i < n/2} Ev_i(q). \end{aligned}$$

By grouping the first and last factors we obtain

$$D_n(q) = D_{n/2}(q) \prod_{1 < i < n} (1 + q^i). \tag{4.1}$$

Therefore, if the number of factors in  $D_{n/2}(q)$  is  $n$ , the polynomial  $D_n(q)$  will have  $n + n = 2n$  factors. This completes the proof of Proposition 3.

As for Proposition 2 let  $d_n = a(n, 1) + \dots + a(n, n)$  for  $n > 1$ . From (1.6) and (1.9) it follows that  $d_{2n} = d_{2n+1}$  for  $n > 1$ . On the other hand, the number of factors in  $AG_{2n+1}(q)$  is  $3n + 1$ . Let  $p = 2^l < 2n < 2^{l+1}$  ( $l > 1$ ). To prove Proposition 2 it suffices to show that  $d_{2n} - (3n + 1) > 2^{l-1} - 1$ , i.e.

$$d_{2n} > 3n + 2^{l-1}. \tag{4.2}$$

From (1.8)

$$D_{2n}(q) = D_p(q) \prod_{p+1 \leq i < 2n} Ev_i(q).$$

But, when  $i$  is even, the polynomial  $Ev_i(q)$  is a product of at least two binomials  $(1 + q^j)$ . Hence

$$d_{2n} \geq d_p + 3(2n - p)/2 = 3n + 2^{l-1},$$

which is inequality (4.2).

**5. Concluding remarks.** Recall that a permutation  $x_1 x_2 \dots x_{2n+1}$  of the sequence  $1 2 \dots (2n + 1)$  is *alternating* if  $x_1 > x_2, x_2 < x_3, \dots, x_{2n} < x_{2n+1}$ . As  $[2k+1]^{2n}$  is the generating polynomial for permutations  $x_1 x_2 \dots x_{2n}$  of  $1^{2k+1} 2^{2n-2k-1}$  by number of inversions (see e.g. [1, p. 41]), it is clear that the running term on the right side of (3.3) is the generating polynomial for alternating permutations  $x_1 x_2 \dots x_{2n+1}$  of  $1 2 \dots (2n + 1)$  with  $x_{2k+2} = 1$ , by number of inversions, a result known to several authors [4]–[7]. The proof of Theorem 1 shows that the latter generating function is itself divisible by  $Ev_1(q)Ev_2(q) \dots Ev_n(q)$ . It would be interesting to have a combinatorial proof of this result by using that alternating permutation set-up.

From (3.1) Andrews and Gessel [2] derived the recurrence formula

$$\begin{aligned} T_{2n+1}(q) + \sum_{1 < j < n} (-q; q)_{2j-1} \begin{bmatrix} 2n+1 \\ 2j \end{bmatrix} (-1)^j T_{2n+1-2j}(q) \\ = (-1)^n (-q; q)_{2n}. \end{aligned} \tag{5.1}$$

It was not possible to use (5.1) directly to prove Theorem 1 because, for instance, when  $n = 2^l$  ( $l \geq 1$ ) the polynomial  $(-q; q)_{2n}$  is not divisible by  $D_n(q)$ . That is why we had to derive the quadratic recurrence formula (3.3).

REFERENCES

1. G. E. Andrews, *The theory of partitions*, Encyclopedia of Mathematics and its Applications, Vol. 2, Addison-Wesley, Reading, Mass., 1976.
2. G. E. Andrews and I. Gessel, *Divisibility properties of the q-tangent numbers*, Proc. Amer. Math. Soc. **68** (1978), 380–384.
3. D. Dumont, *Interprétations combinatoires des nombres de Genocchi*, Duke Math. J. **41** (1974), 305–318.
4. I. Gessel, *Generating functions and enumeration of sequences*, Ph. D. thesis, M.I.T., 1977.
5. D. Rawlings, *Generalized Worpitzky identities*, Univ. of California, San Diego, 1979 (preprint); European J. Combinatorics (to appear).
6. M. P. Schützenberger, Oral communication, Combinatorics Conference, Oberwolfach, 1975.
7. R. P. Stanley, *Binomial posets, Möbius inversion and permutation enumeration*, J. Combinatorial Theory Ser. A **20** (1976), 336–356.

DÉPARTEMENT DE MATHÉMATIQUE, UNIVERSITÉ DE STRASBOURG, 7 RUE RENÉ DESCARTES, F 67084 STRASBOURG, FRANCE