Eulerian Calculus, I : Univariable Statistics†

BY

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Further transformations are derived on the symmetric group and related structures that extend previous works on the equidistributions of several classical univariable statistics.

De nouvelles transformations sur le groupe symétrique et les structures parentes sont explicitées et servent à prolonger des travaux antérieurs sur l'équidistribution de plusieurs statistiques classiques à une variable.

1. Introduction

This paper is the first one of a series of three articles (see [ClFo93b] and [ClFo93c] for the other two), in which we investigate further extensions of the classical Eulerian Calculus that involves, first, the constructions of transformations on the symmetric group and related structures, then, explicit calculations of distributions of order statistics on those structures.

In his treatise on Combinatory Analysis, MacMahon [Mac15] made an extensive study of the distributions of several statistics on permutations, or more generally, on permutations with repeated elements (called *permutations on multisets* by several authors, but referred to as *words* in the present paper). He introduced the notions of *descents*, *excedances*, *major index* for finite sequences of numbers and calculated the first generating functions.

Those three notions can be defined as follows: let X^* be the free monoid generated by a totally ordered alphabet X that for convenience we shall take as being the subset $[\,r\,] = \{1, 2, \ldots, r\}$ $(r \geq 1)$ of the positive integers with its standard ordering. The elements of X^* are finite *words* $w = x_1x_2 \ldots x_m$ with letters x_i taken from X. A word w' is said to be a *rearrangement* of the word w if it can be obtained from w by permuting the letters $x_1, x_2, \ldots, x_m$ in some order. The set of all the rearrangements of a word w will be denoted by $C(w)$. Such a set necessarily contains a unique word $v = y_1y_2 \ldots y_m$ whose letters are in non-decreasing order:

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$y_1 \le y_2 \le \cdots \le y_m$. There is then a one-to-one correspondence between rearrangement classes and non-decreasing words. It will be convenient to denote by $\overline{w}$ the unique non-decreasing word in the class $C(w)$ [also equal to $C(\overline{w})$].

Let $w = x_1 x_2 \ldots x_m$ and let $\overline{w} = v = y_1 y_2 \ldots y_m$ be its non-decreasing rearrangement. The *number of excedances*, $\operatorname{exc} w$, and the *number of descents*, $\operatorname{des} w$, of the word w are classically defined as (see [CaFo69])

$$\operatorname{exc} w = \#\{i : 1 \le i \le m,\, x_i > y_i\},$$
$$\operatorname{des} w = \#\{i : 1 \le i \le m-1,\, x_i > x_{i+1}\},$$

while the *major index*, $\operatorname{maj} w$, is the *sum* of the i's such that $1 \le i \le m-1$ and $x_i > x_{i+1}$.

MacMahon [Mac15, p. 186] proved that for each rearrangement class $C(v)$ and each integer j there are as many words $w \in C(v)$ such that $\operatorname{exc} w = j$ as there are words $w' \in C(v)$ such that $\operatorname{des} w' = j$. To establish the result, he made use of his so-called Master Theorem [Mac15, p. 97], that can be stated as follows (see [CaFo69, chap. V] for a non-commutative version):

Let $B = (b(i,j))$ $(1 \le i,j \le r)$ be a square matrix with coefficients in a commutative ring, let X^* be the free monoid generated by $X = [\,r\,]$ and let $u_1, u_2, \ldots, u_r$ be r commutative variables. If $w = x_1 x_2 \ldots x_m$ is a word in X^* whose non-decreasing rearrangement is $v = y_1 y_2 \ldots y_m = 1^{c_1} 2^{c_2} \ldots r^{c_r}$, define

$$\beta(w) = b(y_1, x_1) b(y_2, x_2) \ldots b(y_m, x_m) \quad \textit{and} \quad u(w) = u_1^{c_1} u_2^{c_2} \ldots u_r^{c_r}.$$

Let also U be the diagonal matrix $U = \operatorname{diag}(u_1, u_2, \ldots, u_r)$. Then the following identity holds

$$\frac{1}{\det(I_r - BU)} = \sum_w \beta(w) u(w), \tag{1.1}$$

where I_r is the identity matrix and where the sum is over all words $w \in X^*$.

If v is the non-decreasing word $1^{c_1} 2^{c_2} \ldots r^{c_r}$, we will also denote the rearrangement class $C(v)$ by $R(\mathbf{c})$ and put $\mathbf{u}^{\mathbf{c}} = u_1^{c_1} u_2^{c_2} \ldots u_r^{c_r}$. The rearrangement classes will then be indexed by sequences of non-negative integers $\mathbf{c} = (c_1, c_2, \ldots, c_r)$ of length r. (A more refined notation will be used in the sequel.)

When $b(i,j) = t$, if $i < j$, and 1 in the other cases, it is readily seen that the monomial $\beta(w)$ is equal to $t^{\operatorname{exc} w}$. Let then

$$A_{\mathbf{c}}(t) = \sum_w t^{\operatorname{exc} w} \qquad (w \in R(\mathbf{c})),$$

be the generating polynomial for $R(\mathbf{c})$ by the number of excedances "exc." The generating function for those polynomials is then directly derived from the Master Theorem

$$\frac{1}{\det(I_r - BU)} = \sum_{\mathbf{c}} \mathbf{u}^{\mathbf{c}} A_{\mathbf{c}}(t), \tag{1.2}$$

a result already obtained by MacMahon.

Now the determinant occurring in (1.2) can be further evaluated. An easy calculation shows that

$$\begin{aligned}\det(I_r - BU) &= \begin{vmatrix} 1-u_1 & -tu_2 & \ldots & -tu_r \\ -u_1 & 1-u_2 & \ldots & -tu_r \\ \vdots & \vdots & \ddots & \vdots \\ -u_1 & -u_2 & \ldots & 1-u_r \end{vmatrix} \\ &= 1 - e_1(\mathbf{u}) + (1-t)e_2(\mathbf{u}) - \cdots + (-1)^r(1-t)^{r-1}e_r(\mathbf{u}),\end{aligned}$$

where the $e_i(\mathbf{u})$'s are the elementary symmetric functions in the u_i's. Taking this calculation into account after replacing each $u_i(1-t)$ by u_i and substituting into (1.2) leads to the identity

$$\sum_{s\ge 0} \frac{t^s}{(1-u_1)^{s+1}\ldots(1-u_r)^{s+1}} = \sum_{\mathbf{c}} \frac{\mathbf{u}^{\mathbf{c}}}{(1-t)^{c+1}} A_{\mathbf{c}}(t), \tag{1.3}$$

where $c = c_1 + \cdots + c_r$. The expression (1.3) is more appropriate for proving that the left-hand side is actually the generating function for the polynomials $A_{\mathbf{c}}(t)$ in the "des"-interpretation. See section 2 in which a calculation is made in a more general case. Those techniques are standard (see, e.g. [Ga79]). In fact, MacMahon proved his result on the equidistribution of "exc" and "des" by obtaining (1.3) in two different ways, in the "exc" and "des" interpretations.

Several years later, a "bijective" proof (as some people say to-day) was devised to prove the MacMahon equidistribution property. The so-called *first fundamental transformation* Φ of X^* onto itself was constructed (see [CaFo69] and also [Lo83, chap. 10]); it maps each rearrangement class $C(v)$ onto itself, in a bijective manner, and has the property that

$$\operatorname{exc} \Phi(w) = \operatorname{des} w \tag{1.4}$$

holds for every $w \in C(v)$.

For each pair of letters (a,b) and each word $w = x_1x_2\ldots x_m$ denote by $[a,b](w)$ the number of subscripts i such that $1 \le i \le m-1$ and $x_i = a$, $x_{i+1} = b$. If $v = y_1y_2\ldots y_m$ is the non-decreasing rearrangement of w, denote by $\begin{bmatrix} b \\ a \end{bmatrix}(w)$ the number of subscripts i such that $1 \le i \le m$ and $y_i = b$, $x_i = a$. As was proved in [CaFo69], the first fundamental transformation has the further property that for each pair (a, b) satisfying $a > b$, the identity

$$[a, b](w) = \begin{bmatrix} b \\ a \end{bmatrix}(\Phi(w)) \tag{1.5}$$

holds. Of course property (1.4) is a consequence of (1.5).

A further analysis of the properties of Φ shows that the restriction $a > b$ assumed in (1.5) can be relaxed under certain conditions to include equality $a = b$. The purpose of this paper is to investigate those conditions. Instead of taking the matrix B as above, having only 1's on and under the diagonal and t above it, choose two non-negative integers j and k such that $j + k = r$. Then form the matrix B_k of order r still having only 1's under the diagonal and only t's above, but let its diagonal, $\operatorname{diag} B_k$, be made of j 1's, followed by k t's, i.e.,

$$\operatorname{diag} B_k = (\underbrace{1, \ldots, 1}_{j \text{ times}}, \underbrace{t, \ldots, t}_{k \text{ times}}) \tag{1.6}$$

The monomial $\beta(w) = b(y_1, x_1) \ldots b(y_m, x_m)$ attached to the word $w = x_1 \ldots x_m$ in the Master Theorem is then a power of t, say, $t^{\operatorname{exc}_k w}$, where

(1.7) $\operatorname{exc}_k w$ *is the number of* i *such that* $1 \le i \le m$ *and either* $x_i > y_i$, *or* $x_i = y_i$ *and* $x_i \ge j + 1$.

A refined definition will be given further in the paper. The following two questions arise: what statistic, say "des_k," must be defined on X^* that has the same distribution as "exc_k" on each rearrangement class $C(v)$ and that reduces to "des" when $k = 0$? Suppose that such a statistic can be defined in simple terms (it can!) and has such a property. The second question is how a transformation Φ_k on X^* can be constructed that maps each rearrangement class onto itself and satisfies the identity

$$\operatorname{des}_k w = \operatorname{exc}_k \Phi_k(w). \tag{1.8}$$

In section 2 we show that a calculation à la MacMahon first suggests the definition of the appropriate statistic "des_k," that can be stated as:

(1.9) $\operatorname{des}_k w$ *is the number of* i *such that* $1 \le i \le m$ *and either* $x_i > x_{i+1}$, *or* $x_i = x_{i+1}$ *and* $x_i \ge j + 1$. *(By convention,* $x_{m+1} = j + \frac{1}{2}$.*)*

Section 2 also provides an answer to the first question by proving that the generating polynomials for each rearrangement class $C(v)$ by either "exc_k," or "des_k," are the same.

In fact, there are several statistics "$\mathrm{des}_{k,D}$" and "$\mathrm{exc}_{k,D}$" that can be defined and for which property (1.8) holds, depending on different total orders D we may impose on the interval $[\,r\,]$. This leads to proving invariance principles that essentially say that the distributions of all those statistics "$\mathrm{des}_{k,D}$" and "$\mathrm{exc}_{k,D}$" are independent of the ordering D. All this is the object of section 3 (see Theorems 3.1 and 3.2).

In section 4 we recall the construction of the first fundamental transformation Φ. Section 5 contains the main result of the paper, i.e., the construction of a transformation Φ_k that satisfies identity (1.8). In fact, the construction can be made for each pair $(\mathrm{des}_{k,D}, \mathrm{exc}_{k,D})$, as long as the ordering D is *compatible* with k, a notion that will be further defined.

In section 6 we specialize our results to the case of *permutations*. When $c_1 = \cdots = c_r = 1$, the polynomial

$$\sum_w t^{\mathrm{des}\, w} = \sum_w t^{\mathrm{exc}\, w} \qquad (w \in R(\mathbf{c}))$$

is the usual *Eulerian polynomial* $A_r(t)$ of order r (see, e.g., [Ri58, FoSc70]). When "des" or "exc" are replaced by "des_k" or "exc_k," the generating polynomial $A_r^k(t)$ thereby obtained appears to be a natural interpolation between $A_r(t)$ and $t\,A_r(t)$. We have managed to give several explicit formulas for those polynomials. Steingrìmsson [Ste93] has also generalized the definitions of "des" and "exc" from permutations to so-called "indexed permutations", i.e., elements of the wreath product $Z_k \wr \mathcal{S}_n$. As shown in section 6, his model coincides partly with ours in this permutation case.

In the two subsequent papers we consider joint statistics, and especially, will study the joint behaviours of "des_k" with a new major index "maj_k," and "exc_k" with another new mahonian statistic "den_k." We will also calculate the distributions of those joint statistics using the algebra of symmetric functions.

The following notations will be used throughout. Let r be a positive integer, let $X = [\,r\,]$ and let k be a fixed integer, $0 \le k \le r$. Put $j = r - k$. Let $\mathbf{c} = (c_1, \ldots, c_j)$ and $\mathbf{d} = (d_1, \ldots, d_k)$ be two vectors with positive integer components. Let $c = c_1 + \cdots + c_j$, $d = d_1 + \cdots + d_k$ and $c + d = m$. We will work with the rearrangement class $R(\mathbf{c}, \mathbf{d})$ of all $m!/(c_1! \ldots c_j!\, d_1! \ldots d_k!)$ rearrangements of the non-decreasing word $v = y_1 y_2 \ldots y_m = 1^{c_1} \ldots j^{c_j} (j+1)^{d_1} \ldots r^{d_k}$. We call the letters $1, \ldots, j$ *small* and the letters $j+1, \ldots, r$ *large*. It is convenient to adjoin to the set X one element $\star$ that is small but greater than any other small letter.

2. A Calculation a La MacMahon

In this section we take the natural ordering on $X = [\,r\,]$. If $w = x_1x_2\ldots x_m$ is a word of the rearrangement class $R(\mathbf{c},\mathbf{d})$, then "exc_k" refers to the definition stated in (1.7). It will be convenient to replace the variables $u_{j+1}, \ldots, u_r$ by $v_1, \ldots, v_k$ and write $\mathbf{u}^{\mathbf{c}}$ for $u_1^{c_1}\ldots u_j^{c_j}$ and $\mathbf{v}^{\mathbf{d}}$ for $v_1^{d_1}\ldots v_k^{d_k}$. However we keep the notation U for the matrix $U = \mathrm{diag}(u_1,\ldots,u_j,v_1,\ldots,v_k)$. Now let

$$A_{\mathbf{c},\mathbf{d}}(t) = \sum_w t^{\mathrm{exc}_k\, w} \qquad (w \in R(\mathbf{c},\mathbf{d})), \tag{2.1}$$

be the generating polynomial for the class $R(\mathbf{c},\mathbf{d})$ by "exc_k." We have seen in the introduction that the generating function for those polynomials was simply

$$\sum_{\mathbf{c},\mathbf{d}} \mathbf{u}^{\mathbf{c}}\mathbf{v}^{\mathbf{d}} A_{\mathbf{c},\mathbf{d}}(t) = \frac{1}{\det(I_r - B_kU)}, \tag{2.2}$$

where B_k is the matrix defined in (1.6).

Lemma 2.1. *The following identity holds*

$$\det(I_r - B_kU) = 1 + \sum_{i=1}^{j}(-1)^i(1-t)^{i-1}e_i(u_1,\ldots,u_j) - \sum_{i=1}^{k} t(1-t)^{i-1}e_i(v_1,\ldots,v_k), \tag{2.3}$$

where the e_i's stand for the elementary symmetric functions in the variables between parentheses.

Proof. First

$$\det(I_r - B_kU) = 1 + \sum_{i=1}^{r}(-1)^i \sum_{1\le l_1<\cdots<l_i\le r} u_{l_1}\ldots u_{l_i} \det\bigl(b(l,l')\bigr)_{(l,l'\in\{l_1,\ldots,l_i\})},$$

by letting $u_{j+1} = v_1, \ldots, u_r = v_k$. Secondly notice that the determinant $\det\bigl(b(l,l')\bigr)_{(l,l'\in\{l_1,\ldots,l_i\})}$ is zero, if $\{l_1,\ldots,l_i\}$ has a non-empty intersection with both the intervals $[1,j]$ and $[j+1,r]$, is equal to $(1-t)^{i-1}$, if $1 \le l_1 < \cdots < l_i \le j$, and to $t(t-1)^{i-1}$, if $j+1 \le l_1 < \cdots < l_i \le r$. $\square$

Now substitute the value (2.3) into identity (2.2), multiply numerator and denominator of the fraction by $(1-t)$ and make the substitution

$u_i(1-t) \leftarrow u_i$ and $v_i(1-t) \leftarrow v_i$. We get (remember that $c = c_1 + \cdots + c_j$ and $d = d_1 + \cdots + d_k$, so that $c + d = m$)

$$\begin{aligned}\sum_{\mathbf{c},\mathbf{d}} \frac{\mathbf{u}^{\mathbf{c}}\mathbf{v}^{\mathbf{d}}}{(1-t)^{c+d+1}} A_{\mathbf{c},\mathbf{d}}(t) &\\ &= \frac{1}{(1-t) + \sum_{i=1}^{j} (-1)^i e_i(u_1,\ldots,u_j) - \sum_{i=1}^{k} t e_i(v_1,\ldots,v_k)}\\ &= \frac{1}{(1-u_1)\ldots(1-u_j) - t(1+v_1)\ldots(1+v_k)},\end{aligned}$$

and finally

$$\sum_{\mathbf{c},\mathbf{d}} \frac{\mathbf{u}^{\mathbf{c}}\mathbf{v}^{\mathbf{d}}}{(1-t)^{c+d+1}} A_{\mathbf{c},\mathbf{d}}(t) = \sum_{s\ge 0} t^s \frac{(1+v_1)^s \ldots (1+v_k)^s}{(1-u_1)^{s+1}\ldots(1-u_j)^{s+1}}. \tag{2.4}$$

Now recurrence relations for the polynomials $A_{\mathbf{c},\mathbf{d}}(t)$ can be derived, following a classical pattern (see, e.g., [Ga79]). Let A be the common value of the previous series. Comparing the derivatives $D_{v_k}A$ and D_tA calculated with the right-hand side of (2.4) leads to the relation

$$(1+v_k)D_{v_k}A = t\,D_tA. \tag{2.5}$$

Now with the help of the left-hand side of (2.4) the latter identity yields

$$\begin{aligned}(d_k+1)&A_{\mathbf{c},\mathbf{d}+1_k}(t)\\ &= (t(c+d+1+d_k) - d_k)A_{\mathbf{c},\mathbf{d}}(t) + t(1-t)A'_{\mathbf{c},\mathbf{d}}(t),\end{aligned} \tag{2.6}$$

where $\mathbf{d}+1_k$ is the sequence $(d_1,\ldots,d_k+1)$ and $A'_{\mathbf{c},\mathbf{d}}(t)$ is the derivative of $A_{\mathbf{c},\mathbf{d}}(t)$. Next let $A_{\mathbf{c},\mathbf{d}}(t) = \sum_s A_{\mathbf{c},\mathbf{d},s}t^s$. Then (2.6) is equivalent to the recurrence relation

$$(d_k+1)A_{\mathbf{c},\mathbf{d}+1_k,s} = (c+d+d_k+2-s)A_{\mathbf{c},\mathbf{d},s-1} + (s-d_k)A_{\mathbf{c},\mathbf{d},s}. \tag{2.7}$$

We could as well look for a relation between $D_{u_j}A$ and D_tA and easily obtain

$$(1-u_j)D_{u_j}A = t\,D_tA + A, \tag{2.8}$$

from which we can derive the relations:

$$\begin{aligned}(c_j+1)&A_{\mathbf{c}+1_j,\mathbf{d}}(t)\\ &= (t(c+d-c_j) + c_j + 1)A_{\mathbf{c},\mathbf{d}}(t) + t(1-t)A'_{\mathbf{c},\mathbf{d}}(t),\end{aligned} \tag{2.9}$$

$$(c_j+1)A_{\mathbf{c}+1_j,\mathbf{d},s} = (c+d+1-s-c_j)A_{\mathbf{c},\mathbf{d},s-1} + (c_j+1+s)A_{\mathbf{c},\mathbf{d},s}, \tag{2.10}$$

where $\mathbf{c}+1_j = (c_1,\ldots,c_{j-1},c_j+1)$.

The two pairs of recurrence relations (2.6), (2.9), on the one hand, and (2.7), (2.10), on the other hand, uniquely determine the polynomials $A_{\mathbf{c},\mathbf{d}}(t)$ using the initial condition $A_{\mathbf{0},\mathbf{0}}(t) = 1$ (or $A_{\mathbf{0},\mathbf{0},0} = 1$ and $A_{\mathbf{0},\mathbf{0},s} = 0$ for $s \ne 0$). Actually, using the two relations (2.5) and (2.8), we can reprove that the left-hand side of (2.4) is precisely equal to the right-hand side.

PROPOSITION 2.2. *Let $A^{\mathrm{des}}_{\mathbf{c},\mathbf{d}}(t)$ be the generating polynomial for $R(\mathbf{c},\mathbf{d})$ by the number of k-descents* "des_k." *Then*

$$A^{\mathrm{des}}_{\mathbf{c},\mathbf{d}}(t) = A_{\mathbf{c},\mathbf{d}}(t).$$

Proof. We already know that the coefficients of $A_{\mathbf{c},\mathbf{d}}(t)$ satisfy (2.7) and (2.10). We only have to verify that the same holds for the coefficients $A^{\mathrm{des}}_{\mathbf{c},\mathbf{d},s}$ of $A^{\mathrm{des}}_{\mathbf{c},\mathbf{d}}(t)$. But $(d_k+1)A^{\mathrm{des}}_{\mathbf{c},\mathbf{d}+1_k,s}$ is the number of words in $R(\mathbf{c},\mathbf{d}+1_k)$ with s k-descents having d_k letters equal to r, plus one letter, say, equal to $\dot{r}$. To generate such a word, we can choose a word in $R(\mathbf{c},\mathbf{d})$ having s k-descents and insert $\dot{r}$ into any one of the $(s-d_k)$ k-descents that do not begin with r. Exactly $(s-d_k)A^{\mathrm{des}}_{\mathbf{c},\mathbf{d},s}$ words can be obtained by such an insertion.

The other possibility is to take a word in $R(\mathbf{c},\mathbf{d})$ having $(s-1)$ k-descents and insert $\dot{r}$ into each of the slots that is not a k-descent (there are exactly $(c+d-(s-1))$ such slots) and also after each letter r and at the end of the word (there are exactly (d_k+1) such slots). Altogether, $(c+d+d_k+2-s)A^{\mathrm{des}}_{\mathbf{c},\mathbf{d},s-1}$ words in $R(\mathbf{c},\mathbf{d}+1_k)$ with a single dotted $\dot{r}$ can be obtained in this way. This shows that the coefficients $A^{\mathrm{des}}_{\mathbf{c},\mathbf{d},s}$ also satisfy (2.8).

The same technique applies to identity (2.10). ∎

REMARK. Let σ (resp. τ) be a permutation of $\{1,2,\ldots,j\}$ (resp. of $\{1,\ldots,k\}$). Denote by $\sigma\mathbf{c}$ and by $\tau\mathbf{d}$ the sequences $(c_{\sigma 1},\ldots,c_{\sigma j})$ and $(d_{\tau 1},\ldots,d_{\tau k})$, respectively, and by $R(\sigma\mathbf{c},\tau\mathbf{d})$ the class of all the rearrangements of the word $1^{c_{\sigma 1}}\ldots j^{c_{\sigma j}}(j+1)^{d_{\tau 1}}\ldots r^{d_{\tau k}}$. As formula (2.4) is symmetric in the u_i's and in the v_i's, the two polynomials $A_{\mathbf{c},\mathbf{d}}(t)$ and $A_{\sigma\mathbf{c},\tau\mathbf{d}}(t)$ are identical. (The result will be proved combinatorially in Theorem 3.1.) In the next table we have then only displayed the values of the polynomials $A_{\mathbf{c},\mathbf{d}}(t)$ when $\mathbf{c}$ and $\mathbf{d}$ are *partitions*, i.e., non-increasing sequences.

A short table for the polynomials $A_{\mathbf{c},\mathbf{d}}(t)$:
$A_{(0),(0)}(t)=1$, $A_{(1),(0)}(t)=1$, $A_{(0),(1)}(t)=t$,
$A_{(2),(0)}(t)=1$, $A_{(1,1),(0)}(t)=1+t$, $A_{(1),(1)}(t)=2t$, $A_{(0),(1,1)}(t)=t+t^2$,
$A_{(0),(2)}(t)=t^2$,
$A_{(3),(0)}(t)=1$, $A_{(2,1),(0)}(t)=1+2t$, $A_{(1,1,1),(0)}(t)=1+4t+t^2$,
$A_{(2),(1)}(t)=3t$, $A_{(1,1),(1)}=4t+2t^2$, $A_{(1),(1,1)}(t)=2t+4t^2$,
$A_{(1),(2)}(t)=3t^2$, $A_{(0),(1,1,1)}(t)=t+4t^2+t^3$, $A_{(0),(2,1)}(t)=2t^2+t^3$,
$A_{(0),(3)}(t)=t^3$.

3. Invariance Principles

Let D be a total ordering on $[\,r\,]$. We say that D is *compatible* with k if, for all x large and y small, we have $x >_D y$. Again let $w = x_1 x_2 \ldots x_m \in X^*$ and $\overline{w} = v = y_1 y_2 \ldots y_m$ be its non-decreasing rearrangement. We say that w has a *k-excedance at i* $(1 \le i \le m)$, if either $x_i > y_i$, or $x_i = y_i$ and x_i is large. We also say that w has a *k-descent at i* $(1 \le i \le m)$, if either $x_i > x_{i+1}$, or $x_i = x_{i+1}$ and x_i is large. (By convention, $x_{m+1} = \star$.) Note that there is a descent at m if x_m is large.

The number of k-excedances and k-descents of a word w will be denoted by $\operatorname{exc}_k w$ and $\operatorname{des}_k w$, respectively. We will sometimes use the notations $\operatorname{exc}_{k,D}$ and $\operatorname{des}_{k,D}$ if we need to draw attention to the ordering D.

For example, if $X = [\,3\,]$, D is the standard ordering and w is the word $w = 3,2,2,1,1,2$ then $\operatorname{des}_0 w = 2 = \operatorname{des}_1 w$, $\operatorname{des}_2 w = 4$, $\operatorname{des}_3 w = 5$. As $\overline{w} = 1,1,2,2,2,3$, $\operatorname{exc}_0 w = \operatorname{exc}_1 w = 2$, $\operatorname{exc}_2 w = \operatorname{exc}_3 w = 3$.

Theorem 3.1. *Let D and E be total orderings of $[\,r\,]$ compatible with k. Then there is a bijection χ of $R(\mathbf{c},\mathbf{d})$ onto itself such that for all $w \in R(\mathbf{c},\mathbf{d})$,*

$$\operatorname{des}_D w = \operatorname{des}_E \chi(w).$$

Theorem 3.2. *Let D and E be total orderings of $[\,r\,]$, not necessarily compatible with k. Then there is a bijection θ of $R(\mathbf{c},\mathbf{d})$ onto itself such that for all $w \in R(\mathbf{c},\mathbf{d})$,*

$$\operatorname{exc}_D w = \operatorname{exc}_E \theta(w).$$

Note that Theorem 3.1 is not true without the assumption of compatibility. For example, let $r = 2$, $k = 1$ and consider $R((1),(1))$, the rearrangement class of $v = 1,2$. Note that 1 is small and 2 is large. Let $2 >_D 1$, $1 >_E 2$. Then D is compatible with k, but E is not. Now

$$\operatorname{des}_D 1,2 = \operatorname{des}_D 2,1 = \operatorname{exc}_D 1,2 = \operatorname{exc}_D 2,1 = 1,$$

but

$$\operatorname{des}_E 1,2 = 2, \quad \operatorname{des}_E 2,1 = 0, \quad \operatorname{exc}_E 1,2 = \operatorname{exc}_E 2,1 = 1.$$

Proof of Theorem 3.1. Suppose first that a, b are letters in $[\,r\,]$ such that $a >_D b$ and that there is no letter x in X such that $a >_D x >_D b$ (i.e., *a covers b*.) We assume that E is obtained from D by reversing the order of a and b. (We may say that E is obtained from D by a *transposition*.) Since both D and E are compatible with k, either both a and b are large or both are small.

Given a word $w \in R(\mathbf{c},\mathbf{d})$, write $w = u_1 v_1 \ldots u_m v_m$, where each u_i contains only the letters a and b, and each v_i does not contain any a or b. Let

u_i' be the word obtained from u_i by reversing the order of its letters. Accordingly, if $u_i = z_1 z_2 \dots z_s$, then $u_i' = z_s \dots z_2 z_1$. Put $w' = u_1' v_1 \dots u_r' v_r$. Since $[b,a](w') = [a,b](w)$, $[a,a](w') = [a,a](w)$ and $[b,b](w') = [b,b](w)$, while for any letters x, y other than a and b, $[a,x](w') + [b,x](w') = [a,x](w) + [b,x](w)$ and $[x,a](w') + [x,b](w') = [x,a](w) + [x,b](w)$ and also $[x,y](w') = [x,y](w)$, we see that

$$\operatorname{des}_D w = \operatorname{des}_E w'.$$

Hence we may define $\chi(w) = w'$. It is clear that χ is a bijection, as one may easily write down its inverse.

In the general case, if D and E are any orderings on X that are compatible with k, one can obtain E from D by a sequence of transpositions. This completes the proof of Theorem 3.1.

Before proving Theorem 3.2, we introduce some more notation. If w and v are words, the two-rowed matrix $\begin{pmatrix} v \\ w \end{pmatrix}$ is called a *biword.* It is a *circuit* if v is a rearrangement of w. When calculating excedances, it is often convenient to consider the circuit $\tilde{w} = \begin{pmatrix} v \\ w \end{pmatrix}$, where v is the nondecreasing rearrangement of w. A *cycle* is a circuit $\begin{pmatrix} \delta w \\ w \end{pmatrix}$ such that δw is the right to left cyclic shift of w, i.e., $w = x_1 \dots x_n$ and $\delta w = x_2 \dots x_n x_1$. We will write this cycle as $[x_1, x_2, \dots, x_n]$. A *true cycle* is a cycle $\begin{pmatrix} \delta w \\ w \end{pmatrix}$ in which no letter of w is repeated.

Multiplication of biwords $\begin{pmatrix} v \\ w \end{pmatrix}$ is defined to be juxtaposition. We assume that the biwords $\begin{pmatrix} v \\ w \end{pmatrix}$ and $\begin{pmatrix} v' \\ w' \end{pmatrix}$ commute if v and v' have no letters in common. With these definitions, it was shown in [CaFo69] that every circuit can be written as a product of true cycles, and that the factorization is unique except for the order of the factors. More precisely (see [CaFo69, chapter 4, proposition 4.1]), if a circuit factorizes into two different true cycle products, say $\alpha_1 \alpha_2 \dots \alpha_s$ and $\beta_1 \beta_2 \dots \beta_t$, then $s = t$ and we can go from the first product to the second one by finitely many steps consisting of permuting in a product two consecutive true cycles having no letters in common. Furthermore, there is an algorithm that transforms each circuit into a product of true cycles. Note of course that there may be more than one way of writing the same cycle. For example, the cycles $[1,2,3] = \begin{pmatrix} 2\ 3\ 1 \\ 1\ 2\ 3 \end{pmatrix}$ and $[2,3,1] = \begin{pmatrix} 3\ 1\ 2 \\ 2\ 3\ 1 \end{pmatrix}$ are the same.

As an example of factorization, let $w = 3,1,1,4,3,2,3,2,5,4,2,3,2,2$.

Then

$$\tilde{w} = \begin{pmatrix} \overline{w} \\ w \end{pmatrix} = \begin{pmatrix} 1\ 1\ 2\ 2\ 2\ 2\ 2\ 3\ 3\ 3\ 3\ 4\ 4\ 5 \\ 3\ 1\ 1\ 4\ 3\ 2\ 3\ 2\ 5\ 4\ 2\ 3\ 2\ 2 \end{pmatrix}$$

$$= \left(\begin{array}{c|c|c|c|c|c} 1\ 2\ 3 & 1 & 2\ 5\ 3\ 4 & 3\ 2\ 4 & 2 & 2\ 3 \\ 3\ 1\ 2 & 1 & 4\ 2\ 5\ 3 & 4\ 3\ 2 & 2 & 3\ 2 \end{array}\right)$$

$$= [3,1,2][1][4,2,5,3][4,3,2][2][3,2],$$

which is a product of true cycles. (The algorithm is explained in detail in [CaFo69].)

Proof of Theorem 3.2. As in the proof of Theorem 3.1, we may assume that a, b are letters in $[\,r\,]$ such that $a >_D b$ and a covers b. We assume that E is obtained from D by reversing the order of a and b. Note that in this case neither D nor E are assumed to be compatible with k, so we do not assume that a and b are both large or both small.

Let w be a word in $C(v)$. Factorize the circuit $\tilde{w}$ into true cycles:

$$\tilde{w} = \alpha_1 \alpha_2 \ldots \alpha_m.$$

Consider the cycle α_i. If the letters a and b do not both occur in this cycle then put $\alpha_i' = \alpha_i$. If the letters a and b do both occur in α_i then let α_i' be the cycle obtained from α_i by interchanging a and b. Let $\tilde{w}'$ be the product of the cycles α_i'. Finally, reorder the columns of $\tilde{w}'$ (remembering that columns $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} x' \\ y' \end{pmatrix}$ commute if $x \neq x'$) to get the circuit $\begin{pmatrix} v' \\ w' \end{pmatrix}$, where v' is the non-decreasing rearrangement of w' (in the ordering E). Then define $\theta(w) = w'$.

The following relations hold :

$$\begin{bmatrix} b \\ a \end{bmatrix}(w') = \begin{bmatrix} a \\ b \end{bmatrix}(w), \quad \begin{bmatrix} a \\ a \end{bmatrix}(w') = \begin{bmatrix} a \\ a \end{bmatrix}(w), \quad \begin{bmatrix} b \\ b \end{bmatrix}(w') = \begin{bmatrix} b \\ b \end{bmatrix}(w);$$

furthermore, for any letters x and y other than a and b, we have

$$\begin{bmatrix} a \\ x \end{bmatrix}(w') + \begin{bmatrix} b \\ x \end{bmatrix}(w') = \begin{bmatrix} a \\ x \end{bmatrix}(w) + \begin{bmatrix} b \\ x \end{bmatrix}(w),$$

$$\begin{bmatrix} x \\ a \end{bmatrix}(w') + \begin{bmatrix} x \\ b \end{bmatrix}(w') = \begin{bmatrix} x \\ a \end{bmatrix}(w) + \begin{bmatrix} x \\ b \end{bmatrix}(w)$$

and

$$\begin{bmatrix} x \\ y \end{bmatrix}(w') = \begin{bmatrix} x \\ y \end{bmatrix}(w).$$

It follows that $\operatorname{exc}_D w = \operatorname{exc}_E w'$. □

For example, let $w = 3,1,2,1,3,3,3,2,2,4,5,2,2,4$. Then as before

$$\tilde{w} = \begin{pmatrix} \overline{w} \\ w \end{pmatrix} = [3,1,2][1][4,2,5,3][4,3,2][2][3,2].$$

Let $a = 2$, $b = 3$. Then

$$\tilde{w}' = [2,1,3][1][4,3,5,2][4,2,3][2][2,3] = \begin{pmatrix} 1\,3\,2\,1\,3\,5\,2\,4\,2\,3\,4\,2\,3\,2 \\ 2\,1\,3\,1\,4\,3\,5\,2\,4\,2\,3\,2\,2\,3 \end{pmatrix}$$
$$= \begin{pmatrix} 1\,1\,3\,3\,3\,3\,2\,2\,2\,2\,2\,4\,4\,5 \\ 2\,1\,1\,4\,2\,2\,3\,5\,4\,2\,3\,2\,3\,3 \end{pmatrix},$$

so $w' = 2,1,1,4,2,2,3,5,4,2,3,2,3,3$.

4. The First Fondamental Transformation

As in [Lo83, chap. 10] the construction of Φ may be described as follows.

(a) Start with a word $w = x_1 x_2 \dots x_m \in C(v)$ and form its *left to right dominated factorization* $w = w_1 w_2 \dots w_p$. The factors $w_1, w_2, \dots, w_p$ of that factorization are *dominated*, in the sense that each w_i starts with its *maximum* letter, say a_i, and no other letter is equal to a_i in the factor. Furthermore, when reading those first letters from left to right, they form a *non-decreasing* sequence: $a_1 \le a_2 \le \dots \le a_p$.

For instance, the left to right dominated factorization of the following word is indicated by vertical bars:

$$w = \,|\,3,1,1,1\,|\,5\,|\,5\,|\,5,2,3,3,2,1,4\,|\,6\,|\,6,1,3\,|$$

Notice that the first letters are increasing: $3 \le 5 \le 5 \le 5 \le 6 \le 6$.

(b) The next step consists of forming a second word $\Delta w = \delta w_1 \dots \delta w_p$, where each δw_i is the right to left cyclic shift of w_i, as defined in the previous section. By means of the words Δw and w we set up the biword $\begin{pmatrix} \Delta w \\ w \end{pmatrix}$. With our example

$$\begin{pmatrix} \Delta w \\ w \end{pmatrix} = \left(\begin{array}{c|c|c|c|c|c} 1\,1\,1\,3 & 5 & 5 & 2\,3\,3\,2\,1\,4\,5 & 6 & 1\,3\,6 \\ 3\,1\,1\,1 & 5 & 5 & 5\,2\,3\,3\,2\,1\,4 & 6 & 6\,1\,3 \end{array}\right)$$

(c) In the following stage the two-row columns of $\begin{pmatrix} \Delta w \\ w \end{pmatrix}$ are rearranged in such a way that the top row is non-decreasing. When making the rearrangement we adopt the previous commutation rules:

$$\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}, \qquad \text{iff} \quad a \ne c. \tag{4.1}$$

In particular, two two-row columns with identical top entries cannot commute. Again working with the same example leads to the biword:

$$\begin{pmatrix} \dots \\ \Phi(w) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 5 & 5 & 5 & 6 & 6 \\ \mathbf{3} & 1 & 1 & \mathbf{2} & \mathbf{6} & \mathbf{5} & \mathbf{3} & 1 & 2 & 3 & 1 & 1 & 5 & 5 & 4 & 6 & 3 \end{pmatrix}$$

(e) By definition the bottom word of this last two-row matrix is the transformed word $\Phi(w)$. Its excedances have been printed in bold-face and they correspond, in a one-to-one manner, with the descents of w that are: 31, 52, 32, 21, 61.

5. The Construction of Φ_k

First, assume that the total ordering on $[\,r\,]$ is the natural ordering. We then construct a *bijection* Φ_k of each rearrangement class $R(\mathbf{c},\mathbf{d})$ onto itself such that for all $w \in R(\mathbf{c},\mathbf{d})$,

$$\operatorname{des}_k w = \operatorname{exc}_k \Phi_k(w). \tag{5.1}$$

When $k=0$, it suffices to take $\Phi_k = \Phi$ (the first fundamental transformation), since $\operatorname{des}_k = \operatorname{des}$ and $\operatorname{exc}_k = \operatorname{exc}$ in such a case. Suppose that $1 \le k \le r$ and consider a word $w = x_1 x_2 \ldots x_m \in C(v) = R(\mathbf{c},\mathbf{d})$. The construction of $\Phi_k(w)$ is made in several steps (the reader is advised to follow the successive steps with the help of the example given just after this present description) :

(a) Add $\star$ at the end of w.

(b) Consider the left to right dominated factorization of $w\star$ and for a reason that will be evident hereafter denote it by

$$w\star = \,|\, a_1u_1 \,|\, \ldots \,|\, a_pu_p \,|\, b_1v_{1,1} \,|\, b_1v_{1,2} \,|\, \ldots \,|\, b_1v_{1,m_1} \,|\, \ldots \,|\, b_qv_{q,1} \,|\, b_qv_{q,2} \,|\, \ldots \,|\, b_qv_{q,m_q} \,|\,,$$

where the a's and b's are letters, the u's and v's words, such that $1 \le a_1 \le \cdots \le a_p \le j$, $j+1 \le b_1 < \cdots < b_q = r$ and where all letters in u_i (resp. $v_{i,l}$) are *less* than a_i (resp. b_i).

(c) Next form the word

$$w' = \,|\, a_1u_1 \,|\, \ldots \,|\, a_pu_p \,|\, b_1v_{1,m_1} \,|\, \ldots \,|\, b_1v_{1,2} \,|\, b_1v_{1,1} \,|\, \ldots \,|\, b_qv_{q,m_q} \,|\, \ldots \,|\, b_qv_{q,2} \,|\, b_qv_{q,1} \,|\,.$$

Notice the difference between w and w': we have only changed the orders of the factors beginning with the same large letter.

(d) Then form the biword

$$\begin{pmatrix} \Delta w' \\ w' \end{pmatrix} = \left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c} u_1a_1 & \ldots & u_pa_p & v_{1,m_1}b_1 & \ldots & v_{1,2}b_1 & v_{1,1}b_1 & \ldots & v_{q,m_q}b_q & \ldots & v_{q,2}b_q & v_{q,1}b_q \\ a_1u_1 & \ldots & a_pu_p & b_1v_{1,m_1} & \ldots & b_1v_{1,2} & b_1v_{1,1} & \ldots & b_qv_{q,m_q} & \ldots & b_qv_{q,2} & b_qv_{q,1} \end{array}\right)$$

(e) Rearrange the columns of the latter biword in such a way that the top row is non-decreasing, still using the previous commutation rules (4.1). Thereby we obtain $\Phi(w')$:

$$\begin{pmatrix} \dots \\ \Phi(w') \end{pmatrix} = \begin{pmatrix} 1^{c_1} & \dots & j^{c_j} & \star & (j+1)^{d_1} & \dots & r^{d_k} \\ w_1 & \dots & w_j & w_\star & z_1 & \dots & z_k \end{pmatrix},$$

where $w_1, \dots, w_j, w_\star, z_1, \dots, z_k$ are words of lengths $c_1, \dots, c_j, 1, d_1, \dots, d_k$, respectively. Note that $w_\star$ is equal to the last letter of w.

(f) Now reverse (i.e., take the mirror-image of) each of the words $z_1, \dots, z_k$ to obtain $\widetilde{z}_1, \dots, \widetilde{z}_k$ within the biword

$$\begin{pmatrix} \dots \\ w'' \end{pmatrix} = \begin{pmatrix} 1^{c_1} & \dots & j^{c_j} & \star & (j+1)^{d_1} & \dots & r^{d_k} \\ w_1 & \dots & w_j & w_\star & \widetilde{z}_1 & \dots & \widetilde{z}_k \end{pmatrix};$$

(g) By construction, $\star$ is the last letter of v_{q,m_q}, then becomes the first letter of z_k, and, finally, appears as the last letter of $\widetilde{z}_k$ and therefore of w''. Write $\widetilde{z}_k = \widetilde{y}_k\star$. We can then drop the last letter of the bottom word w'' of the previous biword and obtain a word of the class $R(\mathbf{c}, \mathbf{d})$, which is, by definition, the word $\Phi_k(w)$. Thus $\Phi_k(w)\star = w''$, so that

$$\begin{pmatrix} \dots \\ \Phi_k(w) \end{pmatrix} = \begin{pmatrix} 1^{c_1} & \dots & j^{c_j} & (j+1)^{d_1} & \dots & r^{d_k} \\ w_1 & \dots & w_j & w_\star\widetilde{z}_1 & \dots & \widetilde{y}_k \end{pmatrix}.$$

Note that $w_\star\widetilde{z}_1$ is of length (d_1+1) and $\widetilde{y}_k$ is of length (d_k-1). Accordingly, the bottom word has the required length.

For example, suppose $j=2$, $k=4$, $r=6$ and let w be the word

$$w = 2,1,1,3,1,3,3,5,5,2,3,3,2,1,4,5,4,6,6,1,3.$$

The steps (a) – (g) for w are the following:

(a) $\quad w\star = 2,1,1,3,1,3,3,5,5,2,3,3,2,1,4,5,4,6,6,1,3,\star;$

(b) $\quad w\star = |\,2,1,1\,|\,3,1\,|\,3\,|\,3\,|\,5\,|\,5,2,3,3,2,1,4\,|\,5,4\,|\,6\,|\,6,1,3,\star\,|\,;$

(c) $\quad w' = |\,2,1,1\,|\,3\,|\,3\,|\,3,1\,|\,5,4\,|\,5,2,3,3,2,1,4\,|\,5\,|\,6,1,3,\star\,|\,6\,|\,;$

(d) $$\begin{pmatrix} \Delta w' \\ w' \end{pmatrix} = \left(\begin{array}{c|c|c|c|c|c|c|c|c} 1\,1\,2 & 3 & 3 & 1\,3 & 4\,5 & 2\,3\,3\,2\,1\,4\,5 & 5 & 1\,3\,\star\,6 & 6 \\ 2\,1\,1 & 3 & 3 & 3\,1 & 5\,4 & 5\,2\,3\,3\,2\,1\,4 & 5 & 6\,1\,3\,\star & 6 \end{array}\right);$$

(e) $$\begin{pmatrix} \dots \\ \Phi(w') \end{pmatrix} = \left(\begin{array}{c|c|c|c|c|c|c} 1\,1\,1\,1\,1 & 2\,2\,2 & \star & 3\,3\,3\,3\,3\,3 & 4\,4 & 5\,5\,5 & 6\,6 \\ 2\,1\,3\,2\,6 & 1\,5\,3 & 3 & 3\,3\,1\,2\,3\,1 & 5\,1 & 4\,4\,5 & \star\,6 \end{array}\right);$$

(f) $$\begin{pmatrix} \dots \\ w'' \end{pmatrix} = \left(\begin{array}{c|c|c|c|c|c|c} 1\,1\,1\,1\,1 & 2\,2\,2 & \star & 3\,3\,3\,3\,3\,3 & 4\,4 & 5\,5\,5 & 6\,6 \\ 2\,1\,3\,2\,6 & 1\,5\,3 & 3 & 1\,3\,2\,1\,3\,3 & 1\,5 & 5\,4\,4 & 6\,\star \end{array}\right);$$

(g) $$\begin{pmatrix} \dots \\ \Phi_4(w) \end{pmatrix} = \begin{pmatrix} 1\,1\,1\,1\,1\,2\,2\,2\,3\,3\,3\,3\,3\,3\,4\,4\,5\,5\,5\,6\,6 \\ 2\,1\,3\,2\,6\,1\,5\,3\,3\,1\,3\,2\,1\,3\,3\,1\,5\,5\,4\,4\,6 \end{pmatrix}.$$

Let us verify that Φ_k is a *bijection* of $R(\mathbf{c},\mathbf{d})$ onto itself. First, steps (a) and (b) make up a bijection of the latter set onto $R(\mathbf{c},\mathbf{d})\star$. Next, step (c) maps this new set onto the set, say R', of all the rearrangements of $1^{c_1}\ldots j^{c_j}\star(j+1)^{d_1}\ldots r^{d_k}$ whose first factor (when read from left to right) in its left to right dominated factorization beginning with r ends with $\star$. This step makes up a true bijection, since the forementioned factorization is unique.

Next steps (d) and (e) are nothing but the definition of the bijection Φ. Therefore, Φ maps R', in a one-to-one manner, onto the set, say $R^{(\mathrm{e})}$, of rearrangements $w^{(\mathrm{e})}$ such that the d_k-th letter of $w^{(\mathrm{e})}$, when read *from right to left* is $\star$. Clearly, step (f) is a bijection of $R^{(\mathrm{e})}$ onto $R(\mathbf{c},\mathbf{d})\star$ and step (g) is straightforward.

Thus Φ_k is a well-defined bijection. All the steps are reversible and rather simple to describe with the exception of the reverse of step (e). However this step is simply the inverse of the first fundamental transformation, a description of which may be found in [Lo83, p. 199].

It remains to verify the property: $\operatorname{des}_k w = \operatorname{exc}_k \Phi_k(w)$. To achieve this goal introduce the following notations. For each word w and each large letter b let $\operatorname{rf}_b(w)=1$, if the single letter word b occurs as a factor in the left to right factorization of w and is *the rightmost* factor starting with b. Otherwise, let $\operatorname{rf}_b(w)=0$. For instance, with the notations of step (b), $\operatorname{rf}_{b_i}(w\star)=1$, if and only if v_{i,m_i} is empty ($1\le i\le q$). Also with our previous example we have $\operatorname{rf}_3(w\star)=1$, $\operatorname{rf}_4(w\star)=\operatorname{rf}_5(w\star)=\operatorname{rf}_6(w\star)=0$.

When a word is written as a biword $\binom{u}{v}$, with a *non-decreasing* top row u (as in steps (e), (f), (g)), let $\operatorname{rf}_b\binom{u}{v}=1$, if the rightmost biletter $\binom{x}{y}$ such that $x=b$ is equal to $\binom{b}{b}$. Otherwise, let $\operatorname{rf}_b\binom{u}{v}=0$. For instance, $\operatorname{rf}_{j+i}\binom{\cdots}{w''}=1$ (in step (f)), if and only if the last letter of the word $\widetilde{z}_i$ is equal to $(j+i)$ ($1\le i\le k$). Also with our previous example we have $\operatorname{rf}_3\binom{\cdots}{w''}=1$, $\operatorname{rf}_4\binom{\cdots}{w''}=\operatorname{rf}_5\binom{\cdots}{w''}=\operatorname{rf}_6\binom{\cdots}{w''}=0$.

Let us examine step by step how the k-descents in w are mapped onto the k-excedences in $\Phi_k(w)$. To stress the fact that the statistic "$\operatorname{exc}_k w$" depends on the "vertical" pairs $\binom{x}{y}$ we will also write $\operatorname{exc}_k\binom{\cdots}{w}$. We also make use of the notation introduced just before (1.5).

(a) $\to$ (b) : first, note that the k-descents of w are the k-descents of $w\star$, as $\star$ cannot create any k-descent at the end. Therefore

$$\operatorname{des}_k w = \operatorname{des}_k(w\star).$$

(b) $\to$ (d) : when $x>y$ all the k-descents of the form xy in $w\star$ (or in w') occur *inside* the factors of the left to right dominated factorization.

Therefore

$$\begin{bmatrix} y \\ x \end{bmatrix} \begin{pmatrix} \Delta w' \\ w' \end{pmatrix} = [x, y](w\star) \qquad \text{(whenever } x > y\text{)}.$$

On the other hand, the number of biletters $\binom{b}{b}$ occurring in $\binom{\Delta w'}{w'}$ is equal to one plus the number of factors bb in $w\star$, if the rightmost factor of $w\star$ starting with b is the single letter word b. Otherwise, the former numbers are equal. Thus

$$\begin{bmatrix} b \\ b \end{bmatrix} \begin{pmatrix} \Delta w' \\ w' \end{pmatrix} = [b, b](w\star) + \mathrm{rf}_b(w\star) \qquad (b \text{ large}).$$

Therefore

$$\mathrm{exc}_k \begin{pmatrix} \Delta w' \\ w' \end{pmatrix} = \mathrm{des}_k(w\star) + \sum_{b \text{ large}} \mathrm{rf}_b(w\star).$$

(d) $\rightarrow$ (f) : as the biwords $\binom{\cdots}{\Phi(w')}$ and $\binom{\cdots}{w''}$ in steps (e) and (f) are derived from $\binom{\Delta w'}{w'}$ by permuting the columns of the biwords, we have :

$$\mathrm{exc}_k \begin{pmatrix} \cdots \\ w'' \end{pmatrix} = \mathrm{exc}_k \begin{pmatrix} \Delta w' \\ w' \end{pmatrix},$$

and therefore

(5.2) $$\mathrm{exc}_k \begin{pmatrix} \cdots \\ w'' \end{pmatrix} = \mathrm{des}_k(w\star) + \sum_{b \text{ large}} \mathrm{rf}_b(w\star).$$

(f) $\rightarrow$ (g) : first, we have the relations

(5.3) $$\mathrm{rf}_b \begin{pmatrix} \cdots \\ w'' \end{pmatrix} = \mathrm{rf}_b(w\star) \qquad (b \text{ large}),$$

because if the rightmost factor of $w\star$ starting with b is a single letter word, the *leftmost* factor starting with b in w' is b. Hence the *leftmost* biletter $\binom{x}{y}$ in $\binom{\Delta w'}{w'}$ such that $x = b$ is $\binom{b}{b}$ (step (d)). It is also true in $\binom{\cdots}{\Phi(w')}$ (step (e)), when the commutation rules (4.1) are applied. Therefore $\mathrm{rf}_b \binom{\cdots}{w''} = 1$ (step (f)). In the same manner $\mathrm{rf}_b \binom{\cdots}{w''} = 0$, if $\mathrm{rf}_b(w\star) = 0$.

It remains to examine the effect of the (g)-shift on the biword $\binom{\cdots}{w''}$. If y is small and different from $\star$, each column of the form $\binom{y}{x}$ remains alike in step (g) (no shift in the first $(c_1 + \cdots + c_j)$ columns), so that all the k-excedences in those columns are preserved.

Next the column $\binom{\star}{w_\star}$ in $\binom{\cdots}{w''}$ is mapped onto the column $\binom{j+1}{w_\star}$ in $\binom{\cdots}{\Phi_k(w)}$. Hence that column provides a k-excedence, iff $\binom{\star}{w_\star}$ was one.

Finally, when the shift is made on the rightmost $(d_1+\cdots+d_k)$ columns, all the k-excedences are preserved, except those of the form $\binom{b}{b}$ followed by a column of the form $\binom{(b+1)}{x}$. The column $\binom{b}{b}$ is transformed into $\binom{(b+1)}{b}$ and the k-excedence is killed by the shift. By definition of "rf_b" the number of those columns is exactly $\sum \mathrm{rf}_b \binom{\cdots}{w''}$ (b large). Therefore

$$\text{(5.4)} \qquad \mathrm{exc}_k \begin{pmatrix} \cdots \\ \Phi_k(w) \end{pmatrix} = \mathrm{exc}_k \begin{pmatrix} \cdots \\ w'' \end{pmatrix} - \sum_{b \text{ large}} \mathrm{rf}_b \begin{pmatrix} \cdots \\ w'' \end{pmatrix}.$$

From (5.2), (5.3) and (5.4) we then conclude that

$$\mathrm{exc}_k \begin{pmatrix} \cdots \\ \Phi_k(w) \end{pmatrix} = \mathrm{des}_k(w\star) = \mathrm{des}_k\, w. \quad \square$$

Let D be a total ordering of $[\,r\,]$ *compatible with* k. By means of the present transformation Φ_k and the bijections derived in Theorems 3.1 and 3.2 we can set up a bijection of $R(\mathbf{c},\mathbf{d})$ that maps the statistic "des_D" onto "exc_D." Thus the following theorem is proved.

Theorem 5.1. *Let D be a total ordering of $[\,r\,]$ compatible with k. Then there is a bijection Φ_k of $R(\mathbf{c},\mathbf{d})$ onto itself such that for all $w \in R(\mathbf{c},\mathbf{d})$,*

$$\mathrm{des}_D\, w = \mathrm{exc}_D\, \Phi_k(w).$$

Remark. By examining the forementioned argument in detail we see that the transformation Φ_k satisfies the identity

$$\text{(5.5)} \qquad \begin{bmatrix} y \\ x \end{bmatrix} \begin{pmatrix} \cdots \\ \Phi_k(w) \end{pmatrix} = [x,y](w\star) \qquad (x > y,\ y \text{ small}),$$

and also the following property :

If the last letter $w_\star$ of w is small, then for every large letter x :

$$\text{(5.6)} \qquad \sum_y \begin{bmatrix} y \\ x \end{bmatrix} \begin{pmatrix} \cdots \\ \Phi_k(w) \end{pmatrix} = \sum_y [x,y](w) \qquad (y \textit{ large} \le x).$$

If $w_\star$ is large, identity (5.7) *always holds for every large letter x different from $w_\star$. When $x = w_\star$,* (5.7) *must be replaced by :*

$$\text{(5.7)} \qquad 1 + \sum_y \begin{bmatrix} y \\ x \end{bmatrix} \begin{pmatrix} \cdots \\ \Phi_k(w) \end{pmatrix} = \sum_y [x,y](w) \qquad (y \textit{ large} \le x).$$

Let us illustrate the latest result by means of the example already used in this section. Define the matrix $D_k(w)$ as the $r \times r$-matrix whose (y,x)-entry (y row index and x column index) is the number $[x,y](w)$, except for the $(y=j+1, x=w_\star)$-entry that is equal to $1+[x,y](w)$. Also define $E_k(w)$ to be the $r \times r$-matrix whose (y,x)-entry (y row index and x column index) is the number $\left[{y \atop x}\right](w)$.

Then (5.5) says that the entries in the cells (y,x) with $1 \le y \le j$ and $y < x$ in the matrices $D_k(w)$ and $E_k(\Phi_k(w))$ coincide (i.e., the entries in the shape). Identities (5.6) and (5.7) say that only the *column sums* in the areas $(j+1 \le y \le x)$ of the two matrices $D_k(w)$ and $E_k(\Phi_k(w))$ coincide (i.e., the entries in the shape).

The two matrices $D_4(w)$ and $E_4(\Phi(w))$ resulting from the previous example are shown below.

$$D_4(w) = \begin{array}{c|cccccc|}
y \backslash x & & & & & & \\
\hline
 & 1 & 2 & 1 & 0 & 0 & 1 \\
 & 0 & 0 & 1 & 0 & 1 & 0 \\
 & 3 & 1 & 3 & 0 & 0 & 0 \\
 & 1 & 0 & 0 & 0 & 1 & 0 \\
 & 0 & 0 & 1 & 1 & 1 & 0 \\
 & 0 & 0 & 0 & 1 & 0 & 1 \\
\hline
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{0} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{1}
\end{array}
\qquad
E_4\big(\Phi_4(w)\big) = \begin{array}{c|cccccc|}
y \backslash x & & & & & & \\
\hline
 & 1 & 2 & 1 & 0 & 0 & 1 \\
 & 1 & 0 & 1 & 0 & 1 & 0 \\
 & 2 & 1 & 3 & 0 & 0 & 0 \\
 & 1 & 0 & 1 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 1 & 2 & 0 \\
 & 0 & 0 & 0 & 1 & 0 & 1 \\
\hline
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{0} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{1}
\end{array}$$

6. The Permutation Case

Suppose $c_1 = \cdots = c_j = 1$, $d_1 = \cdots = d_k = 1$ and still $j+k=r$. Then the rearrangement class $R(\mathbf{c},\mathbf{d})$ becomes the symmetric group $\mathcal{S}_r$ of order r. As the length $l(\mathbf{d})$ of $\mathbf{d}$ is k, we will write

$$A_{\mathbf{c},\mathbf{d}}(t) = A_r^k(t) = \sum_{s=0}^{r} A_{r,s}^k t^s = \sum_w t^{\operatorname{des}_k w} = \sum_w t^{\operatorname{exc}_k w} \quad (w \in \mathcal{S}_r).$$

The statistic "des_k" applied to each permutation $w = x_1 x_2 \ldots x_r \in \mathcal{S}_r$ has the following simple form:

$$\operatorname{des}_k w = \begin{cases} \operatorname{des} w, & \text{if } x_r \le j; \\ 1 + \operatorname{des} w, & \text{if } x_r \ge j+1. \end{cases} \tag{6.1}$$

When $k=0$, the polynomial $A_r^k(t)$ is the usual *Eulerian polynomial*, denoted by $A_r(t)$. Moreover, $A_r^r(t) = t\,A_r(t)$. As it is well-known (see, e.g., [Ri58, p. 211]), the exponential generating functions for those polynomials are given by

$$1+\sum_{r\ge1}\frac{u^r}{r!}A_r(t)=\frac{1-t}{-t+\exp((t-1)u)};\tag{6.2}$$

$$1+\sum_{r\ge1}\frac{u^r}{r!}t\,A_r(t)=\frac{(1-t)\exp(u(t-1))}{-t+\exp((t-1)u)}.\tag{6.3}$$

The "des_k" interpretation given above for $A_r^k(t)$ provides the following recurrence relation for the polynomials $A_r^k(t)$, namely

$$A_r^k(t)=A_r^{k-1}(t)+(t-1)A_{r-1}^{k-1}(t)\qquad(1\le k\le r).\tag{6.4}$$

For the proof we just have to verify that $t\,A_{r-1}^{k-1}(t)$ is the generating polynomial for the permutations ending with $(j+1)$ by "des_k," while $A_r^{k-1}(t)-A_{r-1}^{k-1}(t)$ is the generating polynomials for the other permutations. $\square$

By iteration the recurrence (6.4) leads to:

$$\begin{aligned}A_r^k(t)=A_r^0(t)+\binom{k}{1}(t-1)A_{r-1}^0(t)+\binom{k}{2}(t-1)^2A_{r-2}^0(t)\\ +\cdots+\binom{k}{k}(t-1)^kA_{r-k}^0(t).\end{aligned}\tag{6.5}$$

Using the umbral notation $A^r\equiv A_r\equiv A_r^0(t)$ and $A^0\equiv1$, formula (6.5) may be rewritten as

$$A_r^k(t)=A^{r-k}\big((t-1)+A\big)^k.\tag{6.6}$$

The two complementary exponents k and $r-k$ in (6.6) suggest the introduction of the following polynomials in the three variables x, y, t:

$$A_r(x,y,t)=\sum_{k=0}^{r}\binom{r}{k}x^ky^{r-k}A_r^k(t),\tag{6.7}$$

so that

$$A_r(x,y,t)=\big(x(t-1)+(x+y)A\big)^r,\qquad A^r\equiv A_r.\tag{6.8}$$

Now the familiar exponential generating function (6.2) for the Eulerian polynomials may be rewritten as

$$\exp(uA)=\frac{1-t}{-t+\exp((t-1)u)},$$

so that

$$\sum_{r\geq 0}\frac{u^r}{r!}A_r(x,y,t)=\sum_{r\geq 0}\frac{u^r}{r!}\bigl(x(t-1)+(x+y)A\bigr)^r$$
$$=\exp\bigl(ux(t-1)\bigr)\exp\bigl(u(x+y)A\bigr)$$
$$=\frac{(1-t)\exp(ux(t-1))}{-t+\exp(u(x+y)(t-1))}$$

$$\sum_{r\geq 0}\frac{u^r}{r!}A_r(x,y,t)=\frac{(1-t)\exp(uy(1-t))}{1-t\exp(u(x+y)(1-t))},\tag{6.9}$$

which in its turn specializes to (6.2) and (6.3) for $x=0$, $y=1$, and $x=1$, $y=0$, respectively.

For the sake of completeness, we mention several formulas on the polynomials $A_r(x,y,t)$ that can be derived from (6.9) or even from the polynomials $A_{\mathbf{c},\mathbf{d}}(t)$ of section 2 by various (and classical) methods. First, let

$$A_r(x,y,t)=\sum_{s=0}^{r}A_{r,s}(x,y)t^s,$$

so that

$$A_{r,s}(x,y)=\sum_k\binom{r}{k}x^ky^{r-k}A^k_{r,s}.\tag{6.10}$$

Then, a standard calculation (see, e.g., [Ri58, p. 39]) transforms (6.9) into

$$\frac{A_r(x,y,t)}{(1-t)^{r+1}}=\sum_{s\geq 0}t^s(xs+y(1+s))^r.\tag{6.11}$$

Formula (6.11) has an inverse formula, usually called the *Worpitzky identity* (see, e.g., [Ri58, p. 236])

$$\sum_{i=0}^{r}\binom{z+i}{r}A_{r,i}(x,y)=\bigl(xz+y(1+z)\bigr)^r,\tag{6.12}$$

that can be derived from (6.11) by expanding $(1-t)^{r+1}$:

$$\sum_s A_{r,s}t^s\sum_l\binom{r+l}{r}t^l=\sum_{s\geq 0}t^s(xs+y(1+s))^r$$

and by looking for the coefficients of the same power of t in both members.

Now the recurrence formulas (2.7) and (2.10) for the coefficients of the polynomials $A_{\mathbf{c},\mathbf{d}}(t)$ specialize to

$$A^k_{r,s}=(r-s+1)A^{k-1}_{r-1,s-1}+sA^{k-1}_{r-1,s};\tag{6.13}$$

$$A^k_{r,s}=(r-s)A^k_{r-1,s-1}+(s+1)A^k_{r-1,s}.\tag{6.14}$$

In their turn, those two formulas imply (easy derivation)

$$
\begin{aligned}
(6.15)\quad A_{r,s}(x,y) = ((r-s+1)x + (r-s)y)A_{r-1,s-1}& \\
+ (sx + (s+1)y)A_{r-1,s}(x,y)&,
\end{aligned}
$$

a formula that can also be expressed as:

$$
\begin{aligned}
(6.16)\quad A_r(x,y,t) = \big(y(1+(r-1)t) + xrt\big)A_{r-1}(x,y,t)& \\
+ t(1-t)(x+y)A'_{r-1}(x,y,t)&.
\end{aligned}
$$

Remark 1. Notice that (6.16) was derived from the recurrence formulas (2.7) and (2.10) that were given true combinatorial interpretations in terms of "des_k." It is also possible to derive the exponential generating function for the polynomials $A_r(x,y,t)$, say $A = A(u)$, given in (6.9), from (6.16) by using the same method that was utilized in section 2. In fact, (6.16) is equivalent to the differential equation

$$
yA + t(x+y)D_tA = D_uA,
$$

whose solution in A is precisely given by (6.9).

Remark 2. The exponential generating function (6.9) has already been derived by several authors: Stembridge [Stem91], Steingrìmsson [Ste93] and apparently Brenti [Br93] in a paper to come. They all have worked in a different context from the one discussed in this paper.

Steingrìmsson [Ste93] has recently generalized the definitions of "des" and "exc" from permutations to so-called "indexed permutations", i.e., elements of the wreath product $Z_n \wr \mathcal{S}_r$, where Z_n stands for the group of the integers mod n. The elements of $Z_n \wr \mathcal{S}_r$ are pairs (w,p), where $w \in \mathcal{S}_r$ and p is a mapping of $[\,r\,]$ into Z_n. In his model, a indexed permutation (w,p) has a descent at i, if $1 \le i \le r$, and if, either $p(i) > p(i+1)$, or $p(i) = p(i+1)$ and $x_i > x_{i+1}$. (By convention, $x_{r+1} = r+1$ and $p(r+1) = 0$.) The indexed permutation (w,p) has an excedance at i, if, either $x_i > i$, or $x_i = i$ and $p(i) \ge 1$.

He does construct a bijection of $Z_n \wr \mathcal{S}_r$ onto itself that maps his "des" onto his "exc." For each mapping p let $\mathrm{Pos}\,p = \{x \in Z_n : p(x) \ge 1\}$. Steingrìmsson shows that if $\#\mathrm{Pos}\,p = \#\mathrm{Pos}\,p'$, then the two polynomials $\sum_w t^{\mathrm{des}(w,p)}$ and $\sum_w t^{\mathrm{des}(w,p')}$ $(w \in \mathcal{S}_r)$ are identical. When $p(x) = 0$ for $x \le j = r-k$ and $p(x) \ge 1$ for $j+1 \le x \le r$, the corresponding polynomial $\sum_w t^{\mathrm{des}(w,p)}$ is equal to our polynomial $A_r^k(t)$, since $\mathrm{des}(w,p) = \mathrm{des}_k\, w$ in this case.

The generating polynomial $D_r^n(t)$ for $Z_n \wr \mathcal{S}_r$ by "des" can then be evaluated as follows

$$
\begin{aligned}
D_r^n(t) &= \sum_{w,p} t^{\operatorname{des}(w,p)} = \sum_{k=0}^{r} \sum_{p,\,\#\operatorname{Pos} p=k} \sum_{w} t^{\operatorname{des}(w,p)} \\
&= \sum_{k=0}^{r} A_r^k(t) \times \#\{p : \operatorname{Pos} p = k\} = \sum_{k=0}^{r} A_r^k(t) \binom{r}{k} (n-1)^k \\
&= A(x,y,t)\,\big|_{\{x=n-1,\; y=1\}}\,.
\end{aligned}
$$

Notice that the indexed permutation model coincides partly with ours in the permutation case and has not been extended to the case of arbitrary words (with repetitions). However we give credit to Steingrìmsson to have stated and proved analogues of our theorems 3.1, 3.2 and 5.1 in his indexed permutation set-up.

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