THE SIGNED EULERIAN NUMBERS

BY

JACQUES DÉSARMÉNIEN (*) AND DOMINIQUE FOATA (**)

To John Riordan, in memoriam.

ABSTRACT. — The basic relation between the signed Eulerian Numbers and the classical Eulerian Numbers is derived by using two different methods.

1. Introduction

LODAY [Lo] in his study of the cyclic homology of commutative algebras came across a sequence of numbers $(B_{n,k})$ related to the classical Eulerian numbers $A_{n,k}$ as follows. The latter numbers are defined by the recurrence relation (see, e.g. [Ri, chap. 2])

$$A_{1,1} = 1\,; \qquad A_{1,k} = 0 \text{ for } k \neq 1\,; \tag{1.1}$$

$$A_{n,k} = kA_{n-1,k} + (n-k+1)A_{n-1,k-1}\,; \tag{1.2}$$

for $n \geq 2$ and $1 \leq k \leq n$, so that the first values are easily calculated, as shown in the next table.

$k=$	1	2	3	4	5	6	7
$n=1$	1						
2	1	1					
3	1	4	1				
4	1	11	11	1			
5	1	26	66	26	1		
6	1	57	302	302	57	1	
7	1	120	1191	2416	1191	120	1

The polynomial $A_n(t) = \sum_{k=1}^{n} A_{n,k}t^k$ is called the n-th *Eulerian polynomial.* As it is well-known (see, e.g., [Ri, chap. 7]), $A_n(t)$ is the generating polynomial for the permutation group $\mathfrak{S}_n$ by the *number of descents.* More

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precisely, say that i is a *descent* of a permutation $\sigma=\sigma(1)\ldots\sigma(n)$ of order n, if $1\le i\le n-1$ and $\sigma(i)>\sigma(i+1)$ and let $\operatorname{des}\sigma$ denote the *number of descents* of σ. It is straightforward to verify directly from the foregoing recurrence relation that the following identity

$$A_n(t)=\sum_{\sigma\in S_n} t^{1+\operatorname{des}\sigma}, \tag{1.3}$$

holds, or, equivalently, if $S_{n,k}$ denotes the set of all permutations of order n having $(k-1)$ descents, that

$$A_{n,k}=\operatorname{card} S_{n,k}. \tag{1.4}$$

LODAY [Lo] introduced the numbers

$$B_{n,k}=\sum_{\sigma\in S_{n,k}} \operatorname{sgn}\sigma, \tag{1.5}$$

where $\operatorname{sgn}\sigma$ designates the *signature* of σ, computed the first values and conjectured the following formulas

$$B_{2n,k}=B_{2n-1,k}-B_{2n-1,k-1}\,; \tag{1.6a}$$

$$B_{2n+1,k}=kB_{2n,k}+(2n-k+2)B_{2n,k-1}\,; \tag{1.6b}$$

together with the analogs of the Worpitzky formulas

$$\sum_i \binom{2n+i}{i} B_{2n,k-i}=k^n\ ; \tag{1.7a}$$

$$\sum_i \binom{2n-1+i}{i} B_{2n-1,k-i}=k^n. \tag{1.7b}$$

The first values of the numbers $B_{n,k}$ are shown in the next table.

$k=$	1	2	3	4	5	6	7
$n=1$	1						
2	1	-1					
3	1	0	-1				
4	1	-1	-1	1			
5	1	2	-6	2	1		
6	1	1	-8	8	-1	-1	
7	1	8	-19	0	19	-8	-1

The purpose of this paper is to show what LODAY [Lo] had conjectured, namely that if (1.5) is the definition for the numbers $(B_{n,k})$, then those numbers satisfy the recurrence relations (1.6a) and (1.6b). The Worpitzky formulas (1.7a) and (1.7b) are easily derived from (1.6a) and (1.6b). Let

$$B_n(t) = \sum_{\sigma \in S_n} \operatorname{sgn} \sigma \, t^{1+\operatorname{des} \sigma}. \tag{1.8}$$

Then (1.6a) and (1.6b) can be rewritten as

$$B_{2n}(t) = (1-t) B_{2n-1}(t)\,; \tag{1.9a}$$

$$B_{2n+1}(t) = (2n+1)\, t\, B_{2n}(t) + t(1-t) B'_{2n}(t)\,; \tag{1.9b}$$

where $B'_{2n}(t)$ denotes the derivative of $B_{2n}(t)$. As the recurrence relation (1.1) can also be expressed as

$$A_n(t) = ntA_{n-1}(t) + t(1-t)A'_{n-1}(t),$$

it is straightforward to verify that proving the implication [(1.5) $\Rightarrow$ (1.6a) & (1.6b)] is equivalent to proving the following theorem.

Theorem 1. — *Let $B_n(t)$ be the signed Eulerian polynomial defined in* (1.8)*. Then*

$$B_{2n}(t) = (1-t)^n A_n(t)\,; \tag{1.10a}$$

$$B_{2n+1}(t) = (1-t)^n A_{n+1}(t). \tag{1.10b}$$

The polynomial $B_n(t)$ can also be regarded as a specialization of one of the q-Eulerian polynomial $A_n(t,q)$. If $\operatorname{inv} \sigma$ denotes the usual *number of inversions* of a permutation $\sigma = \sigma(1) \dots \sigma(n)$, that is, the number of pairs (i,j) such that $1 \le i < j \le n$ and $\sigma(i) > \sigma(j)$, then $A_n(t,q)$ is defined to be (see, e.g., [St])

$$A_n(t,q) = \sum_{\sigma \in S_n} t^{1+\operatorname{des} \sigma} q^{\operatorname{inv} \sigma}. \tag{1.11}$$

As $(-1)^{\operatorname{inv} \sigma}$ is precisely equal to the signature $\operatorname{sgn} \sigma$ of the permutation σ, we have

$$B_n(t) = A_n(t,-1).$$

To prove Theorem 1 it then suffices to work out with the various recurrence formulas dealing with the polynomials $A_n(t,q)$, replace q by -1 and try to obtain the two formulas (1.10a) and (1.10b). As it will be seen in the next section, this program is achieved without any major difficulties.

Identities (1.10a) and (1.10b) suggest that there is a signed involution of $\mathfrak{S}_{2n}$ that maps each element on another element with opposite weight, except those permutations which are in bijection with the set $2^{[n]} \times \mathfrak{S}_n$. Such an involution has just been found by Michelle WACHS [Wa].

More interesting is the fact that THEOREM 1 relates to a set of results on congruence properties of the q-Eulerian polynomials modulo a cyclotomic polynomial, a theory that was developed by the first author in [De2] in the context of the q-Kummer congruences. In fact, the forementioned theorem can even be regarded as a consequence of [De2, theorem 10.4]. However to make the present paper self-contained we have preferred to restate and reprove the basic results in a slightly different way.

The *q-Eulerian polynomials* $A_{m,n}(t,q)$ $(m \ge 0,\ n \ge 0)$ *with two indices* were introduced in [De2, § 10]. They may be defined by their generating function

$$(1.12)\qquad \sum_{m\ge 0,\, n\ge 0} A_{m,n}(t,q)\frac{u^n}{(q;q)_n}\frac{v^m}{m!} = \frac{1-t}{1-te\big((1-t)u;q\big)\exp\big((1-t)v\big)},$$

where $(q;q)_n$ and $e(u;q)$ are, respectively, the q-ascending factorial and the first q-exponential whose definitions are recalled in the beginning of section 2. We shall not reprove that formula (1.12) does define a class of generating polynomials for the symmetric groups $\mathfrak{S}_{m+n}$ by a pair of statistics. This result was derived by the first author from a calculation on Schur functions associated with ribbons [De2, § 10]. The two polynomials $A_n(t,q)$ and $A_n(t)$ are the following specializations of $A_{m,n}(t,q)$

$$(1.13)\qquad A_{0,n}(t,q) = A_n(t,q)\,;\qquad A_{m,0}(t,q) = A_m(t),$$

as it can be directly seen from their generating functions shown in (2.3) and (2.4).

For each k let $\Phi_k(q)$ denote the k-th cyclotomic polynomial ($\Phi_1(q) = 1-q\,;\ \Phi_2(q) = 1+q$, etc...). Our purpose is also to prove the following theorem.

THEOREM 2. — *Let n and k be two positive integers and let $n = ka+b$, $0 \le b \le k-1$ be the Euclidean division of n by k. Then the following congruence holds :*

$$A_{m,ka+b}(t,q) \equiv (1-t)^{(k-1)a} A_{m+a,b}(t,q) \bmod \Phi_k(q).$$

For $k=1$ we then have

$$A_{a,1}(t,q) = A_{a+1,0}(t,q) = A_{a+1}(t),$$

for $k=2$, $b=0,1$

$$\begin{aligned} A_{2a+b}(t,q)=A_{0,2a+b}&\equiv(1-t)^a A_{a,b}(t,q) \bmod (1+q)\\ &\equiv(1-t)^a A_{a+b}(t) \bmod (1+q) \end{aligned}$$

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Thus THEOREM 1 is a particular case of THEOREM 2.

2. The q-Eulerian polynomials

As usual let

$$(u;q)_n=\begin{cases}1, & \text{if } n=0;\\ (1-u)(1-uq)\dots(1-uq^{n-1}), & \text{if } n\ge 1;\end{cases}$$

denote the q-ascending factorial and also let

$$(u;q)_\infty=\prod_{n=0}^{\infty}(1-uq^n).$$

The two q-exponentials are then given by (see [An, chap. 2])

$$e(u;q)=\sum_{n\ge 0}\frac{u^n}{(q;q)_n}=(u;q)_\infty^{-1}\,; \tag{2.1}$$

$$E(u;q)=\sum_{n\ge 0}q^{n(n-1)/2}\frac{u^n}{(q;q)_n}=(-u;q)_\infty. \tag{2.2}$$

On the other hand, the exponential generating function for the Eulerian polynomials has the form

$$\sum_{n\ge 0}A_n(t)\frac{u^n}{n!}=\frac{1-t}{1-t\exp((1-t)u)}, \tag{2.3}$$

(see, e.g., [Fo-Sch, p. 68]), while the generating function for the polynomials $A_n(t,q)$, as it was derived probably for the first time by STANLEY [St], reads

$$\sum_{n\ge 0}A_n(t,q)\frac{u^n}{(q;q)_n}=\frac{1-t}{1-t\,e((1-t)u;q)}. \tag{2.4}$$

Notice that (2.4) is obtained from (2.3) by replacing the factorial by the q-ascending factorial and the exponential by the first q-exponential. When the second q-exponential $E(u;q)$ is used instead, we obtain other q-analogs of the Eulerian polynomials that also have a combinatorial interpretation (see, e.g., [De-Fo]).

The right-hand side of (2.4) is also equal to

$$\Bigl(1 - t\sum_{n\ge 1}(1-t)^{n-1}\frac{u^n}{(q;q)_n}\Bigr)^{-1},$$

so that (2.4) is equivalent to the recurrence relation

$$A_n(t,q) = t(1-t)^{n-1} + \sum_{1\le i\le n-1} {n \brack i}_q A_i(t,q)\, t\,(1-t)^{n-1-i}, \tag{2.5}$$

where ${n \brack i}_q$ denotes the q-binomial coefficient $(q;q)_n/(q;q)_i(q;q)_{n-i}$. With $q=1$ we obtain the recurrence relation for the Eulerian polynomials

$$A_n(t) = t(1-t)^{n-1} + \sum_{1\le i\le n-1} \binom{n}{i} A_i(t)\, t\,(1-t)^{n-1-i}. \tag{2.6}$$

Thus to calculate $B_n(t)$ it suffices to let q tend to -1 in (2.5). It is readily seen that

$$\lim_{q\to -1} {2m \brack 2i}_q = \lim_{q\to -1} {2m+1 \brack 2i}_q = \lim_{q\to -1} {2m+1 \brack 2i+1}_q = \binom{m}{i},$$
$$\lim_{q\to -1} {2m \brack 2i+1}_q = 0.$$

Hence (2.5) yields

$$B_{2n}(t) = t(1-t)^{2n-1} + \sum_{1\le i\le n-1} \binom{n}{i} B_{2i}(t)\, t\,(1-t)^{2n-1-2i}; \tag{2.7}$$

$$\begin{aligned} B_{2n+1}(t) = t(1-t)^{2n} &+ \sum_{1\le i\le n} \binom{n}{i} B_{2i}(t)\, t\,(1-t)^{2n-2i} \\ &+ \sum_{0\le i\le n-1} \binom{n}{i} B_{2i+1}(t)\, t\,(1-t)^{2n-2i-1}. \end{aligned} \tag{2.8}$$

By comparing (2.6) and (2.7) we see that the polynomials $(1-t)^n A_n(t)$ satisfy the same recurrence relation as the polynomials $B_{2n}(t)$. Hence (1.10a) holds.

To establish (1.10b) we proceed by induction on n. If (1.10b) holds up to $(2n-1)$, the right-hand side of (2.8) becomes

$$\begin{aligned}
t(1-t)^{2n}+\sum_{1\le i\le n}\binom{n}{i}A_i(t)\,t\,(1-t)^{2n-i}+\sum_{0\le i\le n-1}\binom{n}{i}A_{i+1}(t)\,t\,(1-t)^{2n-1-i}&\\
=t(1-t)^{2n}+\sum_{1\le i\le n}\binom{n}{i}A_i(t)\,t\,(1-t)^{2n-i}&\\
+\sum_{1\le i\le n}\binom{n}{i-1}A_i(t)\,t\,(1-t)^{2n-i}&\\
=(1-t)^n\bigl[t(1-t)^n+\sum_{1\le i\le n}\binom{n+1}{i}A_i(t)\,t\,(1-t)^{(n+1)-1-i}\bigr]&\\
=(1-t)^nA_{n+1}(t),&
\end{aligned}$$

by using recurrence relation (2.6) in the last step. This achieves the proof of Theorem 1.

3. Symmetric functions

Let $x=(x_1,x_2,\dots)$ be an infinite sequence of variables and for each $r=1,2,\dots$ denote by $h_r(x)$ the *homogeneous* symmetric function in the x_j's and by $p_r(x)$ the *power sum* $\sum_j x_j^r$. By convention, $h_0(x)=1$. The generating function $H(u;x)=\sum_{r\ge0}u^r h_r(x)$ can be evaluated in different forms :

$$H(u;x)=\prod_{j\ge1}(1-ux_j)^{-1}=\exp\sum_{r\ge1}u^r\frac{p_r(x)}{r}. \tag{3.1}$$

(See, e.g., [Mac, p. 14 and 17].)

Recall that a partition of an integer n can be expressed as a non-increasing sequence $\lambda=(\lambda_1,\lambda_2,\dots)$ $(\lambda_1\ge\lambda_2\ge\cdots)$, or as a word $\lambda=1^{m_1}2^{m_2}\dots$ (the multiplicative notation) with the meaning that λ has m_1 parts λ_i equal to 1, m_2 parts λ_i equal to 2, etc… As usual, to each partition λ we attach the constant

$$z_\lambda=1^{m_1}2^{m_2}\dots m_1!\,m_2!\,\dots$$

and the *power symmetric function*

$$p_\lambda(x)=p_{\lambda_1}(x)p_{\lambda_2}(x)\dots$$

Also $|\lambda| = n$ means that λ is a partition of n and the notation $l(\lambda)$ stands for the number of parts of λ.

Now let (c_i) $(i = 0, 1, \dots)$ be a sequence of elements belonging to some given field. The relation

$$\sum_{i\ge 0} C_i \frac{u^i}{i!} = \sum_{i\ge 0} c_i (e^u - 1)^i \tag{3.2}$$

defines a sequence (C_i) in a unique manner. Let v be another variable independent from u and the x_j's and expand the series

$$\sum_{i\ge 0} c_i \bigl(H(u;x)e^v - 1\bigr)^i.$$

We may write

$$\sum_{m\ge 0,\, n\ge 0} \mathcal{C}_{m,n}(x) u^n \frac{v^m}{m!} = \sum_{i\ge 0} c_i \bigl(H(u;x)e^v - 1\bigr)^i, \tag{3.3}$$

where each $\mathcal{C}_{m,n}(x)$ is a symmetric function in the x_j's of degree n. Our first goal is to express each $\mathcal{C}_{m,n}(x)$ in the basis $(p_\lambda(x))$ of the ring of the symmetric functions.

Lemma 3.1. — *Let* $\mathcal{C}_{m,n}(x)$ *be the symmetric function defined by* (3.3). *Then*

$$\mathcal{C}_{m,n}(x) = \sum_\lambda C_{m+l(\lambda)} \frac{p_\lambda(x)}{z_\lambda}, \qquad (|\lambda| = n), \tag{3.4}$$

where the C_i *are the coefficients defined in* (3.2).

Proof. — Expand the right-hand side of (3.3)

$$\begin{aligned}
\sum_{m\ge 0,\, n\ge 0} \mathcal{C}_{m,n}(x) u^n \frac{v^m}{m!} &= \sum_{i\ge 0} c_i \bigl(H(u;x)e^v - 1\bigr)^i \\
&= \sum_{i\ge 0} c_i \Bigl(\exp\Bigl(\sum_r u^r \frac{p_r(x)}{r} + v\Bigr) - 1\Bigr)^i \\
&= \sum_{a\ge 0} \frac{C_a}{a!} \Bigl(\sum_r u^r \frac{p_r(x)}{r} + v\Bigr)^a, \qquad (3.5)
\end{aligned}$$

by (3.1) and (3.2). Thus $\mathcal{C}_{m,n}(x)/m!$ is equal to the coefficient of degree m in v and n in u of the expansion of the right-hand of (3.5). In short, write

$$\frac{\mathcal{C}_{m,n}(x)}{m!} = [v^m u^n] \sum_{a\ge 0} \frac{C_a}{a!} \Bigl(\sum_r u^r \frac{p_r(x)}{r} + v\Bigr)^a$$

$$= [v^m u^n] \sum_{b\ge 0} \frac{C_{m+b}}{(m+b)!} \Bigl(\sum_r u^r \frac{p_r(x)}{r} + v\Bigr)^{m+b}$$
$$= [u^n] \sum_{b\ge 0} \frac{C_{m+b}}{m!\, b!} \Bigl(\sum_r u^r \frac{p_r(x)}{r}\Bigr)^{b}.$$

Hence

$$\begin{aligned}\mathcal{C}_{m,n}(x) &= [u^n] \sum_{b\ge 0} \frac{C_{m+b}}{b!} \Bigl(\sum_{r=1}^{n} u^r \frac{p_r(x)}{r}\Bigr)^{b}\\ &= [u^n] \sum_{b\ge 0} \frac{C_{m+b}}{b!} \sum_{m_1,m_2,\ldots} \frac{b!}{1!^{m_1} 2!^{m_2}\ldots} u^n \frac{p_1(x)^{m_1}}{1^{m_1}} \frac{p_2(x)^{m_2}}{2^{m_2}} \cdots\\ &\qquad\qquad [m_1+m_2+\cdots = b\,;\ 1.m_1 + 2.m_2 + \cdots = n.]\\ &= \sum_\lambda C_{m+l(\lambda)} \frac{p_\lambda(x)}{z_\lambda}, \qquad (|\lambda| = n). \quad \blacksquare\end{aligned}$$

Next for each partition $\lambda = 1^{m_1} 2^{m_2} \ldots$ of the integer n let

$$T_\lambda(q) = \frac{(q;q)_n}{\prod_j (1-q^j)^{m_j}} = (q;q)_n\, p_\lambda(1, q, q^2, \ldots). \tag{3.6}$$

It is easy to verify that $T_\lambda(q)$ is a polynomial of degree $n(n-1)/2$. Let

$$K_{m,n}(q) = (q;q)_n\, \mathcal{C}_{m,n}(1, q, q^2, \ldots). \tag{3.7}$$

By (3.4)

$$K_{m,n}(q) = \sum_\lambda C_{m+l(\lambda)} \frac{T_\lambda(q)}{z_\lambda} \qquad (|\lambda| = n), \tag{3.8}$$

so that, by (3.6), $K_{m,n}(q)$ is a polynomial in q of degree at most equal to $n(n-1)/2$. Now replace each variable x_i by q^{i-1} $(i = 1, 2, \ldots)$ in (3.3) and use notation (3.7). We obtain the identity

$$\sum_{m\ge 0,\, n\ge 0} K_{m,n}(q) \frac{u^n}{(q;q)_n} \frac{v^m}{m!} = \sum_i c_i \bigl(e(u;q) e^v - 1\bigr)^i. \tag{3.9}$$

Lemma 3.2. — *Let $n = ka + b$, $0 \le b \le k-1$ and $\lambda = 1^{m_1} 2^{m_2} \ldots$ be a partition of n. Then the following congruences hold :*

(i) if $m_k \ne a$, then $T_\lambda(q) \equiv 0 \bmod \Phi_k(q)$.

(ii) if $m_k = a$, let $\lambda^* = 1^{m_1} \dots (k-1)^{m_{k-1}} (k+1)^{m_{k+1}} \dots$ be the partition obtained from λ by deleting the $m_k = a$ parts equal to k. Then

$$T_\lambda \equiv k^a a!\, T_{\lambda^*}(q) \bmod \Phi_k(q).$$

The proof of the lemma follows from the following congruence already proved in ([De1], lemma 2.1) :

$$(q;q)_n/(1-q^k)^a \equiv k^a a!\, (q;q)_b \bmod \Phi_k(q).$$

Note that when $m_k = a$, we have $l(\lambda) = l(\lambda^*) + a$ and $z_\lambda = z_{\lambda^*}\, k^a a!$ Reporting the foregoing congruences into (3.8) leads to

$$K_{m,n}(q) \equiv \sum_{\lambda^*} C_{m+l(\lambda^*)+a} \frac{T_{\lambda^*}(q)}{z_{\lambda^*}} \bmod \Phi_k(q) \quad (|\lambda^*| = b). \tag{3.10}$$

In particular by (3.8)

$$K_{m+a,b}(q) = \sum_{\lambda^*} C_{m+l(\lambda^*)+a} \frac{T_{\lambda^*}(q)}{z_{\lambda^*}} \quad (|\lambda^*| = b).$$

Therefore

$$K_{m,ka+b}(q) \equiv K_{m+a,b}(q) \bmod \Phi_k(q). \tag{3.11}$$

4. Eulerian polynomials

We now apply the previous results to the sequence of the Eulerian polynomials. Let

$$c_1 = \frac{t}{1-t}, \qquad c_i = (c_1)^i \quad (i \ge 1). \tag{4.1}$$

First, the generating functions (2.3), (2.4) and (1.12) may be rewritten as

$$\sum_{n\ge 0} \frac{A_n(t)}{(1-t)^n} \frac{u^n}{n!} = \sum_{i\ge 0} \Bigl(\frac{t}{1-t}\Bigr)^i (e^u - 1)^i, \tag{4.2}$$

$$\sum_{n\ge 0} \frac{A_n(t,q)}{(1-t)^n} \frac{u^n}{(q;q)_n} = \sum_{i\ge 0} \Bigl(\frac{t}{1-t}\Bigr)^i \bigl(e(u;q) - 1\bigr)^i, \tag{4.3}$$

$$\sum_{n\ge 0} \frac{A_{m,n}(t,q)}{(1-t)^{m+n}} \frac{v^m}{m!} \frac{u^n}{(q;q)_n} = \sum_{i\ge 0} \Bigl(\frac{t}{1-t}\Bigr)^i \bigl(e(u;q)\exp v - 1\bigr)^i, \tag{4.4}$$

so that when the c_i's are given by (4.1), the coefficients C_i of (3.2) are equal to

$$C_i = \frac{A_i(t)}{(1-t)^i} \qquad (i \ge 0). \tag{4.5}$$

When $u = 0$, formula (3.9) becomes

$$\sum_{m\ge 0} K_{m,0}(q)\frac{v^m}{m!} = \sum_{i\ge 0} c_i\bigl(e^v - 1\bigr)^i,$$

and for $v = 0$

$$\sum_{n\ge 0} K_{0,n}(q)\frac{u^n}{(q;q)_n} = \sum_{i\ge 0} c_i\bigl(e(u;q) - 1\bigr)^i.$$

By comparison with (4.2), (4.3) and (4.4) we then have

$$\frac{A_m(t)}{(1-t)^m} = K_{m,0}(q) = C_m, \qquad \frac{A_n(t,q)}{(1-t)^n} = K_{0,n}(q),$$

and

$$\frac{A_{m,n}(t,q)}{(1-t)^{m+n}} = K_{m,n}(q).$$

By (3.11) we conclude that

$$\frac{A_{m,ka+b}(t,q)}{(1-t)^{m+ka+b}} \equiv \frac{A_{m+a,b}(t)}{(1-t)^{m+a+b}} \bmod \Phi_k(q),$$

i.e.,

$$A_{m,ka+b}(t,q) \equiv (1-t)^{(k-1)a} A_{m+a,b}(t,q) \bmod \Phi_k(q).$$

This achieves the proof of Theorem 2.

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Jacques DÉSARMÉNIEN,
Département d'informatique,
Institut universitaire de technologie,
Université Robert-Schumann,
72, route du Rhin,
F-67400 Illkirch-Graffenstaden.
email : A18630@FRCCSC21.BITNET

Dominique FOATA,
Département de mathématique,
Université Louis-Pasteur,
7, rue René-Descartes,
F-67084 Strasbourg Cedex.
email : FOATA@FRCCSC21.BITNET