ADVANCES IN APPLIED MATHEMATICS 2, 250–259 (1981)

Some Hermite Polynomial Identities and Their Combinatorics

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The multilinear extensions of the Mehler formula found by Kibble, Slepian and Louck are shown to be equivalent. They can all be proved by using the combinatorial set-up of involutionary graphs, and so thus the classical Doetsch identity.

1. INTRODUCTION

Let $(H_m(x))$ $(m \geq 1)$ denote the Hermite polynomials defined by the identity

$$1 + \sum_{m\geq 1} (u^m/m!)H_m(x) = \exp(ux - u^2/2), \tag{1.1}$$

or, in an equivalent manner, by

$$H_m(x) = \sum_{0\leq 2k\leq m} (-1)^k x^{m-2k} \frac{m!}{(2!)^k k!\,(1!)^{m-2k}(m-2k)!} \qquad (m \geq 1). \tag{1.2}$$

(In classical books on orthogonal polynomials, Rainville [6], Szegö [8], another normalization for the Hermite polynomials is used. Formula (1.1) is then replaced by

$$1 + \sum_{m\geq 1} (u^m/m!)H_m(x) = \exp(2ux - u^2).$$

However, the notation (1.1) will kept in the whole paper and, accordingly, the classical identities translated with this normalization.)

From (1.2) we get

$$\begin{aligned} H_m(0) &= (-1)^{m/2}\frac{m!}{2^{m/2}(m/2)!} && (m \text{ even}) \\ &= 0 && (m \text{ odd}), \end{aligned} \tag{1.3}$$

0196-8858/81/030250-10$05.00/0

and it will not be surprising that Hermite polynomial identities containing negative powers of 2, as in Louck's identity (see (2.3) below), or factorials $[(m/2)]!$, as in the Doetsch formula (see (4.1) below) be specializations of formulas involving higher products of Hermite polynomials. The purpose of this paper is to explore those specializations. More precisely, remember that the bilinear extension of (1.1), namely,

$$1+\sum_{m\geq 1}(u^m/m!)H_m(y)H_m(z)$$
$$=(1-u^2)^{-1/2}\exp\Big((1/2)\big(y^2-(y-uz)(1-u^2)^{-1}(y-uz)\big)\Big), \tag{1.4}$$

known as the Mehler formula (see, e.g., [1, p. 16; 6, p. 198; 8, p. 380; 9]), has been generalized to the multilinear case by Kibble [4] (as was noted by Askey [2]) and Slepian [7]. Recently, Louck [5] proposed another extension. The purpose of this note is to show that those multilinear extensions are in fact equivalent. As a combinatorial proof of the now called Kibble–Slepian identity was given in [3] by means of the involutionary graph set-up, the combinatorial interpretations of those multilinear extensions are also derived in the paper. Finally, the Doetsch formula is proved by using the same combinatorial techniques.

2. The Multilinear Extensions

Let (s_{ij}) $(i\geq 1,\ j\geq 1)$ be an infinite sequence of indeterminates with the property that

$$s_{ij}=s_{ji}\qquad\text{and}\qquad s_{ii}=0$$

for all i and j. For each $n\geq 1$ denote by S_n the (symmetric) $n\times n$ matrix

$$S_n=(s_{ij})\qquad(1\leq i,\ j\leq n)$$

and by I_n the identity matrix of order n. Now with n being fixed let $y^t=(y_1,y_2,\dots,y_n)$ be a (row) vector with n variables and y be the corresponding column vector. Finally, let $N=(\nu_{ij})$ $(1\leq i,\ j\leq n)$ designate a symmetric $n\times n$ matrix with nonnegative integral entries and a_i, b_i denote the sums

$$\begin{aligned}a_i&=\nu_{i1}+\nu_{i2}+\cdots+\nu_{in}\\ b_i&=\nu_{ii}+a_i\qquad\qquad(1\leq i\leq n).\end{aligned}\tag{2.1}$$

In the first identity (2.2) the summation is over all symmetric $n \times n$ matrices $N = (\nu_{ij})$ with diagonal entries ν_{ii} equal to zero. It reads

$$\sum_N \prod_{i<j} (s_{ij}^{\nu_{ij}}/\nu_{ij}!) \cdot \prod_i H_{a_i}(y_i)$$
$$= (\det(I_n + S_n))^{-1/2} \exp\big((1/2)\big(y^t y - y^t (I_n + S_n)^{-1} y\big)\big), \tag{2.2}$$

with $1 \le i < j \le n$ and $1 \le i \le n$ in the above two products.

In the next two identities (2.2) and (2.3) the matrix $N = (\nu_{ij})$ runs over *all* symmetric $n \times n$ matrices. Furthermore, $D_n = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ is a diagonal matrix, the entries d_i $(1 \le i \le n)$ being indeterminates and $z^t = (z_1, z_2, \ldots, z_n)$ is an n-variable (row) vector. They read

$$\sum_N \prod_{i<j} (s_{ij}^{\nu_{ij}}/\nu_{ij}!) \cdot \prod_i (d_i^{\nu_{ii}}/\nu_{ii}!) H_{b_i}(y_i) \cdot 2^{-\operatorname{Tr} N}$$
$$= (\det(I_n + D_n + S_n))^{-1/2} \exp\big((1/2)\big(y^t y - y^t (I_n + D_n + S_n)^{-1} y\big)\big) \tag{2.3}$$

and

$$\sum_N \prod_{i<j} (s_{ij}^{\nu_{ij}}/\nu_{ij}!) \cdot \prod_i (d_i^{\nu_{ii}}/\nu_{ii}!) H_{a_i}(y_i) H_{\nu_{ii}}(z_i)$$
$$= \big(\det\big(I_n - D_n^2 + S_n\big)\big)^{-1/2} \exp\Big((1/2)\Big(y^t y - (y - D_n z)^t$$
$$\times \big(I_n - D_n^2 + S_n\big)^{-1} (y_n - D_n z)\Big)\Big). \tag{2.4}$$

Identity (2.2) was obtained by Kibble [4] and Slepian [7] and (2.3) is due to Louck [5]. Note that for $n = 2$ (resp. for $n = 1$ and $D_n = 0$) formula (2.2) (resp. (2.4)) reduces to Mehler's formula (1.4).

PROPOSITION 1. *The identities* (2.2) *and* (2.3) *and* (2.4) *are equivalent.*

Proof. Clearly (2.3) implies (2.2) by making all the d_i's equal to 0.

Show next that (2.4) implies (2.3). Let $L(S_n, D_n, y, z)$ be the left-hand side of (2.4) and for convenience replace the running matrix $N = (\nu_{ij})$ by $M = (\mu_{ij})$ so that

$$L(S_n, D_n, y, z) = \sum_M \prod_{i<j} (s_{ij}^{\mu_{ij}}/\mu_{ij}!) \cdot \prod_i (d_i^{\mu_{ii}}/\mu_{ii}!) H_{a_i}(y_i) H_{\mu_{ii}}(z_i)$$

with

$$a_i = \mu_{i1} + \mu_{i2} + \cdots + \mu_{in}.$$

From (1.3) we get

$$L(S_n, D_n, y, 0) = \sum_M \prod_{i<j} (s_{ij}^{\mu_{ij}}/\mu_{ij}!) \cdot \prod_i (d_i^{\mu_{ii}}/\mu_{ii}!) H_{ai}(y_i)$$
$$\times (-1)^{\mu_{ii}/2} \mu_{ii}!/\big(2^{\mu_{ii}/2}(\mu_{ii}/2)!\big)$$

with $M = (\mu_{ij})$ extended over all matrices with μ_{ii} *even* $(1 \leq i \leq n)$. Let $N = (\nu_{ij})$ be defined by

$$\begin{aligned} \nu_{ij} &= \mu_{ij} && \text{for } i \neq j, \\ \nu_{ii} &= \mu_{ii}/2 && \text{for all } i, \end{aligned} \tag{2.5}$$

so that

$$a_i = \mu_{i1} + \mu_{i2} + \cdots + \mu_{in} = \nu_{ii} + (\nu_{i1} + \nu_{i2} + \cdots + \nu_{in}) = b_i.$$

Then

$$L(S_n, D_n, y, 0) = \sum_N \prod_{i<j} (s_{ij}^{\nu_{ij}}/\nu_{ij}!) \cdot \prod_i \left((-d_i^2)^{\nu_{ii}}/\nu_{ii}! \right) H_{b_i}(y_i) \cdot 2^{-\operatorname{Tr} N}$$

Hence, with $z = 0$ identity (2.4) becomes

$$\begin{aligned} &\sum_N \prod_{i<j} (s_{ij}^{\nu_{ij}}/\nu_{ij}!) \cdot \prod_i \left((-d_i^2)^{\nu_{ii}}/\nu_{ii}! \right) H_{b_i}(y_i) \cdot 2^{-\operatorname{Tr} N} \\ &\quad = \left(\det\left(I_n - D_n^2 + S_n \right) \right)^{-1/2} \exp\left((1/2)\left(y^t y - y^t \left(I_n - D_n^2 + S_n \right)^{-1} y \right) \right). \end{aligned}$$

Thus with $z = 0$ and $-D_n^2$ replaced by D_n identity (2.4) yields (2.3).

Show finally that (2.2) implies (2.4). Rewrite (2.2) with n replaced by $2n$ and for $i = 1, 2, \ldots, n$ each y_{n+i} replaced by z_i. Then

$$\begin{aligned} \sum_N \prod_{1 \leq i<j \leq 2n} (s_{ij}^{\nu_{ij}}/\nu_{ij}!) \prod_{1 \leq i \leq n} H_{a_i}(y_i) H_{a_{n+i}}(z_i) &= (\det(I_{2n} + S_{2n}))^{-1/2} \exp\left((1/2)\left((y, z)^t (y, z) \right.\right. \\ &\quad \left.\left. - (y, z)^t (I_{2n} + S_{2n})^{-1} (y, z) \right) \right). \end{aligned}$$

Now map each entry of S_{2n} onto the corresponding entry of the matrix

$$T_{2n} = \begin{pmatrix} S_n & D_n \\ D_n & 0 \end{pmatrix}. \tag{2.6}$$

In other words, leave all the s_{ij} with $1 \leq i, j \leq n$ unchanged, put all the other entries of S_{2n} equal to 0 except the entries $s_{i,n+i}$ and $s_{n+i,i}$ that become equal to d_i $(1 \leq i \leq n)$. The above identity is then transformed into

$$\begin{aligned} \sum_N \prod_{1 \leq i<j \leq n} (s_{ij}^{\nu_{ij}}/\nu_{ij}!) \cdot \prod_{1 \leq i \leq n} (d_i^{\nu_{i,n+i}}/\nu_{i,n+i}!) H_{a_i}(y_i) H_{\nu_{i,n+i}}(z_i) &= (\det(I_n + T_{2n}))^{-1/2} \exp\left((1/2)(y, z)^t (y, z) \right. \\ &\quad \left. - (y, z)^t (I_{2n} + T_{2n})^{-1} (y, z) \right), \end{aligned} \tag{2.7}$$

where the summation is extended over all sequences (ν_{ij}) $(1 \le i, j \le 2n)$ with either $1 \le i < j \le n$ or $1 \le i \le n$ and $j = n + i$. By elementary manipulations on determinants

$$\det\begin{pmatrix} I_n + S_n & D_n \\ D_n & I_n \end{pmatrix} = \det\begin{pmatrix} I_n + S_n - D_n^2 & D_n \\ 0 & I_n \end{pmatrix}$$

so that

$$\det(I_{2n} + T_n) = \det\left(I_n + S_n - D_n^2\right). \tag{2.8}$$

Furthermore, let $\Delta = I_n - D_n^2 + S_n$ and A be the $2n \times 2n$ matrix

$$A = \begin{pmatrix} I_n - \Delta^{-1} & \Delta^{-1}D_n \\ D_n\Delta^{-1} & -D_n\Delta^{-1}D_n \end{pmatrix}. \tag{2.9}$$

Then the argument of exp in (2.4) may be written

$$(1/2)(y, z)^t A(y, z), \tag{2.10}$$

while the corresponding term in (2.7) is equal to

$$(1/2)(y, z)^t\left(I_{2n} - (I_{2n} + T_{2n})^{-1}\right)(y, z). \tag{2.11}$$

To show that

$$A = I_{2n} - (I_{2n} + T_{2n})^{-1}, \tag{2.12}$$

it suffices to prove that

$$(I_{2n} - A)(I_{2n} + T_{2n}) = I_{2n}.$$

But this is a straightforward calculation that follows from the definitions of T_{2n} and A (in (2.6) and (2.9)).

Now replace $\nu_{i,\,n+i}$ by ν_{ii} $(1 \le i \le n)$ in (2.7) and take (2.8), (2.10), (2.11) and (2.12) into account. We then conclude that (2.7) is nothing but a rewriting of (2.4). Q.E.D.

3. Combinatorial Interpretations

In [3] a combinatorial proof of the Kibble–Slepian identity was given. By an *n-involutionary m-graph* there was meant an undirected graph with m

vertices labeled $1, 2, \ldots, m$ and edges and loops colored in such a way that

(i) the colors are taken from the set $\{1, 2, \ldots, n\}$;
(ii) each vertex has valency 2;
(iii) each vertex is incident to two different colors.

Let G be such a graph. For $i < j$ denote by ν_{ij} the number of its vertices incident to colors i and j. Let t_i (resp. f_i) be the number of edges (resp. of loops) colored i. Then, define

$$\mu(G) = \prod_{i<j} s_{ij}^{\nu_{ij}} \cdot \prod_i (-1)^{t_i} y_i^{f_i}. \tag{3.1}$$

The combinatorial proof of (2.2) was made in two steps. First, it was shown that the left-hand side is equal to the generating function for n-involutionary graphs by μ, i.e.,

$$\sum_N \prod_{i<j} (s_{ij}^{\nu_{ij}}/\nu_{ij}!) \cdot \prod_i H_{a_i}(y_i) = \sum_{m \geq 0} (1/m!) \sum \mu(G), \tag{3.2}$$

the last summation being over all n-involutionary m-graphs. Second, the right-hand side member was shown to be the exponential of the generating function for the *connected* n-involutionary graphs by the same μ. Identity (2.2) was simply derived from the exponential formula

$$\sum_{m \geq 0} (1/m!) \sum \mu(G) = \exp \sum_{m \geq 1} (1/m!) \sum \mu(G), \tag{3.3}$$

the last summation being over all the *connected* n-involutionary m-graphs.

Also recall that there are two kinds of connected n-involutionary graphs as shown in Fig. 1, the *cycles*, and the *paths* ending with two loops.

The right-hand side of (2.2) is the exponential of the sum of

$$(1/2) \log \frac{1}{\det(I_n + S_n)} \tag{3.4}$$

and

$$(1/2)\left(y^t y - y^t (I_n + S_n)^{-1} y\right). \tag{3.5}$$

It was then shown that (3.4) and (3.5) were the generating functions for the n-involutionary cycles, and for the n-involutionary paths, respectively, by μ.

Now examine the combinatorial meanings of (2.3) and (2.4). Identity (2.7), which is equivalent to (2.4), will now involve $2n$-involutionary graphs but not all of them. The left-hand side of (2.7) is again

$$\sum_{m \geq 0} (1/m!) \sum \mu(G), \tag{3.6}$$

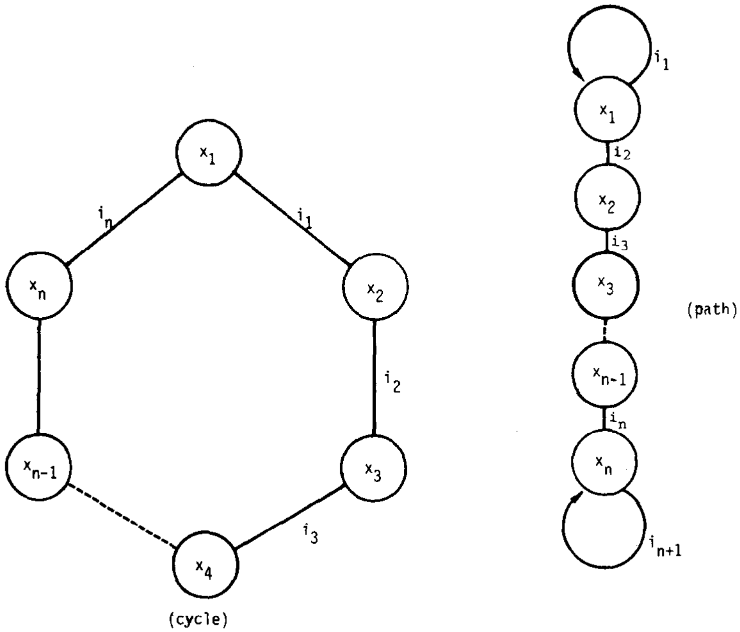

FIGURE 1

but because of the structure of T_{2n} (given in (2.6)), a $2n$-involutionary m-graph G occurs in the latter summation if and only if each of its vertices is incident to two different colors i and j with either $1 \le i < j \le n$, or $1 \le i \le n$ and $j = n + i$. Call *$2n$-bivolutionary m-graph* each $2n$-involutionary m-graph with the latter property. Furthermore, as each variable y_{n+i} was replaced by z_i ($1 \le i \le n$), the monomial $\mu(G)$ is to be replaced by

$$\mu(G) = \prod_{1 \le i < j \le n} s_{ij}^{\nu_{ij}} \cdot \prod_{1 \le i \le n} d_i^{\nu_{ii}} (-1)^{t_i} y_i^{f_i} (-1)^{t_{n+i}} z_i^{f_{n+i}} \tag{3.7}$$

with ν_{ii} denoting the number of vertices of G incident to colors i and $n + i$ ($1 \le i \le n$). Thus, *the left-hand side of* (2.4) *is the generating function for $2n$-bivolutionary graphs by μ with μ given in* (3.7).

Now remember that (2.3) is obtained from (2.4) by putting $z = 0$ and replacing $-D_n^2$ by D_n. Therefore, the left-hand side of (2.3) is the generating function for a subclass of $2n$-bivolutionary graphs and is also of the form (3.6). As $z = 0$, a $2n$-bivolutionary m-graph occurs in the last summation of (3.6) if and only if it has no loop colored $n + i$ ($1 \le i \le n$). As each edge colored $n + i$ is necessarily adjacent with two edges or loops colored i, each integer ν_{ii} is *even*. Because of the substitution of $-D_n^2$ by D_n the monomial

$\mu(G)$ is now

$$\mu(G) = \prod_{1 \le i < j \le n} s_{ij}^{\nu_{ij}} \cdot \prod_{1 \le i \le n} (-d_i)^{\nu_{ii}/2} (-1)^{t_i} y_i^{f_i} (-1)^{t_{n+i}}. \tag{3.8}$$

Thus, *the left-hand side of* (2.3) *is the generating function for the* $2n$*-bivolutionary graphs with no loops colored* $n+i$ $(1 \le i \le n)$ *by* μ *with* μ *given in* (3.8).

4. The Doetsch Identity

Written with notation (1.1) the Doetsch identity [8, p. 380] reads

$$\sum_{m \ge 0} (u^m / [m/2]!) H_m(y)$$
$$= (1+2u^2)^{-3/2} (1 + uy + 2u^2) \exp\left(u^2 y^2 (1+2u^2)^{-1}\right). \tag{4.1}$$

Let $D = (1 + 2u^2)$. Clearly identity (4.1) is the sum of the two identities

$$\sum_{m \ge 0} \left(u^{2m}/(2m)!\right) H_{2m}(y) \cdot \frac{(2m)!}{2^m m!} \cdot 2^m = D^{-1/2} \exp\left(u^2 y^2 D^{-1}\right) \tag{4.2}$$

$$\sum_{m \ge 0} \left(u^{2m+1}/(2m+1)!\right) H_{2m+1}(y) \cdot \frac{(2m+1)!}{2^m m!} \cdot 2^m$$
$$= uyD^{-1} \cdot D^{-1/2} \exp\left(u^2 y^2 D^{-1}\right). \tag{4.3}$$

As it is known and readily verified, both identities (4.2) and (4.3) are consequences of Mehler's formula (1.4). Putting $z = 0$, using (1.3) and replacing u by $(-2)^{1/2} u$ in (1.4) yield (4.2). To obtain (4.3) it suffices to take the partial derivative of (1.4) with respect to z, then put $z = 0$, divide by u, replace u by $(-2)^{1/2} u$ and finally multiply both members by u. Hence, the combinatorial proof of (4.1) can be derived from that of Mehler's formula (1.4) (that is the particular case of (2.2) for $n = 2$.)

With $n = 2$ and $\nu_{12}, s_{12}, y_1, y_2$ replaced by m, u, y, z, respectively, (3.2) becomes

$$\sum_{m \ge 0} (u^m/m!) H_m(y) H_m(z) = \sum_{m \ge 0} (1/m!) \sum \mu(G) \tag{4.4}$$

with

$$\mu(G) = u^m (-1)^{t_1} y^{f_1} (-1)^{t_2} z^{f_2}$$

if G has m vertices. When z is put equal to 0 and u replaced by $(-2)^{1/2}u$ the left-hand side of (4.4) is transformed into the left-hand side of (4.2). As for the right-hand side, it becomes

$$\sum_{m\geq 0}(1/(2m)!)\sum\mu'(G)$$

with the last summation extended over all *Doetsch* $2m$*-graphs*, i.e., the 2-involutionary $2m$-graphs having no loops colored 2, and

$$\mu'(G)=\left((-2)^{1/2}u\right)^{2m}(-1)^{t_1}y^{f_1}(-1)^{t_2}.$$

But if there is no loop colored 2, the number t_2 of edges colored 2 is necessarily equal to m. Therefore

$$\mu'(G)=u^{2m}(-1)^{t_1}y^{f_1}2^{t_2}.$$

In the same manner, when deriving (4.3) from (1.4) as it was mentioned, the left-hand side of (4.4) is transformed into that of (4.2). The right-hand side becomes

$$\sum_{m\geq 0}\left(u^{2m+1}/(2m+1)!\right)\sum\mu'(G)$$

with the last summation extended over all *Doetsch* $(2m+1)$*-graphs*, i.e., the 2-involutionary $(2m+1)$-graphs having a single loop colored 1, and μ' given this time by

$$\mu'(G)=u\left((-2)^{1/2}u\right)^{2m}(-1)^{t_1}y^{f_1}(-1)^{t_2}.$$

Again, $t_2=m$, so that

$$\mu'(G)=u^{2m+1}(-1)^{t_1}y^{f_1}2^{t_2}.$$

Thus, *the left-hand side of the Doetsch identity* (4.1) *is the generating function for the Doetsch graphs, i.e., the* 2*-involutionary graphs having at most one loop colored* 2, *by* μ', *with* μ' *given by*

$$\mu'(G)=u^m(-1)^{t_1}y^{f_1}2^{t_2}, \tag{4.5}$$

if G *has* m *vertices.*

As (4.2) was directly derived from Mehler's formula, as it was explained, the above comment, de facto, provides a combinatorial proof of it. In particular,

$$u^2y^2D^{-1}=u^2y^2(1+2u^2)^{-1},$$

the argument of exp in (4.2), is the generating function for the 2-involutionary paths ending with two loops colored 1, by μ'. It follows from this and the definition of μ' (in (4.5)) that the generating function for the 2-involutionary paths ending with two loops colored 1 and 2, by μ', is equal to

$$uyD^{-1} = uy(1+2u^2)^{-1}.$$

But among the connected components of each Doetsch $(2m+1)$-graph G there is one and only one path that ends with two loops colored 1 and 2. If this path is deleted from G, the remaining graph is a Doetsch $2m'$-graph with $0 \leq 2m' \leq 2m$. Therefore, the generating function for the Doetsch graphs with an odd number of vertices is equal to the product of the generating function for the 2-involutionary paths ending with two loops colored 1 and 2, by the generating function for the Doetsch graphs with an even number of vertices. Thus

$$\begin{aligned}\sum_{m\geq 0} (1/(2m+1)!)\sum \mu'(G) &= uyD^{-1} \sum_{m\geq 0} (1/(2m)!)\sum \mu'(G)\\ &= uyD^{-1}\cdot D^{-1/2}\exp\left(u^2y^2D^{-1}\right).\end{aligned}$$

This completes the combinatorial proof of the Doetsch identity.

Acknowledgments

The author is grateful to Dr. James Louck who kindly sent him his preprint on the Kibble–Slepian identity and invited him to have a combinatorial look at it.

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