CHAPTER 10

Rearrangements of Words

10.1. Preliminaries

When enumerating permutations of finite sequences according to certain patterns (such as with a given number of descents, with a fixed up–down sequence, or with given positions for the maxima) one is frequently led to transfer the counting problem to another class of permutations for which the problem is straightforward or at least easier. Of course there is no general rule to make up those transfers, but we have at our disposal several natural algorithms. The purpose of this chapter is to describe those algorithms and mention several applications.

The typical set-up for describing those algorithms is the following. Let A be a *totally ordered alphabet* and A^* be the free monoid generated by A. A *rearrangement* of a nonempty word $w = a_1 a_2 \cdots a_m$ is a word $w' = a_{i_1} a_{i_2} \cdots a_{i_m}$, where $i_1 i_2 \cdots i_m$ is a permutation of $12 \cdots m$. The set of all the rearrangements of a word w is called a *rearrangement class* (or *abelian class*). Given a subset X of A^* and two integral-valued functions D and E defined on X, the problem is to construct a bijection of X onto itself that maps each word w in X onto a rearrangement w' of w with the subsidiary property that

$$D(w') = E(w). \tag{10.1.1}$$

In most cases the set X is a union of rearrangement classes. It then suffices to give the construction on each such a class. From (10.1.1) it follows that for each integer k and each rearrangement class Y contained in X we have

$$\operatorname{Card}\{w \in Y \mid D(w) = k\} = \operatorname{Card}\{w \in Y \mid E(w) = k\}. \tag{10.1.2}$$

In probabilistic language this simply means that the statistics D and E are *identically distributed* on Y.

One of the reasons for constructing such bijections is to discover further refinement properties of the distributions of the statistics involved (see, for example, Section 10.7 and Problem 10.6.3). As will be seen, the constructions of those bijections, also called *rearrangements*, make use of the classical techniques described in this book (factorizations, cyclic shifts, and so on).

The first example of such a rearrangement is the "first fundamental transformation" (see Section 10.2). It is a bijection of the permutation group of the set of n elements onto itself. Its construction is based upon the fact that each permutation of $12\cdots n$ can be expressed either as a word $a_1 a_2 \cdots a_n$, or as a product of disjoint cycles. It is worth noting that such a simple construction already gives nontrivial results about the distributions of several statistics defined on the permutation group.

The first fundamental transformation is further extended to each arbitrary set of words (having repetitions). There is some algebraic work to do in order to achieve that extension. In particular a substitute for the notion of cycle, which the first fundamental transformation was based upon, has to be found. The algebraic structures to be introduced are first the *flow monoid* (that is, a quotient monoid of $(A\times A)^*$ derived by a set of commutation rules), then a submonoid called the *circuit monoid*. This will be discussed in Sections 10.3 and 10.4. The first fundamental transformation is then described in Section 10.5. Finally Sections 10.6 and 10.7 give a description of the second fundamental transformation, on the one hand, and of the Sparre-Andersen equivalence principle, on the other hand.

10.2. The First Fundamental Transformation

Let n be a (strictly) positive integer and w be a permutation of the set $[n]=\{1,2,\ldots,n\}$. For each $i=1,2,\ldots,n$ let a_i be the image of i under w. The word $a_1 a_2 \cdots a_n$ will be referred to as the *standard* word associated to w and also denoted by w. Assuming that the alphabet A contains the set $\mathbb{N}$ of the natural numbers, the permutation group $\mathfrak{S}_n$ may be regarded as a subset of A^*. Before constructing the first fundamental transformation for each set $\mathfrak{S}_n$ we mention a few notations valid for arbitrary words, not necessarily standard.

Let $w=a_1 a_2 \cdots a_m$ be a nonempty word. Its *first letter* a_1 is denoted by Fw. Rewriting its m letters in nondecreasing order, we obtain its *nondecreasing rearrangement* denoted by $\bar{w}=\bar{a}_1\bar{a}_2\cdots\bar{a}_m$ ($\bar{a}_1\leq\bar{a}_2\leq\cdots\leq\bar{a}_m$). If w is an element of the permutation group $\mathfrak{S}_m$, its nondecreasing rearrangement is $12\cdots m$. If the letters of w are not distinct, containing (say) m_1 letters 1, m_2 letters $2,\ldots,m_n$ letters n, then

$$\bar{w}=1^{m_1}2^{m_2}\cdots n^{m_n}. \tag{10.2.1}$$

Let (a,b) be an ordered pair of integers. Denote by $\nu_{a,b}(w)$ (resp. $\xi_{a,b}(w)$) the number of integers i such that $1\leq i\leq m-1$ and $\bar{a}_i=a, a_i=b$ (resp. $1\leq i\leq m-1$ and $a_i=b, a_{i+1}=a$). Clearly $\nu_{a,b}(w)$ and $\xi_{a,b}(w)$ can only be equal to 0 or 1 if w is standard. The numbers

$$E(w)=\sum_{a<b}\nu_{a,b}(w) \qquad D(w)=\sum_{a<b}\xi_{a,b}(w) \tag{10.2.2}$$

are frequently referred to as being the number of *exceedances* and number of *descents* of w, respectively. Each word w is said to be *initially dominated* if $a_1>a_i$ holds for all i with $2\leqslant i\leqslant n$. Finally, an *increasing factorization* of w is a sequence $(w_1,w_2,\dots,w_p)$ of initially dominated words with the property that

$$w=w_1w_2\cdots w_p$$

and

$$Fw_1\leqslant Fw_2\leqslant\cdots\leqslant Fw_p.$$

For instance, the words $w=563182947$ and $w'=311264622665175$ admit the increasing factorizations (5,631,82,947) and (3112,64,622,6,651,75), respectively.

As shown in the following lemma, increasing factorizations are in fact factorizations of the free monoid in the sense of Chapter 5 (see Problem 5.4.2).

LEMMA 10.2.1. *Every word $w=a_1a_2\cdots a_n$ admits one and only one increasing factorization.*

Proof. Say that the letter a_i is outstanding in w if $i=1$ or $2\leqslant i\leqslant n$ and $a_j\leqslant a_i$ for all $j\leqslant i-1$. When cutting the word w just before each outstanding letter we clearly obtain an increasing factorization. It remains to be shown that it is the only one.

Suppose that there are two such factorizations, say $(v_1,v_2,\dots,v_r)$ and $(w_1,w_2,\dots,w_s)$. Let j be the smallest index such that $v_j\neq w_j$. We can assume that v_j is shorter that w_j, so that $w_j=v_ju$ for some nonempty word u and $Fu=Fv_{j+1}$. As w_j is initially dominated, we have $Fw_j>Fu=Fv_{j+1}$. On the other hand, as $(v_1,v_2,\dots,v_p)$ is an increasing factorization, we get $Fw_j=Fv_j\leqslant Fv_{j+1}$, leading to a contradiction. Thus the factorization is unique. ■

The construction of the first fundamental transformation (in the permutation case) goes as follows: First, let τ be a cyclic permutation of a finite set $B=\{b_1,b_2,\dots,b_m\}$ of m integers. Then, define $q(\tau)$ as the following word of length m:

$$q(\tau)=\tau^m(\max B)\tau^{m-1}(\max B)\cdots\tau(\max B).$$

As τ is cyclic, we have $\tau^m(\max B) = \max B$. Furthermore, $q(\tau)$ is a rearrangement of the m elements of B in some order. Clearly, q is a bijection of the set of cyclic permutations of B onto the set of initially dominated rearrangements of the word $b_1 b_2 \cdots b_m$.

Now let $w = a_1 a_2 \cdots a_n$ be the standard word associated to a permutation of the set $[n]$. If the permutation has r orbits $B_1, B_2, \dots, B_r$, we can assume that those orbits are numbered in such a way that

$$\max B_1 < \max B_2 < \cdots < \max B_r. \tag{10.2.3}$$

Let $\tau_1, \tau_2, \dots, \tau_r$ be the restrictions of w to $B_1, B_2, \dots, B_r$, respectively. As they are all cyclic permutations, we can form the words $q(\tau_1), q(\tau_2), \dots, q(\tau_r)$. We then let $\hat{w}$ be equal to the juxtaposition product,

$$\hat{w} = q(\tau_1)\, q(\tau_2) \cdots q(\tau_r).$$

The sequence $(q(\tau_1), q(\tau_2), \dots, q(\tau_r))$ is precisely the increasing factorization of $\hat{w}$. The mapping that associates $\hat{w}$ to w is a bijection, since there corresponds to w one and only one sequence of cyclic permutations of sets $B_1, B_2, \dots, B_r$, with union $\{1,2,\dots,n\}$ such that (10.2.3) holds. To such a sequence $(\tau_1, \tau_2, \dots, \tau_r)$ there corresponds next one and only one sequence $(w_1, w_2, \dots, w_r)$ of initially dominated words such that $Fw_1 < Fw_2 < \cdots < Fw_r$ and $w_1 w_2 \cdots w_r$ be a rearrangement of $12 \cdots n$. From Lemma 10.2.1 there finally corresponds to $(w_1, w_2, \dots, w_r)$ one and only one permutation $\hat{w}$ admitting $(w_1, w_2, \dots, w_r)$ as its increasing factorization.

Example 10.2.2. Consider the permutation

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 8 & 6 & 9 & 5 & 1 & 4 & 2 & 7 \end{pmatrix}.$$

The orbits written in increasing order according to their maxima are

$$B_1 = \{5\}, \qquad B_2 = \{1,3,6\}, \qquad B_3 = \{2,8\}, \qquad B_4 = \{4,7,9\}.$$

Let τ_j be the restriction of w to B_j $(1 \leq j \leq 4)$. Then

$$\begin{aligned} &q(\tau_1) = 5;\\ &q(\tau_2) = \tau_2^3(6)\tau_2^2(6)\tau_2(6) = 631;\\ &q(\tau_3) = \tau_3^2(8)\tau_3(8) = 82;\\ &q(\tau_4) = \tau_4^3(9)\tau_4^2(9)\tau_4(9) = 947. \end{aligned}$$

Hence

$$\hat{w}=563182947.$$

Going back to the general case the construction of the inverse bijection is made as follows: Start with a permutation v and consider the increasing factorization, say $(w_1, w_2, \ldots, w_r)$ of v. The product (in the group-theoretic sense) of the disjoint cycles $q^{-1}(w_1)q^{-1}(w_2)\cdots q^{-1}(w_r)$ is a permutation of the set $\{1,2,\ldots,n\}$. There corresponds to it a unique standard word w. Then $\hat{w}=v$.

Working again with the foregoing example, with $v=\hat{w}$, note that the increasing factorization of v reads (5,631,82,947). We can then form the product of the disjoint cycles:

$$\begin{pmatrix}5\\5\end{pmatrix}\begin{pmatrix}3 & 1 & 6\\6 & 3 & 1\end{pmatrix}\begin{pmatrix}2 & 8\\8 & 2\end{pmatrix}\begin{pmatrix}4 & 7 & 9\\9 & 4 & 7\end{pmatrix},$$

Erasing the parenthesis and rearranging the columns in such a way that the top row is in increasing order we obtain the permutation

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\3 & 8 & 6 & 9 & 5 & 1 & 4 & 2 & 7\end{pmatrix}.$$

The standard word w such that $\hat{w}=v$ is simply the bottom row of the latter matrix.

The first fundamental transformation is now used to prove the following combinatorial theorem, which essentially says that the number of exceedances E and the number of descents D (see Eqs. (10.2.2)) are identically distributed on each permutations group $\mathfrak{S}_n$.

THEOREM 10.2.3. *For each pair of integers (a,b) with $a<b$ and each standard word w we have*

$$\nu_{a,b}(w)=\xi_{a,b}(\hat{w}). \tag{10.2.3}$$

In particular

$$E(w)=D(\hat{w}).$$

Proof. Let $w=a_1a_2\cdots a_n$. If $\nu_{a,b}(w)=1$, then $a=i<a_i=b$ for some i. But i and a_i belong to the same orbit, say I_j. Let τ_j be the restriction of w to I_j. Then, the dominated word $q(\tau_j)$ contains the factor $a_i i$—that is, $b\,a$—and so $\xi_{a,b}(\hat{w})=1$. Conversely, let $\hat{w}=b_1b_2\cdots b_n$. If $\xi_{a,b}(\hat{w})=1$, there is one and only one factor b_ib_{i+1} of $\hat{w}$ that is equal to $b\,a$. Let $(w_1,w_2,\ldots,w_p)$ be the increasing sequence of $\hat{w}$. As $b_i>b_{i+1}$, the letter b_i cannot be the last letter of a word w_j and b_{j+1} be the first letter of w_{j+1}. Hence $b_i b_{i+1} = b\,a$ is a proper factor of some word w_j. With $\tau_j = q^{-1}(w_j)$ we then have $b = \tau_j(a)$. Thus b is the image of a under w; that is, $\nu_{a,b}(w) = 1$. The second part of the theorem is an immediate consequence of the first part and definition (10.2.2.). ■

It follows from Theorem 10.2.3 that for each integer k we have

$$\mathrm{Card}\{w \in \mathfrak{S}_n \mid E(w) = k\} = \mathrm{Card}\{w \in \mathfrak{S}_n \mid D(w) = k\}$$

Their common value is the *Eulerian number* denoted by $A_{n,k}$. (See Problems 10.2.1–10.2.3.)

10.3. The Flow Monoid

Denote by $M(A)$ the free monoid generated by the cartesian product $A \times A$. It will be convenient to consider the elements of $M(A)$ as two-row matrices $W = \begin{pmatrix} w' \\ w \end{pmatrix}$ with w and w' two words of A^* of the same length. Two elements W_1 and W_2 of $M(A)$ are said to be *adjacent* if there exist U and V in $M(A)$ and two one-column matrices $\begin{pmatrix} a' \\ a \end{pmatrix}, \begin{pmatrix} b' \\ b \end{pmatrix}$ with a, a', b, b' in A, having the property that

$$a' \neq b' \tag{10.3.1}$$

and

$$W_1 = U\begin{pmatrix} a' \\ a \end{pmatrix}\begin{pmatrix} b' \\ b \end{pmatrix}V, \qquad W_2 = U\begin{pmatrix} b' \\ b \end{pmatrix}\begin{pmatrix} a' \\ a \end{pmatrix}V. \tag{10.3.2}$$

Notice condition (10.3.1). The commutation rule refers only to the *top* rows of the matrices. Moreover, two adjacent matrices differ by two adjacent columns whose top elements are *distinct*. Next two elements W_1 and W_2 are said to be *equivalent* if they are equal or if there exist an integer $p \geq 1$ and a sequence of elements $V_0, V_1, \ldots, V_p$ of $M(A)$ such that $W_1 = V_0, W_2 = V_p$ and V_{i-1} and V_i are adjacent for $1 \leq i \leq p$. The equivalence relation just defined is compatible with the juxtaposition product in $M(A)$. The quotient monoid of $M(A)$ derived by this equivalence relation is called the *flow monoid* and denoted by $F(A)$. Its elements are called *flows*. The equivalence class of an element $W = \begin{pmatrix} w' \\ w \end{pmatrix}$ will be denoted by $[W] = \begin{bmatrix} w' \\ w \end{bmatrix}$. The map $\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{bmatrix} a \\ b \end{bmatrix}$ is an injection of $A \times A$ into $F(A)$, and $F(A)$ is generated by the set of all $\begin{bmatrix} a \\ b \end{bmatrix}$ with a, b in A. If v is a word of length m (where $m \geq 1$) and a an element of A, the flow $\begin{bmatrix} a^m \\ v \end{bmatrix}$ has a single representative, namely $\begin{pmatrix} a^m \\ v \end{pmatrix}$.

In the next lemma is determined an invariant of a flow. Let $W=\binom{w'}{w}$ be the two-row matrix with $w=a_1a_2\cdots a_m$ and $w'=a'_1a'_2\cdots a'_m$. For each a in A let $(i_1,i_2,\ldots,i_p)$ be the increasing sequence of integers i such that $1\leqslant i\leqslant m$ and $a'_i=a$. Then W^a will denote the subword $a_{i_1}a_{i_2}\cdots a_{i_p}$ of w.

For instance, for

$$W=\binom{w'}{w}=\begin{pmatrix}3&1&1&2&4&1&5\\2&\mathbf{3}&\mathbf{1}&2&2&\mathbf{4}&1\end{pmatrix},$$

we obtain

$$W^1=314.$$

Of course W^a is the empty word if a does not occur in w'.

LEMMA 10.3.1. *Let* $W_1=\binom{w'_1}{w_1}$ *and* $W_2=\binom{w'_2}{w_2}$ *be two equivalent elements of* $M(A)$. *Then*

(i) $W_1^a=W_2^a$ *holds for every* a *in* A;
(ii) *The word* w_2 (*resp.* w'_2) *is a rearrangement of* w_1 (*resp.* w'_1).

Proof. Properties (i) and (ii) trivially hold when W_1 and W_2 are adjacent. Hence, they are also true for any two equivalent elements of $M(A)$. ∎

THEOREM 10.3.2. (i) *Each nonempty flow* f *has a unique factorization of the form*

$$f=\begin{bmatrix}a_1^{m_1}\\v_1\end{bmatrix}\begin{bmatrix}a_2^{m_2}\\v_2\end{bmatrix}\cdots\begin{bmatrix}a_n^{m_n}\\v_n\end{bmatrix},\tag{10.3.3}$$

with $a_1,a_2,\ldots,a_n$ *in* A *satisfying* $a_1<a_2<\cdots<a_n$ *and* $m_1\geqslant 1, m_2\geqslant 1,\ldots,m_n\geqslant 1$.

(ii) *If* $W=\binom{w'}{w}$ *is any representative of* f, *then the word* $a_1^{m_1}a_2^{m_2}\cdots a_n^{m_n}$ *is the nondecreasing rearrangement of* w'.

(iii) *Finally*

$$W^{a_i}=v_i\quad \textit{for each}\quad i=1,2,\ldots,n.\tag{10.3.4}$$

Proof. Clearly (ii) and (iii) are consequences of (i) and the previous lemma. Let us establish (i).

As each flow of the form $\begin{bmatrix}a^m\\v\end{bmatrix}$ with a in A and v in A^* has a single representative $\binom{a^m}{v}$, it suffices to prove that each nonempty element of

$M(A)$ is equivalent to exactly one product of the form

$$\begin{pmatrix} a_1^{m_1} \\ v_1 \end{pmatrix}\begin{pmatrix} a_2^{m_2} \\ v_2 \end{pmatrix}\cdots\begin{pmatrix} a_n^{m_n} \\ v_n \end{pmatrix},$$

that is, to a matrix $\begin{pmatrix} v' \\ v \end{pmatrix}$ with v' nondecreasing.

Existence. Let $W=\begin{pmatrix} w' \\ w \end{pmatrix}$ be an element of $M(A)$ and assume that the nondecreasing rearrangement of w' is $a_1^{m_1}a_2^{m_2}\cdots a_n^{m_n}$. There is nothing to prove when W has length 1. Assume now that the length of W is at least 2. Denote by w_2' the longest right factor of w' having no letter equal to a_n. Then $w'=w_1'a_nw_2'$ for some word w_1'. Also

$$W=\begin{pmatrix} w_1' \\ w_1 \end{pmatrix}\begin{pmatrix} a_n \\ b \end{pmatrix}\begin{pmatrix} w_2' \\ w_2 \end{pmatrix}$$

for some words w_1, w_2, and some letter b. But

$$\begin{pmatrix} w_1' & w_2' & a_n \\ w_1 & w_2 & b \end{pmatrix}$$

is equivalent to

$$W=\begin{pmatrix} w_1' & a_n & w_2' \\ w_1 & b & w_2 \end{pmatrix}.$$

By induction the matrix

$$\begin{pmatrix} w_1' & w_2' \\ w_1 & w_2 \end{pmatrix}$$

is equivalent to an element $\begin{pmatrix} u' \\ u \end{pmatrix}$ with u' nondecreasing. As u' is a rearrangement of $w_1'\,w_2'$ the word $u'a_n$ is nondecreasing. Thus the element $\begin{pmatrix} u' & a_n \\ u & b \end{pmatrix}$ has the desired property. Moreover, it is equivalent to

$$\begin{pmatrix} w_1' & w_2' & a_n \\ w_1 & w_2 & b \end{pmatrix}$$

and, a fortiori, to $\begin{pmatrix} w' \\ w \end{pmatrix}$ for the equivalence relation is compatible with the product.

Unicity. Let $W^{a_i}=v_i$ for $i=1,2,\ldots,n$. Further, let

$$\begin{pmatrix} v' \\ v \end{pmatrix}=\begin{pmatrix} a_1^{m_1} & a_2^{m_2} & \cdots & a_n^{m_n} \\ v_1 & v_2 & \cdots & v_n \end{pmatrix}.$$

We show that each element $T=\begin{pmatrix} t' \\ t \end{pmatrix}$ equivalent to $\begin{pmatrix} w' \\ w \end{pmatrix}$, with t' nondecreasing, is necessarily equal to $\begin{pmatrix} v' \\ v \end{pmatrix}$. It follows from Lemma 10.3.1 (ii) that t' is equal to $a_1^{m_1}a_2^{m_2}\cdots a_n^{m_n}$. Hence $\begin{pmatrix} t' \\ t \end{pmatrix}$ is equal to the product

$$\begin{pmatrix} a_1^{m_1} \\ t_1 \end{pmatrix}\begin{pmatrix} a_2^{m_2} \\ t_2 \end{pmatrix}\cdots\begin{pmatrix} a_n^{m_n} \\ t_n \end{pmatrix}$$

for some words $t_1, t_2,\ldots,t_n$. But $T^{a_i}=t_i$ (where $1\leqslant i\leqslant n$). As T is equivalent to W, Lemma 10.3.1 (i) implies that $W^{a_i}=T^{a_i}$; that is, $v_i=t_i$ for each $i=1,2,\ldots,n$. Hence $t=t_1t_2\cdots t_n=v_1v_2\cdots v_n$. ■

COROLLARY 10.3.3. *The cancellation law holds in the flow monoid $F(A)$. In other words, for any flows f, f', f'' the equality $ff''=f'f''$ (resp. $f''f=f''f'$) implies $f=f'$.*

Proof. Let f be a flow and a, b two letters. Consider the equation in g

$$f=g\begin{bmatrix} a \\ b \end{bmatrix}. \tag{10.3.5}$$

Using the factorization of f given in Theorem 10.3.2 we have

$$\begin{bmatrix} a_1^{m_1} \\ v_1 \end{bmatrix}\begin{bmatrix} a_2^{m_2} \\ v_2 \end{bmatrix}\cdots\begin{bmatrix} a_n^{m_n} \\ v_n \end{bmatrix}=g\begin{bmatrix} a \\ b \end{bmatrix}.$$

But this equation has a solution only if a is equal to some a_i $(1\leqslant i\leqslant n)$. If $a=a_i$ (say), then $v_i=wb$ for some word w. Let

$$h=\begin{bmatrix} a_1^{m_1} \\ v_1 \end{bmatrix}\cdots\begin{bmatrix} a_i^{m_i-1} \\ w \end{bmatrix}\cdots\begin{bmatrix} a_n^{m_n} \\ v_n \end{bmatrix}.$$

Then

$$f=h\begin{bmatrix} a \\ b \end{bmatrix},$$

so that h is a solution of (10.3.5). Any other solution is of the form

$$g=\begin{bmatrix} a_1^{m_1} \\ u_1 \end{bmatrix}\cdots\begin{bmatrix} a_i^{m_i-1} \\ u_i \end{bmatrix}\cdots\begin{bmatrix} a_n^{m_n} \\ u_n \end{bmatrix}.$$

As

$$h\begin{bmatrix} a \\ b \end{bmatrix} = g\begin{bmatrix} a \\ b \end{bmatrix}$$

we conclude from Theorem 10.3.2 that necessarily $u_1 = v_1, \ldots, u_i b = wb, \ldots, u_n = v_n$. Therefore (10.3.5) has at most one solution.

Consider now three flows f, f', f''. As

$$f\begin{bmatrix} a \\ b \end{bmatrix} = f'\begin{bmatrix} a \\ b \end{bmatrix}$$

implies $f = f'$, we have by induction on the length of f'' that $ff'' = f'f''$ implies $f = f'$. The "resp." part is proved in the same manner. ■

10.4. The Circuit Monoid

Note that in each element $\begin{pmatrix} w' \\ w \end{pmatrix}$ of $M(A)$ the word w' is not necessarily a rearrangement of w (although it is of the same length). When it is a rearrangement, the equivalence class $\begin{bmatrix} w' \\ w \end{bmatrix}$ is called a *circuit*. Clearly, the set $C(A)$ of all circuits form a submonoid of $F(A)$, called the *circuit monoid*. It follows from Theorem 10.3.2 that each circuit c has one and only one representative in the form $\begin{bmatrix} \bar{v} \\ v \end{bmatrix}$ with v a word and $\bar{v}$ the nondecreasing rearrangement of v. Let

$$\bar{c} = \bar{v} \tag{10.4.1}$$

$$\Pi(c) = v. \tag{10.4.2}$$

Conversely, to each word v of A^* there corresponds one and only one two-row matrix of the form $\begin{bmatrix} \bar{v} \\ v \end{bmatrix}$ with $\bar{v}$ the nondecreasing rearrangement of v. Define the *circuit* $\Gamma(v)$ by

$$\Gamma(v) = \begin{bmatrix} \bar{v} \\ v \end{bmatrix}. \tag{10.4.3}$$

We then have $\Gamma\Pi(c) = c$ and $\Pi\Gamma(v) = v$ for each c in $C(A)$ and v in A^*. Thus the two maps

$$\Pi: C(A) \to A^* \quad \text{and} \quad \Gamma: A^* \to C(A) \tag{10.4.4}$$

are *bijective* and *inverses* of each other. Moreover

$$\overline{\Pi(c)} = \bar{c} \quad \text{and} \quad \overline{\Gamma(v)} = \bar{v}, \tag{10.4.5}$$

denoting again by $\bar{v}$ the nondecreasing rearrangement of v.

The definition of Γ (given in (10.4.3)) is straightforward. As for obtaining $\Pi(c)$ (whose definition is shown in (10.4.2)) we can proceed as follows: Take any representative $W=\binom{w'}{w}$ of c and let $a_1<a_2<\cdots<a_n$ be the distinct elements of A that occur in w' (or w). Then $\Pi(c)$ is the word $v_1 v_2 \cdots v_n$ with

$$W^{a_i}=v_i \; (i=1,2,\ldots,n). \tag{10.4.6}$$

The final step is to define another bijection Δ: $A^* \to C(A)$, and the fundamental transformation $w \mapsto \hat{w}$ will be the functional product $\Delta^{-1} \circ \Gamma$. For each word $w=a_1 a_2 \cdots a_m$ denote by δw the *cyclic shift*

$$\delta w = a_2 a_3 \cdots a_m a_1.$$

Remember that a word is said to be (initially) dominated if its first letter is (strictly) greater than all its other letters. In the same manner, a circuit c will be said to be *dominated* if

$$c=\begin{bmatrix} \delta w \\ w \end{bmatrix} \tag{10.4.7}$$

with w dominated. Clearly, for each circuit c there is at most one dominated word w such that (10.4.7) holds. When w is dominated, we will denote by $\gamma(w)$ the circuit

$$\gamma(w)=\begin{bmatrix} \delta w \\ w \end{bmatrix}. \tag{10.4.8}$$

Thus γ is a bijection of the set of all dominated words onto the set of dominated circuits. In (10.4.7) the first letter of the dominated word w, previously denoted by Fw, will also be written Fc.

By definition a *dominated circuit factorization* of a circuit c is a sequence $(d_1, d_2, \ldots, d_r)$ of dominated circuits with the property that

$$c = d_1 d_2 \cdots d_r$$

and

$$Fd_1 \leqslant Fd_2 \leqslant \cdots \leqslant Fd_r. \tag{10.4.9}$$

THEOREM 10.4.1. *Each nonempty circuit admits exactly one dominated circuit factorization.*

The proof of Theorem 10.4.1 actually gives the construction of the factorization. The dominated circuits are to be sorted one by one out of the initial circuit. Let us first prove the following lemma:

LEMMA 10.4.2. *Let v be a nonempty word and $\bar{v}$ be its nondecreasing rearrangement. If there exists an integer $i \geq 1$, a sequence $(a''_0, a''_1, \ldots, a''_i)$ of letters and an element $\begin{pmatrix} v'_i \\ v_i \end{pmatrix}$ of $M(A)$ with the properties that*

(i) *a''_0 is different from each of the letters $a''_1, \ldots, a''_i$;*
(ii) *$\binom{\bar{v}}{v}$ is equivalent to*

$$\begin{pmatrix} v'_i \\ v_i \end{pmatrix}\begin{pmatrix} a''_{i-1} & a''_{i-2} & \cdots & a''_1 & a''_0 \\ a''_i & a''_{i-1} & \cdots & a''_2 & a''_1 \end{pmatrix},$$

then a letter a''_{i+1} and a matrix $\begin{pmatrix} v'_{i+1} \\ v_{i+1} \end{pmatrix}$ of $M(A)$ can be found so that condition (ii) holds when i is replaced by $i+1$.

Proof. Define a''_{i+1} as the bottom element of the rightmost one-column submatrix of $\begin{pmatrix} v'_i \\ v_i \end{pmatrix}$ whose top element is equal to a''_i. This definition makes sense, for conditions (i) and (ii) imply that v'_i contains a letter equal to a''_i. Hence, the following factorization holds:

$$\begin{pmatrix} v'_i \\ v_i \end{pmatrix} = \begin{pmatrix} u'_1 \\ u_1 \end{pmatrix}\begin{pmatrix} a''_i \\ a''_{i+1} \end{pmatrix}\begin{pmatrix} u'_2 \\ u_2 \end{pmatrix}.$$

with no letter equal to a''_i in u'_2. Then put

$$\begin{pmatrix} v'_{i+1} \\ v_{i+1} \end{pmatrix} = \begin{pmatrix} u'_1 & u'_2 \\ u_1 & u_2 \end{pmatrix}.$$

The following two elements of $M(A)$:

$$\begin{pmatrix} v'_i \\ v_i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v'_{i+1} \\ v_{i+1} \end{pmatrix}\begin{pmatrix} a''_i \\ a''_{i+1} \end{pmatrix}$$

are equivalent. Thus condition (i) also holds when i is replaced by $i+1$. ∎

We are now ready to complete the proof of Theorem 10.4.1. Let $v = b_1 b_2 \cdots b_m$ be a word with a nondecreasing rearrangement given by $\bar{v} = a'_1 a'_2 \cdots a'_m = a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n}$ (where $a_1 < a_2 < \cdots < a_n$; $m_1 \geq 1, m_2 \geq 1, \ldots, m_n \geq 1$). Consider the circuit

$$c = \Gamma(v) = \begin{bmatrix} \bar{v} \\ v \end{bmatrix} = \begin{bmatrix} a'_1 & a'_2 & \cdots & a'_m \\ b_1 & b_2 & \cdots & b_m \end{bmatrix}.$$

If v is of length one, then

$$c=\begin{bmatrix}\bar{v}\\ v\end{bmatrix}=\begin{bmatrix}\delta v\\ v\end{bmatrix}$$

and the theorem is proved. Assume $m \geq 2$. If $b_m = a'_m$ $(=a_n)$, let

$$c'=\begin{bmatrix}a'_1 & a'_2 & \cdots & a'_{m-1}\\ b_1 & b_2 & \cdots & b_{m-1}\end{bmatrix} \quad\text{and}\quad d=\begin{bmatrix}a_n\\ a_n\end{bmatrix}.$$

Then

$$c=c'd.$$

By induction on m, the circuit c' admits the unique dominated circuit factorization

$$c'=d_1 d_2 \cdots d_r.$$

As d is trivially dominated and a_n is the maximum letter of v, we also have

$$Fd_1 \leq Fd_2 \leq \cdots \leq Fd_r \leq Fd.$$

Therefore c has the following dominated circuit factorization

$$c=d_1 d_2 \cdots d_r d.$$

If $b_m \neq a'_m$, let $a''_1 = b_m$, $a''_0 = a'_m$, $v'_1 = a'_1 a'_2 \cdots a'_{m-1}$, and $v_1 = b_1 b_2 \cdots b_{m-1}$.Then conditions (i) and (ii) of Lemma 10.4.2 both hold for $i=1$. By applying Lemma 10.4.2 inductively we can form a sequence $(a''_0, a''_1, \ldots)$ of letters. Let $i+1$ be the first integer for which Lemma 10.4.2 does not apply. Such an integer exists since the sequence is necessarily finite. We then have $a''_{i+1} = a''_0$, but still a''_0 different from each of the letters $a''_1, \ldots, a''_i$. If $i+1 \leq m-1$, let

$$c'=\begin{bmatrix}v'_{i+1}\\ v_i\end{bmatrix} \quad\text{and}\quad d=\begin{bmatrix}a''_i & a''_{i-1} & \cdots & a''_1 & a''_0\\ a''_0 & a''_i & \cdots & a''_2 & a''_1\end{bmatrix}.$$

Then d is dominated (by the maximum letter $a''_0 = a_n$). If $d_1 d_2 \cdots d_r$ is the dominated circuit factorization of c', then we conclude, as before, that

$$d_1 d_2 \cdots d_r d$$

is a dominated circuit factorization of c.

It remains for us to prove the unicity of the factorization. Let $c_1 c_2 \cdots c_q$ and $d_1 d_2 \cdots d_r$ be two dominated circuit factorizations of a circuit $c=\Gamma(v)$.

Let $v_q = b'_1 b'_2 \cdots b'_s$ and $w_r = b''_1 b''_2 \cdots b''_t$ be two dominated words defined by $\gamma(v_q) = c_q$ and $\gamma(w_r) = d_r$ (see (10.4.8)). Thus

$$c_q = \begin{bmatrix} \delta v_q \\ v_q \end{bmatrix} = \begin{bmatrix} b'_2 & \cdots & b'_s & b'_1 \\ b'_1 & \cdots & b'_{s-1} & b'_s \end{bmatrix}$$

and

$$d_r = \begin{bmatrix} \delta w_r \\ w_r \end{bmatrix} = \begin{bmatrix} b''_2 & \cdots & b''_t & b''_1 \\ b''_1 & \cdots & b''_{t-1} & b''_t \end{bmatrix}.$$

But b'_1 and b''_1 are both equal to the maximum letter of v. Therefore $b'_s = b''_t$. Assume $s \leqslant t$. By induction $b'_{s-1} = b''_{t-1}, \ldots, b'_1 = b''_{t-s+1}$. But $b'_1 = b''_1$. As w_r is dominated, the equation $b''_1 = b''_{t-s+1}$ can hold only if $t = s$. Hence $v_q = w_r$ and $c_q = d_r$. As the cancellation law holds in $F(A)$ (see Corollary 10.3.3), we obtain $c_1 c_2 \cdots c_{q-1} = d_1 d_2 \cdots d_{r-1}$. The unicity follows by induction on the length. ■

10.5. The First Fundamental Transformation for Arbitrary Words

We are now ready to define the second bijection $\Delta\colon A^* \to C(A)$. Let $(w_1, w_2, \ldots, w_r)$ be the increasing factorization of a word w (see Lemma 10.2.1). Remember that each factor w_i is dominated so that we can form the dominated circuit

$$\gamma(w_i) = \begin{bmatrix} \delta w_i \\ w_i \end{bmatrix}, \quad (i = 1, 2, \ldots, r).$$

Taking their product in $C(A)$ we obtain the circuit

$$\Delta(w) = \gamma(w_1)\gamma(w_2) \cdots \gamma(w_r). \tag{10.5.1}$$

By construction

$$\overline{\Delta(w)} = \bar{w} \tag{10.5.2}$$

(see definition (10.4.1)). On the other hand, γ is a bijection of the set of dominated words onto the set of dominated circuits with the property that $\overline{\gamma(w)} = \bar{w}$ (see (10.4.8)). The map that associates $(w_1, w_2, \ldots, w_r)$ to $(\gamma(w_1), \gamma(w_2), \ldots, \gamma(w_r))$ transforms the increasing factorization of w into the dominated circuit factorization of $\Delta(w)$. It then follows from Lemma 10.2.1 and Theorem 10.4.1 that Δ is *bijective*.

The essential property of Δ is that the adjacent letters ba of w with $a<b$ are transformed into vertical pairs $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\Delta(w)$. We make this definition more precise by introducing a function $\eta_{a,b}$ on circuits as follows: Let

$$c=\begin{bmatrix} a_1' & a_2' & \cdots & a_m' \\ a_1 & a_2 & \cdots & a_m \end{bmatrix}=\begin{bmatrix} a_1' \\ a_1 \end{bmatrix}\begin{bmatrix} a_2' \\ a_2 \end{bmatrix}\cdots\begin{bmatrix} a_m' \\ a_m \end{bmatrix}$$

be a circuit. Then $\eta_{a,b}(c)$ is defined as the number of *vertical occurrences* of $\begin{bmatrix} a \\ b \end{bmatrix}$ in c, that is, the number of integers i with $1 \leqslant i \leqslant m$ and $a_i'=a$, $b_i'=b$. Remember that $\xi_{a,b}(w)$ is the number of two-letter factors of the word w that are equal to ba.

THEOREM 10.5.1. *For each nonempty word w of A^* and each ordered pair (a,b) of letters satisfying $a<b$, we have*

$$\xi_{a,b}(w)=\eta_{a,b}(\Delta(w)). \tag{10.5.3}$$

Proof. Let $(w_1, w_2, \ldots, w_r)$ be the increasing factorization of a word w and $a<b$. First, b cannot be the last letter of w_j, while a is the first letter of the successive factor w_{j+1}. Second, each word $\begin{pmatrix} \delta w_j \\ w_j \end{pmatrix}$ of $M(A)$ cannot end with the letter $\begin{pmatrix} a \\ b \end{pmatrix}$. Hence the horizontal pairs ba and vertical pairs $\begin{pmatrix} a \\ b \end{pmatrix}$ can occur only as shown schematically in the next equation

$$\Delta(w)=\begin{bmatrix} \cdots \\ \cdots \end{bmatrix}\cdots\begin{bmatrix} \cdots a \cdots \\ \cdots ba \cdots \end{bmatrix}\cdots\begin{bmatrix} \cdots \\ \cdots \end{bmatrix}.$$

Thus Eq. (10.5.3) holds. ∎

Denote by Δ^{-1} the inverse of Δ that maps $C(A)$ onto A^*. If w is a word, let

$$\hat{w}=\Delta^{-1}(\Gamma(w)). \tag{10.5.4}$$

THEOREM 10.5.2. *The mapping $w \mapsto \hat{w}$ is a bijection of A^* onto itself having the following properties*:

(i) *$\hat{w}$ is a rearrangement of w;*
(ii) *for each pair (a,b) with $a<b$ then*

$$\nu_{a,b}(w)=\xi_{a,b}(\hat{w}). \tag{10.5.5}$$

Moreover

$$E(w)=D(\hat{w}). \tag{10.5.6}$$

Proof. As both Γ and Δ^{-1} are bijective, their composition product is also bijective. Property (i) follows from (10.4.5) and (10.5.2). Now let $w = a_1 a_2 \cdots a_m$ be a word and $\bar{w} = a'_1 a'_2 \cdots a'_m$ be its nondecreasing rearrangement. Remember that $\nu_{a,b}(w)$ is the number of integers i with $a'_i = a$ and $a_i = b$. The relation $\nu_{a,b}(w) = \eta_{a,b}(\Gamma(w))$ for $a < b$ is a trivial consequence of the definitions of $\nu_{a,b}$ and $\eta_{a,b}$. From (10.5.3) we deduce that

$$\xi_{a,b}(\hat{w}) = \eta_{a,b}(\Delta(\hat{w})) = \eta_{a,b}(\Gamma(w)) = \nu_{a,b}(w).$$

As for (10.5.6) it follows from (10.5.5) and definition (10.2.2). ■

Example 10.5.3. Let us illustrate with an example the construction of the bijection $w \mapsto \hat{w}$ and its inverse. Consider the word

$$w = 3\,1\,5\,1\,4\,2\,2\,6\,6\,7\,2\,6\,1\,5.$$

Its nondecreasing rearrangement reads

$$\bar{w} = 1\,1\,1\,2\,2\,2\,3\,4\,5\,5\,6\,6\,6\,7.$$

Then $\Gamma(w)$ (see (10.4.3)) is the circuit

$$\Gamma(w) = \begin{bmatrix} \bar{w} \\ w \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 4 & 5 & 5 & 6 & 6 & 6 & 7 \\ \mathbf{3} & 1 & \mathbf{5} & 1 & \mathbf{4} & 2 & 2 & \mathbf{6} & \mathbf{6} & \mathbf{7} & 2 & 6 & 1 & 5 \end{bmatrix}.$$

To compute $\Delta^{-1}(\Gamma(w))$ we have to determine the dominated circuit factorization of $\Gamma(w)$ as indicated in Theorem 10.4.1. This is achieved by sorting out the successive dominated circuits from right to left:

$$\begin{aligned}
\Gamma(w) &= \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 4 & 5 & 6 & 6 & 6 \\ 3 & 1 & 5 & 1 & 4 & 2 & 2 & 6 & 6 & 2 & 6 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 7 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 4 & 6 & 6 \\ 3 & 1 & 1 & 4 & 2 & 2 & 6 & 2 & 6 \end{bmatrix} \begin{bmatrix} 5 & 1 & 6 \\ 6 & 5 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 7 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 4 & 6 \\ 3 & 1 & 1 & 4 & 2 & 2 & 6 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 6 \end{bmatrix} \begin{bmatrix} 5 & 1 & 6 \\ 6 & 5 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 7 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 & 6 \\ 6 & 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 6 \end{bmatrix} \begin{bmatrix} 5 & 1 & 6 \\ 6 & 5 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 7 & 5 \end{bmatrix}.
\end{aligned}$$

The word $\hat{w} = \Delta^{-1}(\Gamma(w))$ is then the juxtaposition product of the words occurring in the bottom row in the last product, namely

$$\hat{w} = \widehat{3\,1}\,1\,2\,\widehat{6\,4\,2}\,2\,6\,\widehat{6\,5\,1}\,\widehat{7\,5}.$$

Conversely, to obtain w from $\hat{w}$—that is, $w = \Gamma^{-1}(\Delta(\hat{w}))$—we first form the increasing factorization of $\hat{w}$

$$(3112, 6422, 6, 651, 75),$$

then define $\Delta(\hat{w})$ (see (10.4.1)), namely

$$\Delta(\hat{w})=\begin{bmatrix}1 & 1 & 2 & 3 & 4 & 2 & 2 & 6 & 6 & 5 & 1 & 6 & 5 & 7\\ 3 & 1 & 1 & 2 & 6 & 4 & 2 & 2 & 6 & 6 & 5 & 1 & 7 & 5\end{bmatrix}.$$

Then we reshuffle all the columns of $\Delta(\hat{w})$ so that the top row is in increasing order (see (10.4.2)). We find again $\Gamma(w)$, and finally w occurs in the bottom row.

In the example boldface type in $\Gamma(w)$ marks the letters of w that are greater than the corresponding letters of $\bar{w}$ above them. We have the vertical pairs

$$\binom{1}{3}, \binom{1}{5}, \binom{2}{4}, \binom{4}{6}, \binom{5}{6}, \binom{5}{7}.$$

In $\hat{w}$ we have the horizontal pairs

$$31, 64, 42, 65, 51 \quad \text{and} \quad 75.$$

Thus $E(w)=6=D(\hat{w})$.

10.6. The Second Fundamental Transformation

Let $w=a_1a_2\cdots a_m$ be a word of length $m\geqslant 1$. Its *inversion number*, denoted by INV w, is defined as the number of ordered pairs (i, j) with $1\leqslant i<j\leqslant m$ and $a_i>a_j$. The *down set* of w, denoted by DOWN w, is defined by

$$\text{DOWN}\, w=\{i \mid 1\leqslant i\leqslant m-1, \quad a_i>a_{i+1}\}, \tag{10.6.1}$$

and the *major index* of w, denoted by MAJ w, as the sum (possibly zero) of the elements in DOWN w.

For instance, with

$$\begin{matrix} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\ w= & 4 & 4 & 2 & 3 & 4 & 1 & 3 & 2 & 3,\end{matrix}$$

we have INV $w=20$, DOWN $w=\{2,5,7\}$, so that MAJ $w=2+5+7=14$.

MacMahon (1913, 1915) introduced the function MAJ in the study of ordered partitions. Let X be a rearrangement class of A^* (see Section 10.1) and let

$$I(q)=\sum q^{\text{INV}\, w},\ M(q)=\sum q^{\text{MAJ}\, w}\ (w\in X) \tag{10.6.2}$$

be the generating functions for X by number of inversions and major index, respectively. MacMahon obtained (1916) the surprising result that $I(q)$ and

$M(q)$ have the same expression (see Problem 10.10). Schützenberger (private communication 1966) raised the problem of finding a bijection Φ of A^* onto itself with the property that for every word w

(i) $\Phi(w)$ is a rearrangement of w;
(ii) $\mathrm{INV}\,\Phi(w) = \mathrm{MAJ}\,w$.

The bijection Φ described below was found in Foata (1968) and later referred to as the "second fundamental transformation." When Φ is restricted to the permutation group, it has further interesting properties (see Problem 10.6.3).

The construction of Φ goes as follows: Let a be an element of A and w a nonempty word. If the last letter of w is smaller than or equal to (resp. is greater than) a, the word w clearly admits the unique factorization

$$(v_1 b_1, v_2 b_2, \ldots, v_p b_p)$$

called its *a-factorization* having the following properties:

(i) Each b_i $(1 \leqslant i \leqslant p)$ is a letter satisfying $b_i \leqslant a$ (resp. $b_i > a$);
(ii) Each v_i $(1 \leqslant i \leqslant p)$ is a word that is either empty or has all its letters greater than (resp. smaller than or equal to) a.

Then let

$$\gamma_a(w) = b_1 v_1 b_2 v_2 \cdots b_p v_p.$$

(Note that $w = v_1 b_1 v_2 b_2 \cdots v_p b_p$.) The bijection will be defined by induction on the length of the words as follows:

If w has length 1, let

$$\Phi(w) = w; \tag{10.6.3}$$

If $|w| \geqslant 2$, write $w = va$ with a the last letter of w. By induction determine the word $v' = \gamma_a(\Phi(v))$ and let $\Phi(w)$ be the juxtaposition product

$$\Phi(w) = v'a(= \gamma_a(\Phi(v))a). \tag{10.6.4}$$

Let us describe the effective algorithm for Φ.

ALGORITHM 10.6.1. *Let* $w = a_1 a_2 \cdots a_m$ *be a word;*

1. *Let* $i = 1, w_1' = a_1$;
2. *If* $i = m$, let $\Phi(w) = w_i'$ *and stop*; *else continue*;

3. *If the last letter of* w_i' *is smaller than or equal to (resp. greater than)* a_{i+1}, *split* w_i' *after each letter smaller than or equal to (resp. greater than)* a_{i+1};
4. *In each compartment of* w_i' *determined by the splits move the last letter to the beginning; let* v' *be the word obtained after making those moves; let* $w_{i+1}' = v' a_{i+1}$; *replace* i *by* $i+1$; *go to* 2.

For instance, the image under Φ of the word $w = 442341323$ is obtained as follows.

$$\begin{aligned} w_1' &= 4| \\ w_2' &= 4|4| \\ w_3' &= 442| \\ w_4' &= 2|4|4|3| \\ w_5' &= 2|4|4|3|4| \\ w_6' &= 2|443|41| \\ w_7' &= 23|4|4|14|3| \\ w_8' &= 3|2|4441|3|2| \\ \Phi(w) = w_9' &= 321444323. \end{aligned}$$

The algorithm can be reversed (see Problem 10.6.1).

THEOREM 10.6.2. *The map* Φ *is bijective. Furthermore, the image* $\Phi(w)$ *of each word* w *is a rearrangement of* w. *Finally, the following identity holds*

$$\mathrm{INV}\,\Phi(w) = \mathrm{MAJ}\,w. \tag{10.6.5}$$

Proof. The first two statements are easy to prove. As for the last one, let $w = a_1 a_2 \cdots a_m$ and for each a in A let $l_a(w)$ (resp. $r_a(w)$) be the number of subscripts i with $1 \leq i \leq m$ and $a_i \leq a$ (resp. $a_i > a$). Of course, $l_a(w) + r_a(w) = |w|$. Furthermore,

$$\mathrm{INV}\,wa = \mathrm{INV}\,w + r_a(w).$$

Now if the last letter of w is less than or equal to a, we have

$$\begin{aligned} \mathrm{INV}\,\gamma_a(w) &= \mathrm{INV}\,w - r_a(w) \\ \mathrm{MAJ}\,wa &= \mathrm{MAJ}\,w. \end{aligned}$$

When the last letter of w is greater than a, we have this time

$$\begin{aligned} \mathrm{INV}\,\gamma_a(w) &= \mathrm{INV}\,w + l_a(w) \\ \mathrm{MAJ}\,wa &= \mathrm{MAJ}\,w + l_a(w) + r_a(w). \end{aligned}$$

Property (10.6.5) is then a consequence of these five relations together with (10.6.4). First

$$\begin{aligned}\mathrm{INV}\,\Phi(wa) &= \mathrm{INV}\,\gamma_a(\Phi(w))a\\ &= \mathrm{INV}\,\gamma_a(\Phi(w)) + r_a(\gamma_a(\Phi(w)))\\ &= \mathrm{INV}\,\gamma_a(\Phi(w)) + r_a(w),\end{aligned}$$

since $\gamma_a(\Phi(w))$ is only a rearrangement of w. Then, if the last letter of w is less than or equal to a, we get (by induction):

$$\begin{aligned}\mathrm{INV}\,\Phi(wa) &= \mathrm{INV}\,\gamma_a(\Phi(w)) + r_a(w)\\ &= (\mathrm{INV}\,\Phi(w) - r_a(w)) + r_a(w)\\ &= \mathrm{MAJ}\,w\\ &= \mathrm{MAJ}\,wa.\end{aligned}$$

Finally, if the last letter of w is greater than a, we have (by induction)

$$\begin{aligned}\mathrm{INV}\,\Phi(wa) &= \mathrm{INV}\,\gamma_a(\Phi(w)) + r_a(w)\\ &= \mathrm{INV}\,\Phi(w) + l_a(w) + r_a(w)\\ &= \mathrm{MAJ}\,w + |w|\\ &= \mathrm{MAJ}\,wa.\end{aligned}$$

■

Working with the foregoing example $w = 4\ 4\ 2\ 3\ 4\ 1\ 3\ 2\ 3$ and $\Phi(w) = 3\ 2\ 1\ 4\ 4\ 4\ 3\ 2\ 3$, we obtain

$$\mathrm{MAJ}\,w = 14 = \mathrm{INV}\,\Phi(w).$$

10.7. The Sparre-Andersen Equivalence Principle

The Sparre-Andersen equivalence principle was presented in Proposition 5.2.9. Let us recall that if $w = a_1 a_2 \cdots a_m$ is a word of A^* with A the field of the real numbers, we let

$$\sigma(w) = a_1 + a_2 + \cdots + a_m \tag{10.7.1}$$

be the *total sum* and $\sigma_0(w) = 0, \sigma_1(w) = \sigma(a_1) = a_1, \sigma_2(w) = \sigma(a_1 a_2) = a_1 + a_2, \ldots, \sigma_m(w) = \sigma(a_1 a_2 \cdots a_m) = a_1 + a_2 + \cdots + a_m$ be the *partial sums* of w. For each $k = 0, 1, \ldots, m$ let $\Pi_{m,k}(w)$ be the number of subscripts i for which

- Either $0 \leqslant i \leqslant k-1$ and $\sigma_i(w) \geqslant \sigma_k(w)$,
- Or $k+1 \leqslant i \leqslant m$ and $\sigma_i(w) > \sigma_k(w)$. (10.7.2)

Thus $\Pi_{m,k}(w)$ is the *number of partial sums greater than or equal to* $\sigma_k(w)$ (with the convention that whenever two partial sums are equal the left-hand one is counted before the right-hand one with $\sigma_k(w)$ itself not being counted.)

For instance, with $m=8, k=5$, and $w=1,-2,0,3,-1,1,-2,1$, the sequence of partial sums is $(0,1,-1,-1,2,1,2,0,1)$, as graphically represented in Figure 10.1. As $\sigma_k(w)=\sigma_5(w)=1$, we obtain $\Pi_{8,5}(w)=3$.

Clearly $\Pi_{m,0}(w)$ is the *number of* (strictly) *positive partial sums* that was denoted by $L(w)$ in Chapter 5. Now the *index of the first maximum* in $(\sigma_0(w), \sigma_1(w), \ldots, \sigma_m(w))$ is equal to k—a quantity denoted by $\Pi(w)$ in Chapter 5—if and only if $\Pi_{m,k}(w)=0$. The Sparre-Andersen equivalence principle expresses the fact that in each rearrangement class X there are for each integer k as many words with $L(w)\ (=\Pi_{m,0}(w))=k$ as words w' with $\Pi(w')=k$; that is, $\Pi_{m,k}(w')=0$.

Later Sparre-Andersen (1962) obtained a further extension of his principle as follows:

THEOREM 10.7.1. *Let j, k be two integers with $0 \leqslant j, k \leqslant m$. Then in each rearrangement class X of words of length m there are as many words w with $\Pi_{m,k}(w)=j$ as words w' with $\Pi_{m,j}(w')=k$.*

Theorem 10.7.1 will be proved by constructing a bijection ρ_k of X onto itself with the property that

$$\Pi_{m,k}(w)=j \Leftrightarrow \Pi_{m,j}(\rho_k(w))=k. \tag{10.7.3}$$

The following bijection ρ was defined in Chapter 5. If w is the empty word 1, let $\rho(w)=w$, while if a is a letter of A, define by induction on the length m of w

$$\begin{aligned}\rho(wa) &= \rho(w)a \quad \text{if} \quad \sigma_{m+1}(wa) \leqslant 0\\ &= a\rho(w) \quad \text{otherwise.}\end{aligned} \tag{10.7.4}$$

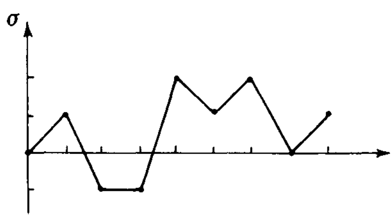

Figure 10.1. Graph associated with a sequence of partial sums.

It was shown in Proposition 5.2.9 that ρ was a bijection of each rearrangement class onto itself and also

$$L(w)=\Pi(\rho(w)),$$

that is,

$$\Pi_{m,0}(w)=j\Leftrightarrow\Pi_{m,j}(\rho(w))=0. \tag{10.7.5}$$

Clearly (10.7.5) is a particular case of (10.7.3) obtained for $k=0$. But how is ρ_k to be defined for the other values of k? This is the purpose of this section.

In Example 5.2.8 it was noted that the two sets

$$R=\{r\in A^*|r=uv\Rightarrow\sigma(v)>0\}$$
$$S=\{s\in A^*|s=uv\Rightarrow\sigma(u)\leqslant 0\}$$

were submonoids of A^* and each word w' has a unique factorization

$$w'=rs,\quad r\in R,\quad s\in S. \tag{10.7.6}$$

Moreover, it was shown that the length of r is equal to the index of the first maximum in the sequence of the partial sums of w', a result that can also be expressed, if $|w|=m$, by

$$|r|=j\Leftrightarrow\Pi_{m,j}(w')=0.$$

It follows from (10.7.5) that if

$$\rho(w)=rs,\quad r\in R,\quad s\in S, \tag{10.7.7}$$

then

$$\Pi_{m,0}(w)=|r|. \tag{10.7.8}$$

CONSTRUCTION OF THE BIJECTION ρ_k. Let k be a fixed integer with $0\leqslant k\leqslant m$ and $w=a_1a_2\cdots a_m$ be a word. To obtain $\rho_k(w)$ calculate successively

1. w_1 and w_2 by

$$w=w_1w_2,\quad |w_1|=k;$$

2. r_1, r_2, s_1, s_2 by

$\rho(\widetilde{w}_1)=r_1s_1, \rho(w_2)=r_2s_2$, with $r_1, r_2\in R$ and $s_1, s_2\in S$ and ρ defined in (10.7.4);

3. u_1 and u_2 by

$$u_1 = \rho^{-1}(r_2 s_1), \qquad u_2 = \rho^{-1}(r_1 s_2);$$

4. $\rho_k(w) = \widetilde{u}_1 u_2$.

THEOREM 10.7.3. *The mapping ρ_k defined previously maps each rearrangement class onto itself and satisfies property* (10.7.3).

Proof. Clearly ρ_k is a rearrangement. As ρ is bijective and the factorization given in (10.7.6) is unique, the map ρ_k is also bijective. On the other hand, $\Pi_{m,k}(w)$ is also the number of subscripts i for which

- Either $0 \leqslant i \leqslant k-1$ and $\sigma(a_{i+1}a_{i+2}\cdots a_k) \leqslant 0$;
- Or $k+1 \leqslant i \leqslant m$ and $\sigma(a_{k+1}a_{k+2}\cdots a_i) > 0$.

Thus

$$\Pi_{m,k}(w) = \Pi_{k,k}(w_1) + \Pi_{m-k,0}(w_2).$$

As the reverse image $\widetilde{w}_1$ of w_1 has the same total sum as w_1, we deduce that

$$\Pi_{k,k}(w_1) = k - \Pi_{k,0}(\widetilde{w}_1).$$

Therefore

$$\Pi_{m,k}(w) = k - \Pi_{k,0}(\widetilde{w}_1) + \Pi_{m-k,0}(w_2). \qquad (10.7.9)$$

Let $|u_1| = |\widetilde{u}_1| = j$. In the same manner

$$\Pi_{m,j}(\rho_k(w)) = j - \Pi_{j,0}(u_1) + \Pi_{m-j,0}(u_2).$$

On the other hand, (10.7.7) and (10.7.8) applied to $\widetilde{w}_1, w_2, u_1$, and u_2 yield

$$\Pi_{k,0}(\widetilde{w}_1) = |r_1|, \qquad \Pi_{m-k,0}(w_2) = |r_2|,$$
$$\Pi_{j,0}(u_1) = |r_2|, \qquad \Pi_{m-j,0}(u_2) = |r_1|.$$

Hence

$$\begin{aligned}\Pi_{m,k}(w) &= k - |r_1| + |r_2| \\ &= |s_1| + |r_2| = |u_1| = j,\end{aligned}$$

and

$$\begin{aligned}\Pi_{m,j}(\rho_k(w)) &= j - |r_2| + |r_1| \\ &= |s_1| + |r_1| = |w_1| = k.\end{aligned}$$

Thus property (10.7.3) is verified. ■

Example 10.7.4. The construction of ρ_k can be illustrated with the example shown in Figure 10.1. There $m = 8$, $k = 5$, $w = 1, -2, 0, 3, -1, 1, -2, 1$ and $\Pi_{8,5}(w) = j = 3$. Again consider the four steps of the construction of ρ_k

1. $w_1 = 1, -2, 0, 3, -1$; $\quad w_2 = 1, -2, 1$;
2. $\rho(\widetilde{w}_1) = 1, 0, 3, -1, -2$; $\quad \rho(w_2) = 1, -2, 1$.

As the indices of the first maxima of the partial sums of $\rho(\widetilde{w}_1)$ and $\rho(w_2)$ are equal to 3 and 1, respectively, we have

$$r_1 = 1, 0, 3; \qquad s_1 = -1, -2; \qquad r_2 = 1; \qquad s_2 = -2, 1;$$

3. $u_1 = \rho^{-1}(r_2 s_1) = 1, -1, -2$;
   $u_2 = \rho^{-1}(r_1 s_2) = -2, 1, 3, 0, 1$.
4. $\rho_k(w) = \widetilde{u}_1 u_2 = -2, -1, 1, -2, 1, 3, 0, 1$, which corresponds to the partial sum graph drawn in Figure 10.2.

The number of partial sums of $\rho_k(w)$ that are greater than or equal to $\sigma_j(\rho_k(w)) = \sigma_3(\rho_5(w)) = -2$ is equal to $k = 5$.

Notes

The first fundamental transformation for permutations is already implicit in Riordan (1958, Chapter 8). It is an essential tool in the study of Eulerian polynomials, as shown in Foata and Schützenberger (1970). The extension of the first fundamental transformation to arbitrary words was obtained by Foata (1965). Then Cartier and Foata (1969) derived a convenient set-up to describe it first by introducing the monoids subject to commutation rules, second by developing the study of the flow and circuit monoids. Lallement (1977) took up again this study in one chapter of his book. The circuit monoid was also used by Foata (1979, 1980), in particular to derive a

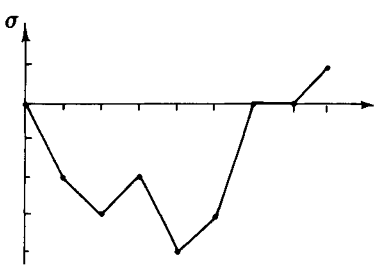

Figure 10.2. Graph associated with a sequence of partial sums.

noncommutative version of the matrix inversion formula. Möbius inversion identities can be obtained for commutation rule monoids (see Cartier and Foata 1969). Content, Lemay, and Leroux (1980) proposed a general setting for Möbius inversion that includes locally finite partially ordered sets and commutation rule monoids. The second fundamental transformation was derived by Foata (1968) and further used in Foata (1977) and Foata and Schützenberger (1978). See MacMahon (1913, 1915, 1916) for the first studies of the major index. Several multivariate distributions on $\mathfrak{S}_n$ involving the major index and inversion number have been calculated, particularly by Stanley (1976), Gessel (1977), Garsia and Gessel (1979), Rawlings (1981). The extension of the equivalence principle is due to Sparre-Andersen (1962). Other combinatorial constructions have been found that basically involve rearrangements of sequences. See for example Dumont and Viennot (1980), Dumont (1981) and Strehl (1981).

Problems

Section 10.2

10.2.1. For $0 \leqslant k \leqslant n$ let $A_{n,k}$ denote the number of permutations in $\mathfrak{S}_n$ having k descents. Take a permutation $w = a_1 a_2 \cdots a_{n-1}$ $(n \geqslant 2)$ and insert n before w, after w or between two letters. The number of descents remains alike or increases by one. This provides the recurrence relation for the *Eulerian numbers* $A_{n,k}$, that reads

$$A_{1,0} = 1, \qquad A_{1,k} = 0 \quad \text{for} \quad k \neq 1$$

and for $n \geqslant 2$ and $0 \leqslant k \leqslant n-1$

$$A_{n,k} = (k+1)A_{n-1,k} + (n-k)A_{n-1,k-1}.$$

(See Foata and Schützenberger 1970).

10.2.2. For each permutation $w = a_1 a_2 \cdots a_n$ the *number of rises* of w, denoted by $R(w)$, is defined to be the number of integers j with $0 \leqslant j \leqslant n-1$ and $a_j < a_{j+1}$ (by convention $a_0 = 0$), while the *number of* 0-*exceedances* of w, denoted by $E_0(w)$, is the number of integers j with $1 \leqslant j \leqslant n$ and $a_j \geqslant j$. Note that $E_0 \neq E + 1$.

Consider the following sequence

$$w = a_1 a_2 \cdots a_n,$$
$$w_1 = a_2 a_3 \cdots a_n a_1$$
$$w_2 = \hat{w}_1 \text{ ("first fundamental transformation")}$$
$$w_3 = \widetilde{w}_2 \text{ (reverse image).}$$

The mappings $w \mapsto w_3$ and $w_2 \mapsto w_3$ are bijections of $\mathfrak{S}_n$ onto itself with the property that

$$E_0(w) = R(w_3) = (1+D)(w_2).$$

(See Foata and Schützenberger 1970.)

10.2.3. For each positive integer n let

$$A_n(t) = \sum A_{n,k} t^k \; (0 \leq k \leq n-1)$$

be the nth *Eulerian polynomial.* From Problem 10.2.1 it follows that

$$A_n(t) = \sum t^{D(w)} \; (w \in \mathfrak{S}_n),$$

and from Problem 10.2.2

$$tA_n(t) = \sum t^{R(w)} \; (w \in \mathfrak{S}_n).$$

By classifying the permutations according to the position of the letter n we have

$$A_n(t) = A_{n-1}(t) + t \sum \binom{n-1}{m} A_m(t) A_{n-1-m}(t) \qquad (0 \leq m \leq n-2; \quad n \geq 1)$$

and

$$tA_n(t) = \sum \binom{n-1}{m} tA_m(t) \cdot tA_{n-1-m}(t) \qquad (0 \leq m \leq n-1; \quad n \geq 1)$$

The former identity is equivalent to

$$1 + \sum A_n(t) u^n/n! = \exp\Big(u + \sum tA_{m-1}(t) u^m/m!\Big) \qquad (n \geq 1; \quad m \geq 2),$$

whereas the latter one is equivalent to

$$1 + \sum tA_n(t) u^n/n! = \exp \sum tA_{n-1}(t) u^n/n! \quad (n \geq 1).$$

The last two identities form a system of two equations with two unknowns. Solving this system yields

$$1 + \sum A_n(t) u^n/n! = (1-t)/(-t + \exp(ut - u))$$
$$1 + \sum tA_n(t) u^n/n! = (1-t)/(1 - t\exp(u - ut)) \quad (n \geq 1).$$

Section 10.4

10.4.1. Let Π be the bijection of the circuit monoid $C(A)$ onto A^* and Γ be the inverse bijection, as they were defined in Section 10.4. For each pair of words w, w' in A^* the formula $w\tau w' = \Pi(\Gamma(w)\Gamma(w'))$ defines a new product in A^*, called the *intercalation product*. The ordered pair $C'(A) = (A^*, \tau)$ is called the *intercalation monoid*. It is isomorphic to $C(A)$. Let w, w' be two words and denote by $a_1, a_2, \ldots, a_n$ the increasing sequence of the letters occurring in either w or w'. Let $m_i = |w|_{a_i}$ (resp. $m'_i = |w'|_{a_i}$) be the number of occurrences of a_i in w (resp. in w') and $(w_1, w_2, \ldots, w_n)$ (resp. $(w'_1, w'_2, \ldots, w'_n)$) be the factorization of w defined by $|w_i| = m_i$ (resp. of w' defined by $|w'_i| = m'_i$). Then

$$w\tau w' = w_1 w'_1 w_2 w'_2 \cdots w_n w'_n.$$

For instance, with $w = 311454$ and $w' = 52243$ we have $w\tau w' = 31521245443$. (See Cartier and Foata 1969.)

10.4.2. A *cycle* is defined to be a nonempty circuit

$$c = \begin{bmatrix} \delta w \\ w \end{bmatrix} = \begin{bmatrix} a_2 a_3 \cdots a_m & a_1 \\ a_1 a_2 \cdots a_{m-1} & a_m \end{bmatrix} \quad \text{with} \quad w = a_1 a_2 \cdots a_m$$

standard. Two cycles $c = \begin{bmatrix} \delta w \\ w \end{bmatrix}$ and $c' = \begin{bmatrix} \delta w' \\ w' \end{bmatrix}$ are said to be *disjoint* if w and w' have no letter in common. The circuit monoid $C(A)$ is generated by the set of all cycles submitted to the following commutation rule that $cc' = c'c$ whenever c and c' are disjoint. (See Cartier and Foata 1969.)

10.4.3. Let n be a positive integer and A be the finite alphabet $\{1, 2, \ldots, n\}$. Construct the circuit monoid $C(A)$. If a circuit c is a product of exactly $p(c)$ disjoint cycles, let

$$\mu(c) = (-1)^{p(c)}.$$

In the other cases, let $\mu(c) = 0$. The *characteristic series* of $C(A)$ is given by

$$\sum c = \left(\sum \mu(c) c\right)^{-1} \quad (c \in C(A)).$$

(See Cartier and Foata 1969.)

10.4.4. Let $B = (b(i, j))$ $(1 \leq i, j \leq n)$ be an $n \times n$ matrix. We assume that the n^2 entries $b(i, j)$ are indeterminates subject to the following commutation rule that $b(i, j)$ and $b(i', j')$ commute whenever i and

i' are distinct. Let $\mathbf{Z}[[B]]$ denote the $\mathbf{Z}$-algebra of formal power series in the variables $b(i, j)$s (still subject to the foregoing commutation rule). The polynomial $\det(I-B)$ (with I the identity matrix of order n) belongs to $\mathbf{Z}[[B]]$. For each nonempty circuit

$$c=\begin{bmatrix} a_1' & a_2' & \cdots & a_m' \\ a_1 & a_2 & \cdots & a_m \end{bmatrix}$$

let

$$\beta(c)=b(a_1', a_1)\,b(a_2', a_1)\cdots b(a_m', a_m)$$

and $\beta(c)=1$ if c is the empty circuit. The following identity holds:

$$\det(I-B)=\sum \mu(c)\beta(c).$$

By extending β to a homomorphism of the large algebra of $C(A)$ into $\mathbf{Z}[[B]]$ we deduce from Problem 10.4.3 that

$$(\det(I-B))^{-1}=\sum \beta(c) \qquad (c\in C(A)).$$

(See Cartier and Foata 1969.)

10.4.5. Let $X_1, X_2, \ldots, X_n$ be n commuting variables, and let $B'=(b'_{ij})$ $(1\leqslant i, j\leqslant n)$ be a matrix with real entries. Let $\alpha(b(i, j))=b'_{ij}X_j$ and extend the definition of α to all of $\mathbf{Z}[[B]]$ by linearity. The image under α of the latter identity is

$$\begin{vmatrix} 1-b'_{11}X_1 & \cdots & -b'_{1n}X_n \\ \cdots & \cdots & \cdots \\ -b'_{n1}X_1 & \cdots & 1-b'_{nn}X_n \end{vmatrix}^{-1} =$$

$$\sum a(m_1, m_2, \ldots, m_n) X_1^{m_1} X_2^{m_2} \cdots X_n^{m_n}$$

$$(m_1\geqslant 0, m_2\geqslant 0, \ldots, m_n\geqslant 0),$$

where $a(m_1, m_2, \ldots, m_n)$ is the coefficient of the monomial $X_1^{m_1}X_2^{m_2}$ $\cdots X_n^{m_n}$ in the expansion of

$$\Big(\sum_j b'_{1j}X_j\Big)^{m_1}\Big(\sum_j b'_{2j}X_j\Big)^{m_2}\cdots\Big(\sum_j b'_{nj}X_j\Big)^{m_n}.$$

This identity constitutes the essence of the MacMahon Master Theorem. (See Cartier and Foata 1969.)

Section 10.6

10.6.1. The algorithm for the inverse Φ^{-1} of the second fundamental transformation can be described as follows:

Let $w = a_1 a_2 \cdots a_m$ be a word.

1. Let $i = m$, $v' = a_1 a_2 \cdots a_m$;
2. Let b_i be the last letter of v'; if $i = 1$, let $\Phi^{-1}(w) = b_1 b_2 \cdots b_m$, else let v_{i-1} be defined by $v' = v_{i-1} b_i$;
3. If the first letter of v_{i-1} is greater than (resp. smaller than or equal to) b_i, split v_{i-1} before each letter greater than (resp. smaller than or equal to) b_i;
4. In each compartment of v_{i-1} determined by the splits move the first letter to the end; let v' be the word obtained after making those moves; replace i by $i-1$ and go to 2.

(See Foata 1968; Foata and Schützenberger 1978.)

10.6.2. Let q be a real or complex variable and for each positive integer m let $[m] = 1 + q + q^2 + \cdots + q^{m-1}$ and $[m] = 1$ if $m = 0$. Also let $[m]! = [m][m-1] \cdots [2][1]$. Let X be the rearrangement class of the word $1^{m_1} 2^{m_2} \cdots n^{m_n}$. Then

$$\frac{[m_1 + m_2 + \cdots + m_n]!}{[m_1]![m_2]! \cdots [m_n]!} = \sum q^{\mathrm{INV}\, w} = \sum q^{\mathrm{MAJ}\, w} \quad (w \in X).$$

(See Andrews 1976, Chapter 3).

10.6.3. For each $w = a_1 a_2 \cdots a_n$ in the permutation group $\mathfrak{S}_n$ let IDOWN w be the down set of the inverse w^{-1} (in the group $\mathfrak{S}_n$). Clearly, the integer i belongs to IDOWN w if and only if in the word $a_1 a_2 \cdots a_n$ the letter $i+1$ occurs to the left of the letter i. Let IMAJ w be the sum of the elements in IDOWN w. The second fundamental transformation, *restricted to the permutation group* $\mathfrak{S}_n$, preserves IDOWN; that is:

$$\mathrm{IDOWN}\, \Phi(w) = \mathrm{IDOWN}\, w.$$

Denote by $\mathbf{i}(w)$ the inverse w^{-1} of the permutation w and consider the transformation

$$\Psi = \mathbf{i} \Phi \mathbf{i} \Phi^{-1} \mathbf{i}.$$

Then Ψ is a bijection of $\mathfrak{S}_n$ onto itself with the property that

$$\mathrm{MAJ}\, \Psi(w) = \mathrm{INV}\, w \quad \text{and} \quad \mathrm{INV}\, \Psi(w) = \mathrm{MAJ}\, w.$$

In particular, the six ordered pairs (MAJ, INV), (IMAJ, INV), (IMAJ, MAJ), (MAJ, IMAJ), (INV, IMAJ), and (INV, MAJ) have the same bivariate distribution on $\mathfrak{S}_n$. (See Foata and Schützenberger 1978.)

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