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Combinatorics of Laguerre Polynomials

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Abstract. Classical formulas on Laguerre polynomials, especially the bilinear Hille-Hardy-Erdélyi identity are derived by combinatorial methods. A multilinear extension of the latter identity is also proposed and proved by the same combinatorial techniques.

1. Introduction

Let $L_n^{(\alpha)}(x)$ $(n \geq 0)$ be the Laguerre polynomials defined by the Rodrigues formula

$$L_n^{(\alpha)}(x) = (e^x x^{-\alpha}/n!)(d/dx)^n(e^{-x}x^{-n+\alpha}) \tag{1.1}$$

For each complex number a, b, let

$$(a)_0 = 1 \quad \text{and} \quad (a)_n = a(a+1)...(a+n-1)\ (n \geq 1)$$

and for each variable u let

$$ {}_0F_1\Bigl({- \atop a}; u\Bigr) = \sum_n \frac{1}{(a)_n}\frac{u^n}{n!} \quad (n \geq 0) \tag{1.2}$$

and

$$ {}_1F_1\Bigl({a \atop b}; u\Bigr) = \sum_n \frac{(a)_n}{(b)_n}\frac{u^n}{n!} \quad (n \geq 0) \tag{1.3}$$

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be the usual hypergeometric series. The formula

$$L_n^{(\alpha)}(x) = \sum_k \frac{1}{k!\,(n-k)!}(\alpha+k+1)_{n-k}(-x)^k \quad (0 \le k \le n) \tag{1.4}$$

is readily derived from (1.1) by using Leibniz's rule, whereas the generating function

$$\sum_n \frac{u^n}{(\alpha+1)_n} L_n^{(\alpha)}(x) = e^u\,{}_0F_1\Big({- \atop \alpha+1}; -xu\Big) \quad (n \ge 0) \tag{1.5}$$

is a straightforward rewriting of (1.4).

The other two generating functions

$$\sum_n u^n L_n^{(\alpha)}(x) = (1-u)^{-\alpha-1}\exp(-xu/(1-u)) \quad (n \ge 0) \tag{1.6}$$

$$\sum_n \frac{u^n}{(\alpha+1)_n}(\beta)_n L_n^{(\alpha)}(x) = (1-u)^{-\beta}\,{}_1F_1\Big(\frac{\beta}{\alpha+1}; \frac{-xu}{1-u}\Big) \quad (n \ge 0) \tag{1.7}$$

as well as the bilinear formula, usually referred to as the Hille-Hardy identity,

$$\sum_n \frac{u^n}{(\alpha+1)_n} n!\, L_n^{\alpha}(x) L_n^{\alpha}(y) \tag{1.8}$$

$$= (1-u)^{-\alpha-1}\exp\Big(\frac{-(x+y)u}{(1-u)}\Big)\,{}_0F_1\Big({- \atop \alpha+1}; \frac{xyu}{(1-u)^2}\Big) \quad (n \ge 0)$$

require more elaborate methods. (See, e.g,. Rainville [27, chap. 13], Askey [2,3].)

Let k be a positive integer and $a_1, \ldots, a_k, c$ be $(k+1)$ complex numbers. Also let $x_1, \ldots, x_k$ be complex variables. The *k-variable confluent hypergeometric series* [1, 8 ,9] are defined by

$$\Phi_k\,(a_1, \ldots, a_k; c; x_1, \ldots, x_k)$$
$$= \sum_{n_1,\ldots,n_k} \frac{(a_1)_{n_1}\cdots(a_k)_{n_k}}{(c)_{n_1+\ldots+n_k}} \frac{x_1^{n_1}}{n_1!}\cdots\frac{x_k^{n_k}}{n_k!} \quad (n_1 \ge 0, \ldots, n_k \ge 0)$$

For $k = 1$ we obtain the confluent hypergeometric series Φ_1, simply denoted by Φ, or ${}_1F_1$ as in (1.3) so that we have

$$\Phi_1(a;c;u) = \Phi(a;c;u) = {}_1F_1\Big({a \atop c}; u\Big) = \sum_n \frac{(a)_n}{(c)_n}\frac{u^n}{n!} \quad (n \ge 0). \tag{1.9}$$

The Laguerre polynomial as defined in (1.4) can then be rewritten as

$$L_n^{(\alpha)}(x) = ((a+1)_n/n!)\,\Phi(-n;\alpha+1;x). \tag{1.10}$$

Erdélyi [9] found the following extension of the Hille-Hardy formula

$$\sum_n (u^n/n!)\,(\beta)_n \Phi(a-n;\gamma;x)\Phi(b-n;\delta;y) \tag{1.11}$$
$$= (1-u)^{-\beta} \sum \frac{(\beta)_r}{r!}\Big(\frac{xyu}{(1-u)^2}\Big)^r \frac{1}{(\gamma)_r\,(\delta)_r}\,\Phi_2\Big(a,\beta+r;\gamma+r;x,-\frac{xu}{1-u}\Big)$$
$$\times\,\Phi_2\Big(b,\beta+r;\delta+r;y,-\frac{yu}{1-u}\Big) \quad (n\ge 0, r\ge 0).$$

Finally, a multilinear extension of (1.1) was obtained by the authors [16] in the following manner. Let $k \ge 2$ and assume that the two summations below are over all sequences n_{ij} (resp. (r_{ij}) $(1 \le i < j \le k)$ of $k(k-1)/2$ nonnegative integers. For $1 \le i < j \le k$ let $n_{ji} = n_{ij}, r_{ji} = r_{ij}$ and for each $i = 1, 2, \ldots, k$ let $n_{i\bullet} = \sum_j n_{ij}, r_{i\bullet} = \sum_j r_{ij} (1 \le j \le k; j \ne i)$. Further, let y_i $(1 \le i \le k), u_{ij} (1 \le i < j \le k)$ be variables, $a_i, \gamma_i (1 \le i \le k), \beta_{ij}$ $(1 \le i < j \le k)$ be complex numbers and $u_{ji} = u_{ij}, \beta_{ji} = \beta_{ij}$ $(1 \le i < j \le k)$. The multilinear extension of (1.11), as it appears in [16], reads

$$\sum_{(n_{ij})} \prod_{i<j} (u_{ij}^{n_{ij}}/n_{ij}!)(\beta_{ij})_{n_{ij}} \prod_i \Phi(a_i - n_{i\bullet};\gamma_i;y_i) \tag{1.12}$$
$$= \prod_{i<j}(1-u_{ij})^{-\beta_{ij}} \sum_{(r_{ij})} \prod_{i<j} \frac{(\beta_{ij})_{r_{ij}}}{r_{ij}!}\Big(\frac{y_i y_j u_{ij}}{(1-u_{ij})^2}\Big)^{r_{ij}}$$
$$\times \prod_i \frac{1}{(\gamma_i)_{r_{i\bullet}}}\Phi_k\Big(a_i, \beta_{i1}+r_{i1}, \ldots, \beta_{ik}+r_{ik}; \gamma_i + r_{i\bullet}; y_i, \frac{-y_i u_{i1}}{1-u_{i1}}, \ldots, \frac{-y_i u_{ik}}{1-u_{ik}}\Big).$$

Note that all the diagonal terms $n_{ii}, r_{ii}, \beta_{ii}, u_{ii}$ are nonexistent, so that Φ_k above does have k arguments. When $k = 2$, the summation on the left-hand side of (1.12) is a single summation with respect to $n = n_{1\bullet} = n_{12} = n_{21} = n_{2\bullet}$ and (1.12) reduces to (1.11).

It is interesting to write down formulas (1.11) and (1.12) when the confluent series $\Phi(a-n;\gamma;x)$, $\Phi(b-n;\delta;y)$, $\Phi\,(a_I - n_{i\bullet};\gamma_i;y_i)$ terminate, namely when a, b and a_i are equal to 0. From (1.10) we have, for instance,

$$\Phi(-n;\gamma;x) = (n!/(\gamma)_n)\,L_n^{(\gamma-1)}(x). \tag{1.13}$$

Formula (1.11) becomes

$$\sum_n (u^n/n!)\,(\beta)_n \Phi(a-n;\gamma;x)\Phi(-n;\delta;y) \tag{1.14}$$
$$= (1-u)^{-\beta} \sum_r \frac{(\beta)_r}{r!}\Big(\frac{xyu}{(1-u)^2}\Big)^r \frac{1}{(\gamma)_r\,(\delta)_r}\,\Phi_2\Big(\beta+r;\gamma+r;x,-\frac{xu}{1-u}\Big)$$
$$\times\,\Phi_2\Big(\beta+r;\delta+r;y,-\frac{yu}{1-u}\Big) \quad (n\ge 0, r\ge 0).$$

and formula (1.12) reduces to

$$\sum_{(n_{ij})}\prod_{i<j}(u_{ij}^{n_{ij}}/n_{ij}!)(\beta_{ij})_{n_{ij}}\prod_i \Phi(-n_{i\bullet};\gamma_i;y_i) \tag{1.15}$$
$$=\prod_{i<j}(1-u_{ij})^{-\beta_{ij}}\sum_{(r_{ij})}\prod_{i<j}\frac{(\beta_{ij})_{r_{ij}}}{r_{ij}!}\Big(\frac{y_iy_ju_{ij}}{(1-u_{ij})^2}\Big)^{r_{ij}}$$
$$\times\prod_i\frac{1}{(\gamma_i)_{r_{i\bullet}}}\Phi_{k-1}\Big(\beta_{i1}+r_{i1},\ldots,\beta_{ik}+r_{ik};\gamma_i+ri_\bullet;\frac{u_{i1}}{1-u_{i1}},\ldots,\frac{-y_iu_{ik}}{1-u_{ik}}\Big).$$

The following arrow diagram indicates the relations between the foregoing formulas. The arrows stand for straightforward implications by assigning special values to some parameters. For instance, (1.7) implies (1.6) by letting $\beta=\alpha+1$ or (1.14) implies (1.7) with $y=0$ and $\gamma=\alpha+1$

$$(1.12)\Rightarrow\{(1.11),(1.15)\}\Rightarrow(1.14)\Rightarrow\{(1.7),(1.8)\}\Rightarrow(1.6).$$

The present work fits into a series of papers (see [12, 13, 14, 15]) whose purposes were to set up new proofs of orthogonal polynomial identities by means of combinatorial methods. As indicated by Askey and Wilson [5] those polynomials can be handily described in a chart with arrows going from polynomial P to polynomial Q whenever Q can be obtained from P by an appropriate limit. In that hierarchy Hermite polynomials involve the least number of parameters. It was then not surprising that they were among the first ones to be studied from a combinatorial point of view (see [12, 13, 14]).

Along those lines the next polynomials to be studied are the Laguerre polynomials. They are adjacent to the Hermite polynomials in Askey and Wilson's hierarchy. Furthermore, some identities involving the latter polynomials can be directly derived from the corresponding identities on the Laguerre polynomials by using the classical connecting relations between the two classes of polynomials. For instance, the Hille-Hardy formula (1.8) implies the Mehler identity for the Hermite polynomials, as shown in Watson [33].

The purpose of this paper is then to pursue the program with *Laguerre polynomials* and so derive identities (1.6), (1.7), (1.8), (1.14), (1.15) by combinatorial methods. Referring to the preceding diagram it would suffice to prove (1.15). However, we have preferred to give the combinatorial proof of Erdélyi's formula (1.14) (section 4) and sketch the derivation of its multilinear extension (1.15) (section 5.) As expected, a combinatorial proof of an orthogonal polynomial identity requires the discovery of an appropriate discrete model for the polynomial under study − this is done in section 3− which has enough symmetries to provide two ways of summing the generating functions for those structures. Those two ways yield the two members of the identity to be proved. The principle will be illustrated in section 4 where identity (1.14) is proved.

Other authors, such as Mullin and Rota [25], Garsia and Remmel [18], Leroux [23], have proposed combinatorial models for the Laguerre polynomials. However, those interpretations require α to be treated as a nonnegative integer. In the model used below α is an arbitrary parameter and $n!L_n^{(\alpha)}(x)$ is a generating polynomial for discrete structures by a weight which depends on the variables α and x. Those structures consist of finite paths and cycles which are counted by the variables x and $\alpha+1$, respectively. If α were only a nonnegative integer, the cycles would vanish and there would not be any possibility left for proving the foregoing identities.

In the sequel $|A|$ denotes the cardinality of a finite set A and $[n]$ the interval $\{1,2,\ldots,n\}$ for each positive integer n.

2. Three Combinatorial Lemmas

Let (A,B) be an ordered partition of a finite set S and consider an injection f of A into S. The map f can be identified with its *graph* whose *vertices* are the elements of S and whose *arcs* are the arrows going from each vertex v to vertex v' whenever $f(v)=v'$. Clearly, the *connected* components of the graph are *cycles* whose vertices are all in A and *paths* which go through vertices of A and end with vertices in B. (See Fig. 1 where an injection of $A=\{1,2,3,4,5\}$ into $S=A+B=\{1,2,\ldots,9\}$ is represented.)

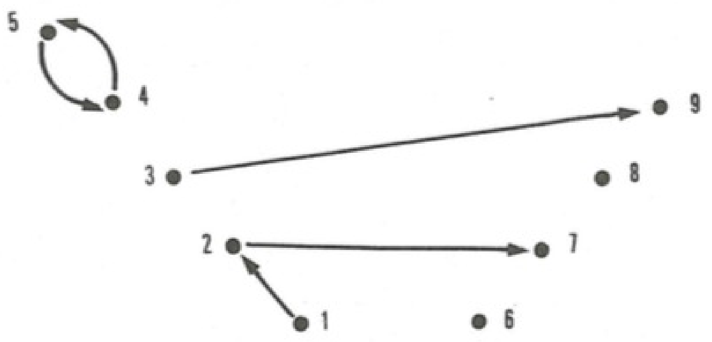

Fig. 1 : an injection of A into $A+B$

Each vertex in $B\backslash f(A)$ is regarded as a path reduced to a single vertex. Thus $|B|$ is also the *number of paths* of f. Let $\mathrm{cyc}(f)$ denote the number of cycles of f and with β designating a variable let the weight of f be defined by

$$w_\beta(f)=\beta^{\mathrm{cyc}(f)}. \qquad (2.1)$$

Lemma 2.1. *Let $|A| = i, |B| = j$ and $i + j = n$. Then $(\beta + j)_i (= (\beta + j)(\beta + j + 1) \cdots (\beta + j + i - 1))$ is the generating polynomial for the injections f of A into $S = A + B$ by w_β.*

Proof. The lemma is obvious when $|A| = i = 1$, as there is only one injection of A into S that consists of the loop around the unique vertex of A, with weight β, and the $(n - 1)$ paths connecting that vertex with the $j = n - 1$ vertices of B, each with weight 1.

Consider an ordered partition (A', B') of S such that $|A'| = i-1, |B'| = j + 1$ and let s be an element of B'. For constructing an injection f of $A = A' \cup \{s\}$ into S start with an injection f' of A' into S and make either one of the following two operations (i) or (ii) :

(i) if s is not the end of a path of f', form a loop around s; in the other case, join s with the origin (necessarily in A') of that path by a new arc;

(ii) Let t be any vertex of $B' \backslash \{s\}$; if t is not the end of a path of f', join s with t by a new arc; in the other case, join s with the origin (necessarily in A') of that path by a new arc.

Clearly, all the injections of A into S can be so obtained. Procedure (i) adds a new cycle to f', whereas the number of cycles remains invariant in procedure (ii). As the latter procedure can be applied to each of the j vertices in $B = B' \backslash \{s\}$ it follows by induction on i that the generating polynomial for the injections of A into S is equal to $\beta(\beta + j + 1)_{i-1} + j(\beta + j + 1)_{i-1} = (\beta + j)_i$. ∎

Remark 2.2. When $\beta = 1$, there is no counter left for the cycles and $(1 + j)_i = n!/j!$ is simply the number of injections of A (of cardinality i) into S (of cardinality $n = i+j$.) When $j = 0$, we do count the permutations of S according to the number of cycles. As it is well-known (see e.g. [28 p. 71]), the generating polynomial is $(\beta)_n$. The next lemma is fundamental in the proofs of identities (1.14) and (1.15).

Lemma 2.3. *Let (I, J, R) be an ordered partition of a finite set with $|I| = i, |J| = j, |R| = r$. Then $(\beta)_r(\beta + r)_i(\beta + r)_j$ is the generating polynomial for permutations θ of the set $I + J + R$ with the property that $\theta(J) \cap I = \emptyset$ by w_β.*

Proof. From lemma 2.1 the expression $(\beta + r)_j$ is the generating polynomial for the injections f of J into $R + J$. On the other hand, $(\beta)_{r+i} (= (\beta)_r(\beta + r)_i)$ is the generating polynomial for permutations ρ of $R + I$. Whenever $\rho(v) = v'$ with $v' \in R$, join v with the *origin* of the path of f whose end is v'. (Remember that f has exactly r paths, some possibly reduced to single vertices.) The graph so constructed is a permutation θ of $I + J + R$. Moreover, $\mathrm{cyc}(f) + \mathrm{cyc}(\rho) = \mathrm{cyc}(\theta)$ and the mapping $(f, \rho) \mapsto \theta$ is one-to-one. By construction f is the restriction of θ to J. Therefore, there is no arc of θ going from J to I. ∎

The next lemma deals with the enumeration of graphs whose connected components are only paths. Let (I,J,R,S) be a fixed ordered partition of a finite set. Three kinds of paths whose vertex labels are taken from $I+J+R+S$ are now defined. As shown in Fig. 2, the *x-paths* (resp. *y-paths*) have all their vertex labels in S except the path end that belongs to I (resp. J).

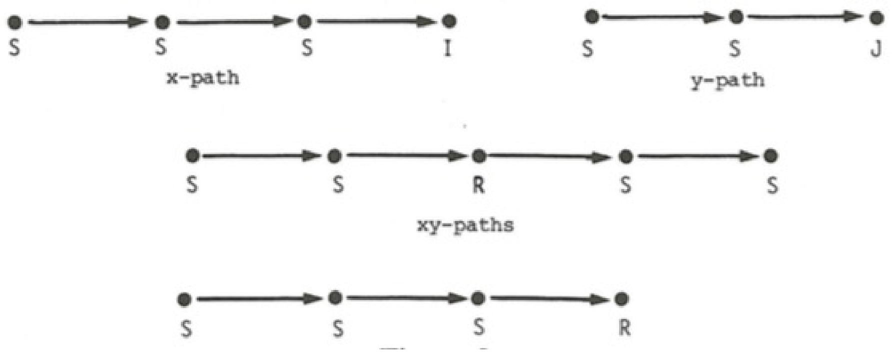

Fig. 2 : x-paths, y-paths, xy-paths

In an *xy-path* all vertex labels except one are in S. The only exception belongs to R and is *not* necessarily at the end of the path. The unique label of the x-path (resp. y-path, resp. xy-path) that belongs to I (resp. J, resp. R) is called the *tag* of the path. Note that a path may be reduced to its tag.

A graph G whose vertex set is $I+J+R+S$ is said to be *Erdélyian on* (I,J,R,S) if its connected components are only x -paths, y-paths and xy -paths. (See Fig. 3). Let $|I|=i, |J|=j, |R|=r$, $|S|=s$. Then an Erdélyian graph on (I,J,R,S) has exactly i x-paths, j y-paths and r xy-paths. The tags of the paths may be used to rearrange all the paths of G in some fixed order. That remark will be essential in the proof of the next lemma.

Lemma 2.4. *Let $\epsilon(i,j,r,s)$ denote the number of Erdélyian graphs on (I,J,R,S) with $|I|=i, |J|=j, |R|=r, |S|=s$. Then,*

$$\sum_s \frac{u^s}{s!}\epsilon(i,j,r,s)=(1-u)^{-i-j-2r} \quad (s\geq 0). \tag{2.2}$$

Proof. Let v (resp. v', resp. v'') be a given element of I (resp. J, resp. R) and $\{v_1,v_2,\ldots,v_n\}$ be a given subset of S. The number of x-paths (resp. y-paths, resp. xy-paths) whose tag is v (resp. v', resp. v'') and other vertices are $v_1,v_2,\ldots,v_n$ in some order, is equal to $n!$ (resp. $n!$, resp. $n!(n+1)$.) Therefore, the exponential generating paths, namely for Erdélyian graphs on $(I,J,R,[s])(s\geq 0)$.

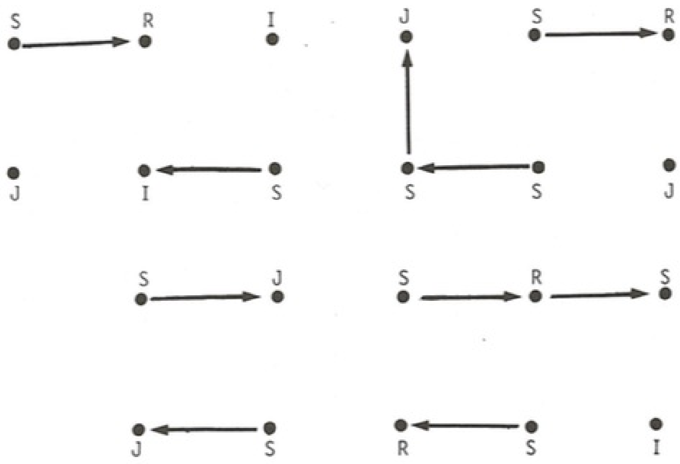

Fig. 3 : Erdélyian graph on (I, J, R, S)

Remark 2.5. The binomial theorem applied to (2.2) implies that $\epsilon(i,j,r,s) = (i+j+2r)_s$. However it will be more convenient in the sequel to have the result on Erdélyian graph counting in the form (2.2).

3. Laguerre Configurations

Let S be a finite set. A *Laguerre configuration on* S is defined to be an ordered pair $\phi = ((A,B), f)$ where (A,B) is an ordered partition of S and f an injection of A into $S = A + B$. Let $\mathrm{Lag}(S)$ be the set of all Laguerre configurations on S. With X and Y designating two variables let the weight of a Laguerre configuration $\phi = ((A,B), f)$ be defined as (see (2.1))

$$w(\beta, X, Y; \phi) = w_\beta(f)\, X^{|A|} Y^{|B|}. \tag{3.1}$$

It follows from Lemma 2.1 that the generating polynomial for all the Laguerre configurations on a set S of cardinality n by the weight w is given by

$$\sum_\phi w(\beta, X, Y; \phi) = \sum_{i,j} \binom{n}{i,j} (\beta+j)_i X^i Y^j \ (\phi \in \mathrm{Lag}(S); i+j=n). \tag{3.2}$$

Proposition 3.1 (Combinatorial interpretation of the Laguerre polynomials). *When* $|S| = n, \beta = \alpha+1, X = 1, Y = -x$, *we have*

$$n! L_n^{(\alpha)}(x) = \sum_\phi w(\alpha+1, 1, -x; \phi) \quad (\phi \in \mathrm{Lag}(S)). \tag{3.3}$$

Proof. When replacing β, X and Y by $\alpha+1, 1, -x$, respectively, on the righthand side of (3.2) we obtain the expression for the Laguerre polynomial shown in (1.4) multiplied by $n!$ $\square$

As an application of Proposition 3.1 let us establish identity (1.6). The *connected* Laguerre configurations are the *cycles* and the *paths*. For each finite set S let $C(S)$ (resp. $P(S)$) denote the set of all distinct cycles (resp. paths) on all the elements of S. If $|S| = n$, then $|C(S)| = (n-1)!$ and $|P(S)| = n!$ Of course, each element ϕ of $C(S)$ (resp. $P(S)$) is a particular configuration on S with weight $w(\alpha+1, 1, -x; \phi) = \alpha+1$ (resp. $= -x$.) Therefore, the exponential generating function for the cycles (resp. paths) defined by $\sum_n (u^n/n!) \sum_\phi w(\alpha+1, 1, -x; \phi)$ $(n \geq 1; \phi \in C[n])$ (resp. $(n \geq 1; \phi \in P[n])$) is equal to

$$\sum_n (u^n/n!)\,(\alpha+1)(n+1)! = (\alpha+1)\log(1/(1-u))$$

$$\text{(resp.}\quad \sum_n (u^n/n!)\,(-x)n! = -xu/(1-u).\quad)$$

As the Laguerre configurations form a *partitional complex* [11, 17] or an *espèce* (in the terminology dear to our Québécois friends [10, 22]), the exponential generating function by w for all the Laguerre configurations whose connected components are cycles (resp. paths) is

$$\exp((\alpha+1)\log(1/(1-u))) = (1-u)^{-\alpha-1} \quad \text{(resp. } \exp(-xu/(1-u)).$$

Finally, $\sum_n (u^n/n!)\,(n!L_n^{(\alpha)}(x))$ is equal to the product of the latter two series since the Laguerre configurations have connected components of both kinds.

4. The Erdélyi Identity.

We now have all the ingredients to prove the Erdélyi identity (1.14). From (1.13) the left-hand member of (1.14) may be rewritten

$$\sum_n \frac{u^n}{n!\,(\gamma)_n(\delta)_n}(\beta)_n n!\,L_n^{(\gamma-1)}(x) n!\,L_n^{(\delta-1)}(y). \tag{4.1}$$

From Remark 2.2 and Proposition 3.1 we have

$$(\beta)_n n! L_n^{(\gamma-1)}(x) n!\,L_n^{(\delta-1)}(y) = \sum_{\sigma,\phi,\psi} w_\beta(\sigma) w(\gamma, 1, -x; \phi) w(\delta, 1, -y; \psi) \tag{4.2}$$

where the summation is over all triplets (σ, ϕ, ψ) with σ a permutation of $[n]$ and $\phi = ((A, B), f), \psi = ((C, D), g)$ two Laguerre configurations on $[n]$. When superimposing the graphs of those three configurations on a set of n labelled vertices, we first have the cycles of σ – call them *β-cycles* – then, the cycles and paths of both ϕ and ψ. The elements of B (resp.

of D) are the path ends of ϕ (resp. of ψ). Depending on whether a vertex v belongs of $B\backslash D, D\backslash B$, or $B \cap D$, it is said to be *x-marked*, *y-marked* or *xy-marked*. The other vertices are unmarked.

In Fig. 4 only the β -cycles of a triplet (σ, ϕ, ψ) are drawn and the marks are indicated by the letters x, y and xy. The unmarked vertices are simply dots. The meanings of the dotted arrows and the letters Q, R, S, I, J will be further explained.

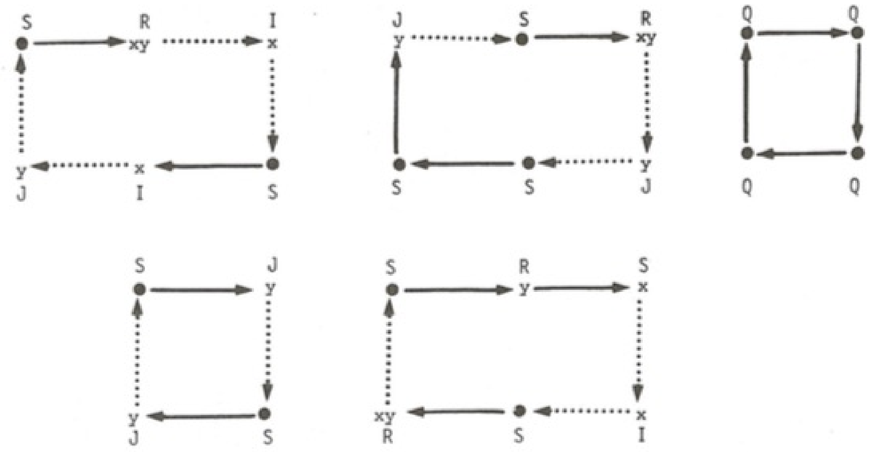

Fig. 4 : β -cycles of a triplet (σ, ϕ, ψ)

Two distinct vertices v and v' are said to be *linked* if the following three properties hold

(i) v and v' are y -marked and x -marked, respectively;

(ii) v and v' belong to the same β-cycle;

(iii) when going from v to v' along that β-cycle, there is no marked vertex between them (see Fig. 5).

Now the n vertices of the graph (σ, ϕ, ψ) may be partitioned into five disjoint classes Q, R, S, I, J as follows. If all the vertices of a β cycle are unmarked, all its vertices fall into class Q. If a vertex is either xy -marked, or y -marked and linked, it belongs to R. If a vertex is x -marked (resp. y -marked) and not linked, it goes into I (resp. J). The class S consists of all the remaining vertices. In particular, S contains the vertices which are x-marked and linked.

Let $|Q| = q$, $|R| = r$, $|S| = s$, $|I| = i$, $|J| = j$. In the example shown in Fig. 4 those numbers are $q = 4$, $r = 4$, $s = 10$, $i = 3$, $j = 5$. Furthermore, each vertex has been assigned one of the letters Q, R, S, I or J according to the foregoing rule. The dotted arrows are the arcs going out of vertices which are either xy-marked, or x-marked, or y-marked but not linked.

Fig. 5 : no marked vertex between two linked vertices

Now remove all the dotted arrows from the graph drawn in Fig. 4. What remains is a collection of unmarked cycles (there is only one such a cycle in the foregoing example) and an Erdélyian graph G on (I, J, R, S) (actually the one drawn in Fig. 3). As was noticed in section 2, each x-path (resp. y-path, resp. xy-path) of G has one and only one tag in I (resp. in J, resp. in R.) Hence, the dotted arrows may be viewed as forming the graph of a permutation θ of the set $I + J + R$ As there is no dotted arrow going out of any linked y-marked vertex, the permutation θ has the further property that

$$\theta(J) \cap I = \emptyset. \tag{4.3}$$

On the other hand, let σ' and σ'' be the restrictions of σ to Q and $[n]\backslash Q$, respectively. Clearly, σ' and σ'' are both permutations and

$$w_\beta(\sigma) = w_\beta(\sigma')\, w_\beta(\sigma'')\,. \tag{4.4}$$

But as σ'' and θ have the same number of cycles, we also have

$$w_\beta(\sigma) = w_\beta(\sigma')\, w_\beta(\theta). \tag{4.5}$$

Thus, starting with a triplet (σ, ϕ, ψ) with $\phi = ((A, B), f)$ and $\psi = ((C, D), g)$ we can associate in a bijective manner :

(i) an ordered partition (Q, R, S, I, J) of $[n]$;
(ii) a permutation σ' of Q and a permutation θ of $I + J + R$ satisfying (4.3) and (4.5) ;
(iii) an Erdélyian graph G on (I, J, R, S); let R' be the set of the ends of the xy-paths of G ;
(iv) two injections f and g – namely, the two injections underlying the Laguerre configurations ϕ and ψ – with path end sets $R' + I$ and $R + J$, respectively.

Let $|Q| = q$, $|R| = r$, $|S| = s$, $|I| = i$, $|J| = j$, and so $|R'| = r$. Because of (3.1) and (4.5) we have :

$$w_\beta(\sigma)\, w(\gamma, 1, -x; \phi)\, w(\delta, 1, -y; \psi) \tag{4.6}$$
$$= w_\beta(\sigma')w_\beta(\theta)(-x)^{r+i}(-y)^{r+j}w_\gamma(f)w_\delta(g).$$

Therefore, the summations in (4.2) can also be made by first fixing the ordered partition (Q, R, S, I, J) and summing the products (4.6) over all sequences $(\sigma', \theta, G, f, g)$ satisfying conditions (i), (ii), (iii), (iv) above. Thus,

$$(\beta)_n n! L_n^{(\gamma-1)}(x) n! L_n^{(\delta-1)}(y) \tag{4.6}$$
$$= \sum \binom{n}{q, r, s, i, j} (-x)^{r+i}(-y)^{r+j} \sum w_\beta(\sigma')\, w_\beta(\theta) w_\gamma(f) w_\delta(g).$$

But

$$\sum_{\sigma'} w_\beta\,(\sigma') = (\beta)_q \qquad \text{(by Remark 2.2.)}$$

$$\sum_{\theta} w_\beta(\theta) = (\beta)_r(\beta+r)_i(\beta+r)_j \qquad \text{(by Lemma 2.3.)}$$

$$\sum_{G} 1 = \epsilon(i,j,r,s) \qquad \text{(in the notation of Lemma 2.4.)}$$

$$\sum_{f} w_\gamma(f) = (\gamma+r+i)_{n-r-i} \qquad \text{(by Lemma 2.1.)}$$

$$\sum_{g} w_\delta(g) = (\delta+r+j)_{n-r-j} \qquad \text{(by Lemma 2.1.)}$$

Therefore,

$$\begin{aligned}
\sum_n \frac{u^n}{n!\,(\gamma)_n(\delta)_n}(\beta)_n\, n!\, L_n^{(\gamma-1)}(x)\, n!\, L_n^{(\delta-1)}(y) \\
&= \sum_q \frac{u^q}{q!}(\beta)_q \sum_{r,i,j} \frac{1}{r!\,i!\,j!\,(\gamma)_{r+i}(\delta)_{r+j}}(\beta)_r(\beta+r)_i(\beta+r)_j \\
&\qquad \times (xyu)^r(-ux)^i(-uy)^j \sum \frac{u^s}{s!}\epsilon(i,j,r,s) \\
&= (1-u)^{-\beta} \sum_{r,i,j} \frac{(\beta)_r}{(\gamma)_r(\delta)_r r!}\left(\frac{xyu}{(1-u)^2}\right)^r \\
&\qquad \times \frac{(\beta+r)_i}{(\gamma+r)_i\, i!}\left(\frac{-xu}{1-u}\right)^i \frac{(\beta+r)_j}{(\gamma+r)_j\, j!}\left(\frac{-yu}{1-u}\right)^j
\end{aligned}$$

from the binomial theorem and Lemma 2.4. Finally, the latter expression can also be rewritten

$$\begin{aligned}
(1-u)^{-\beta} \sum_r \frac{(\beta)_r}{(\gamma)_r(\delta)_r\, r!}\left(\frac{xyu}{(1-u)^2}\right)^r \\
\times \Phi\left(\beta+r;\gamma+r;-\frac{xu}{1-u}\right)\Phi\left(\beta+r;\delta+r;-\frac{yu}{1-u}\right),
\end{aligned}$$

which is the right-hand side of identity (1.14). This completes the proof of that identity.

5. The Multilinear Extension

As explained in [16] the search for a multilinear extension of the Hille-Hardy formula was motivated by the following facts. There exists a multilinear extension of the Mehler formula for the Hermite polynomials found by Kibble [21] and Slepian [29]. (See also [24].) That extension can also be proved by combinatorial methods (see [14]).) As the Hermite polynomials $H_n(x)(n \geq 0)$ appear to be limit cases of the Laguerre polynomials in the formula

$$H_n(x) = n! \lim(2/\alpha)^{n/2} L_n^{(\alpha)} \left(\alpha - x(2a)^{1/2}\right) \quad (a \to +\infty)$$

(see [31 p. 389, 4, 6, 26, 32]), it was natural to build up a series of Laguerre polynomials that reduced to the Kibble-Slepian series when passing from the Laguerre to the Hermite polynomials. The multilinear model used for the Kibble-Slepian identity was to be applied to the Laguerre configurations with an appropriate normalization. This gave the series written in (5.1), another version of the left hand side of (1.15).

On the other hand, identity (1.14) says that in the bilinear case that series can be expressed as a product of $(1-u)^{-\beta}$ by a series in the arguments $xyu/(1-u)^2, xu/(1-u)$ and $yu/(1-u)$. The problem was then to find an analogous expression in the multilinear case, namely an expression that was equal to the product of $\prod(1-u_{ij})$ by a series in the arguments $y_i y_j u_{ij}/\left(1-u_{ij}\right)^2 (i<j)$ and $y_i u_{ij}/\left(1-u_{ij}\right)(i \neq j)$. This is precisely what was obtained in (1.15).

In this section we derive the combinatorial proof of (1.15). Note that one of the two analytical proofs of (1.12) was sketched in [16]. As indicated by our colleague Josef Hofbauer [19], the same analytical techniques can be applied to other generalized confluent series studied by Erdélyi [10] and Carlitz [7] to obtain analogous multilinear identities. Finally, by multilinear extension it was meant a formula that could not be derived from the bilinear case by an inductive argument as it was in [30] – but needed a direct derivation.

For the combinatorial proof of (1.15) proceed as follows. First, rewrite its left-hand side as

$$\sum_{(n_{ij})} \prod_{i<j} \frac{u_{ij}^{n_{ij}}}{n_{ij}!} \prod_i \frac{1}{(\gamma_i)_{n_i}} \prod_{i<j} (\beta)_{n_{ij}} \prod_i n_{i\bullet}! L_{n_{i\bullet}}^{(\gamma_i - 1)}(y_i). \tag{5.1}$$

For a given sequence (n_{ij}) $(1 \leq i < j \leq k$) let $(N_{ij})_{i<j}$ be a fixed partition of the interval $\left\{1, 2, \ldots, \sum_{i<j} n_{ij}\right\}$ such that $|N_{ij}| = n_{ij}(i<j)$ For $i<j$ let $N_{ji} = N_{ij}$ and for $i = 1, 2, \ldots, k$ let $N_{i\bullet} = \sum N_{ij}$ $(1 \leq j \leq k; j \neq i)$. Thus $N_{i\bullet} \cap N_{j\bullet} = N_{ij}$ for $i \neq j$

As in section 4 we have

$$\prod_{i<j}(\beta)_{n_{ij}} \prod_{i} n_{i\bullet}! L_{n_{i\bullet}}^{(\gamma_i-1)}(y_i) = \sum \prod_{i<j} w_{\beta_{ij}}(\sigma_{ij}) \prod_{i} w(\gamma_i, 1, -y_i; \phi_i),$$

where the summation is over all sequences $((\sigma_{ij})\,(i<j),\ \phi_i\,(1 \le i \le k))$ with the following properties :

(i) σ_{ij} is the permutation of $N_{ij}(i<j)$;

(ii) $\phi_i = ((A_i B_i)\,, f_i)$ is a Laguerre configuration on the set $N_{i\bullet}$ $(1 \le i \le k)$.

In each compartment N_{ij} the n_{ij} vertices can be partitioned into five classes $Q_{ij}, R_{ij}, S_{ij}, E_{ij}, E_{ji}$, the variables y_i and $y_j (i<j)$, on the one hand, the sets E_{ij} and E_{ji}, on the other hand, playing the roles of the variables x and y, and of the sets I and J, in section 4, respectively. Let R'_{ij} be the set of the ends of the $y_i y_j$-paths in N_{ij}. Also, let $|Q_{ij}| = q_{ij}$, $|R_{ij}| = \left|R'_{ij}\right| = r_{ij}$, $|S_{ij}| = s_{ij}, |E_{ij}| = e_{ij}, |E_{ji}| = e_{ji}$.

As in section 4 with each sequence $((\sigma_{ij})(i<j),\ \phi_i = ((A_i, B_i)), f_i))$ $(1 \le i \le k))$ there corresponds in a bijective manner for each $i<j$

(i) an ordered partition $(Q_{ij}, R_{ij}, s_{ij}, E_{ji})$ of N_{ij};

(ii) a permutation σ'_{ij} of Q_{ij} and a permutation θ_{ij} of $E_{ij} + E_{ji} + R_{ij}$;

(iii) an Erdéyian graph G_{ij} on $(E_{ij}, E_{ji}, R_{ij}, S_{ij})$;

(iv) a sequence of k injections $f_i (1 \le i \le k)$, each f_i mapping A_i into $A_i + B_i$; having the following properties :

(v) $\theta_{ij}\,(E_{ji}) \cap E_{ij} = \emptyset$;

(vi) $w_{\beta_{ij}}\,(\sigma_{ij}) = w_{\beta_{ij}}\left(\sigma'_{ij}\right) w_{\beta_{ij}}\,(\theta_{ij})$;

(vii) $B_i \cap N_{ij} = R'_{ij} + E_{ij}$ and $B_j \cap N_{ij} = R_{ij} + E_{ji}$.

Now the following summation can be made

$$\begin{aligned} &\prod_{i<j}(\beta_{ij})_{n_{ij}} \prod_{i} n_{i\bullet}!\, L_{n_{i\bullet}}^{(\gamma_i-1)}(y_i) \qquad (5.2)\\ &\quad = \sum \prod_{i<j} \binom{n_{ij}}{q_{ij}, r_{ij}, s_{ij}, e_{ij}, e_{ji}} (-y_i)^{r_{ij}+e_{ij}} (-y_i)^{r_{ij}+e_{ji}}\\ &\qquad \times \sum \prod_{i<j} w_{\beta_{ij}}(\sigma_{ij})\, w_{\beta_{ij}}(\theta_{ij}) \prod_{i} w_{\gamma_i}(f_i), \end{aligned}$$

the first sigma sign being for each $i<j$ over all sequences

$$(q_{ij}, r_{ij}, s_{ij}, e_{ij}, e_{ji})$$

of sum n_{ij} and the second one over all sequences

$$\big((\sigma'_{ij}, \theta_{ij}, G_{ij})\ (i<j);\quad f_i\ (1 \le i \le k)\big)$$

satisfying conditions (i) – (vii) above.

In particular, f_i is an injection of $N_{i\bullet}$ into itself with path end set

$$B_i = \sum_l (R_{li} + E_{li}) + \sum_m (R'_{im} + E_{im}) \quad (1 \le l \le i-1; i+1 \le m \le k).$$

For $i < j$ let $r_{ji} = r_{ij}$, and for each i let $r_{i\bullet} = \sum r_{ij}$, $e_{i\bullet} = \sum e_{ij}, n_{i\bullet} = \sum n_{ij}$ $(1 \le j \le k; j \ne i), m_{i\bullet} = n_{i\bullet} - r_{i\bullet} - e_{i\bullet}$. Then

$$\sum_{f_i} w_{\gamma_i}(f_i) = (\gamma_i + r_{i\bullet} + e_{i\bullet})_{m_i}$$

Therefore,

$$\begin{aligned}
&\prod_{i<j} (\beta_{ij})_{n_{ij}} \prod_i n_{i\bullet}!\, L_{n_{i\bullet}}^{(\gamma_i - 1)}(y_i) \\
&\quad = \sum \prod_{i<j} \binom{n_{ij}}{q_{ij}, r_{ij}, s_{ij}, e_{ij}, e_{ji}} (-y_i)^{r_{ij}+e_{ij}} (-y_i)^{r_{ij}+e_{ji}} (\beta_{ij})_{q_{ij}} (\beta_{ij})_{r_{ij}} \\
&\qquad \times (\beta_{ij} + r_{ij})_{e_{ij}} (\beta_{ij} + r_{ij})_{e_{ji}} \epsilon(e_{ij}, e_{ji}, r_{ij}, s_{ij}) \prod_i (\gamma_i + r_{i\bullet} + e_{i\bullet})_{m_{i\bullet}}
\end{aligned}$$

Finally,

$$\begin{aligned}
&\sum_{(n_{ij})} \prod_{i<j} \frac{u_{ij}^{n_{ij}}}{n_{ij}!} \prod_i \frac{1}{(\gamma_i)_{n_{i\bullet}}} \prod_{i<j} (\beta_{ij})_{n_{ij}} \prod_i n_{i\bullet}!\, L_{n_{i\bullet}}^{(\gamma_i - 1)}(y_i) \\
&\quad = \sum \prod_i \frac{1}{(\gamma_i)_{r_{i\bullet}}} \prod_{i<j} (\beta)_{q_{Ij}} \frac{u_{ij}^{q_{ij}}}{q_{ij}!} (\beta_{ij})_{r_{ij}} (y_i y_j)^{r_{ij}} \frac{u_{ij}^{r_{ij}}}{r_{ij}!} \\
&\qquad \times \epsilon(e_{ij}, e_{ji}, r_{ij}, s_{ij}) \frac{u_{ij}^{s_{ij}}}{s_{ij}!} \prod_i \frac{1}{(\gamma_i + r_{i\bullet})_{e_{i\bullet}}} \prod_{j \ne i} (\beta_{ij} + r_{ij})_{e_{ij}} \frac{(-y_i u_{ij})^{e_{ij}}}{e_{ij}!}
\end{aligned}$$

where the summation is over all sequences $(q_{ij}, r_{ij}, s_{ij}, e_{ij}, e_{ji})$ $(i < j)$. The latter expression is also equal to

$$\begin{aligned}
&\prod_{i<j} (1 - u_{ij})^{-\beta_{ij}} \sum_{(r_{ij})} \prod_{kj} \frac{(\beta_{ij})_{r_{ij}}}{r_{ij}!} \left(\frac{y_i y_j u_{ij}}{(1 - u_{ij})^2} \right)^{r_{ij}} \\
&\qquad \times \prod_i \frac{1}{(\gamma_i)_{r_{i\bullet}}} \sum \frac{1}{(\gamma_i + r_{i\bullet})_{e_i}} \prod_{j+i} (\beta_{ij} + r_{ij})_{e_{ij}} \frac{1}{e_{ij}!} \left(-\frac{y_i u_{ij}}{1 - u_{ij}} \right)^{e_{ij}},
\end{aligned}$$

the second summation being over all sequences

$$(e_{i,1}, \ldots, e_{i,i-1,} e_{i,i+1}, \ldots, e_{ik}).$$

This is precisely the right-hand side of (1.15).

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