# Steinberg symbol and tau function.

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22 October 2025

# Steinberg symbol.

$R$ — ring, $\mathcal{A}$ — Abelian group, $a, b, c \in R^\times$.

$$\{\cdot, \cdot\} \to \mathcal{A}$$

1. $\{ab, c\} = \{a, c\}\{b, c\}$,
2. $\{a, b\}\{b, a\} = 1$,
3. $\{a, b\} = 1$ if $a + b = 1$ or $a + b = 0$.
4. $\{a, a\}^2 = 1$,

# Main example (Beilinson).

$R = C^\infty(\mathbb{R}/\mathbb{Z} \to \mathbb{C})$, $\mathcal{A} = \mathbb{C}^\times$.

$$\{f, g\} = \left( \exp \frac{1}{2\pi i} \int_x^{x+1} \ln f \, d \ln g \right) g(x)^{-\deg f}$$

$f, g$ — functions on Riemann surface $\Sigma$ with isolated singularities, $\gamma$ — closed path on $\Sigma$, $p \in \Sigma$, $\gamma_p$ small path around $p$:

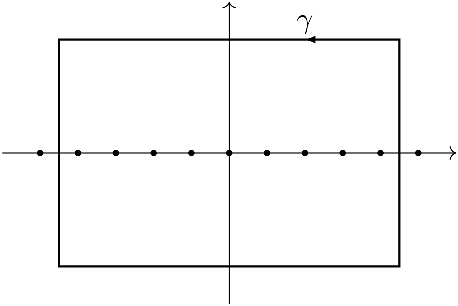
$$\{f, g\}_\gamma := \{f|_\gamma, g|_\gamma\}, \quad \{f, g\}_p := \{f, g\}_{\gamma_p}$$

Claim: $f, g$ meromorphic then

$$\{f, g\}_p = (-1)^{\deg_p f \deg_p g} \left. \frac{f^{\deg_p g}}{g^{\deg_p f}} \right|_p$$

$$\{f, g\}_\gamma = \prod_{p \text{ inside } \gamma} \{f, g\}_p$$

# Example – complex analysis




$$1 \leftarrow \{1 - \frac{a}{z}, \sin z\}_\gamma =$$

$$= \{1 - \frac{a}{z}, sinz\}_0 \{1 - \frac{a}{z}, sinz\}_a \prod_{k \neq 0} \{1 - \frac{a}{z}, sinz\}_{\pi k} =$$

$$= \frac{1}{\sin a} a \prod_{k \neq 0} (1 - \frac{a}{\pi k})$$

# Heisenberg group

$\gamma$ - unit circle,

$\boldsymbol{B} = R^\times$: functions $\gamma \to \mathbb{C}^\times$,
$\boldsymbol{B}^+$ (resp. $\boldsymbol{B}^-$): restrictions of functions holomorphic inside $\gamma$ (resp. holomorphic outside $\gamma$ with $z^k f \to 1$ for $z \to \infty$).
$\boldsymbol{B}_0 \subset \boldsymbol{B}$: $\boldsymbol{B}_0 = \{f |\deg f = 0\}$.

Any function $f$ can be represented as a product $f = f^+ f^-$ with $f^+ \in \boldsymbol{B}^+$ and $f_-$ in $\boldsymbol{B}^-$.

Heisenberg group $\hat{\boldsymbol{B}} = \mathbb{C}^\times \times \boldsymbol{B} = \{x\hat{f} | x \in \mathbb{C}^\times, f \in \boldsymbol{B}\}$,

$$\hat{f}\hat{g} = \{g^-, f^+\}\widehat{fg}$$

$$\langle\rangle : \widehat{\boldsymbol{B}} \to \mathbb{C}, \qquad \langle x\hat{f}\rangle = x\delta_{\deg f}, \qquad \langle \widehat{f^+} A \widehat{f^-}\rangle = \langle A\rangle\delta_{\deg f}$$

# Fermionic algebra

$F$: 1/2-differentials on a unit circle,

$F^{\pm}$: 1/2-differentials extensible inside/outside of the unit circle.

$\boldsymbol{F}$: External algebra of $F + F^* = \{(\varepsilon, \varepsilon^*)\}$.

$\hat{\boldsymbol{F}}$: Clifford algebra of $F + F^*$.

$\langle\rangle : \hat{\boldsymbol{F}} \to \mathbb{C}, \quad \langle 1 \rangle = 1$, $\langle v^- A v^+ \rangle = 0$ for any $v^{\pm} \in F^{\pm} + (F^{\pm})^{\perp}$.

# $\hat{\boldsymbol{B}}_0 \to \hat{\boldsymbol{F}}$

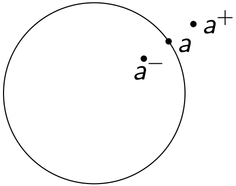
Action of $\boldsymbol{B}$ on $F$ by orthogonal transformations:

$$f : (\varepsilon, \varepsilon^*) \mapsto (f\varepsilon, f^{-1}\varepsilon^*).$$

Induced map $\hat{\boldsymbol{B}}_0 \to \hat{\boldsymbol{F}}$ (Spinor representation):

$$f \mapsto \exp \int \ln f \; \varepsilon \wedge \varepsilon^*,$$

## $\hat{F} \to \hat{B}$




$$\psi_a = \lim_{a^+, a^- \to a} \frac{z - a^+}{z - a^-}, \qquad \psi_a^* = \lim_{a^+, a^- \to a} \frac{z - a^-}{z - a^+}.$$

1. $\hat{\psi}_a \hat{\psi}_b + \hat{\psi}_b \hat{\psi}_a = 0$
2. $\hat{\psi}_a^* \hat{\psi}_b^* + \hat{\psi}_b^* \hat{\psi}_a^* = 0$
3. $\hat{\psi}_a \hat{\psi}_b^* + \hat{\psi}_b^* \hat{\psi}_a = 2\pi i \delta_{a-b}$
4. $\hat{f} \hat{\psi}_a \hat{f}^{-1} = \widehat{f(a)\psi_a},$
5. $\hat{f} \hat{\psi}_a^* \hat{f}^{-1} = \widehat{f^{-1}(a)\psi_a^*}.$

# Tau function

$L \in F$ such that $L + F^+ = F$.

$\langle\rangle_L : \hat{\boldsymbol{F}} \to \mathbb{C}$, $\langle 1 \rangle = 1$, $\langle v^- A v^+ \rangle_L = 0$ for any $v^+ \in F^+ + (F^+)^\perp$, $v^- \in L + (L)^\perp$,

$$\tau_L(f^-) = \langle \hat{f^-} \rangle_L,$$

where $f^- \in \boldsymbol{B}^-$.

$\boldsymbol{B}_L = \{\hat{g} | gL = L\}$,

## Proposition

*For any* $g \in \boldsymbol{B}_L$

$$\tau_L(f^-) = \{g^+, f^-\}\tau_L(g^- f^-).$$

## Proof.

$\langle \hat{f^-} \rangle_L = \langle \hat{g} \hat{f^-} \rangle_L = (-1)^{\deg f^- \deg g}\{f^-, g\}\langle \hat{f^-} \hat{g} \rangle_L = \{g^+, f^-\}\langle \widehat{g^- f^-} \rangle_L.$ $\square$

# Corollary (automorphic property of the tau function)

$\Sigma$: closed Riemann surface,
$p_1, \ldots, p_s \in \Sigma$,
$\boldsymbol{B}$: collection of functions on punctured vicinities of $p_i$'s,
$L$: holomorphic $1/2$-differentials on $\Sigma \backslash \{p_1, \ldots, p_s\}$.
$\boldsymbol{B}_L$: holomorphic nonzero function on $\Sigma \backslash \{p_1, \ldots, p_s\}$.
$\tau_\Sigma(f_1, \ldots, f_s) = \langle f_1, \ldots, f_s \rangle_L$

For any $g \in \boldsymbol{B}$

$$\tau_\Sigma(g_1^- f_1^-, \ldots, g_s^- f_s^-) = \prod_\alpha \{f_\alpha^-, g_\alpha^+\}_{p_\alpha} \tau_\Sigma(f_1^-, \ldots, f_s^-)$$

Where $g_\alpha$ is the restriction of $g$ to the vicinity of $p_\alpha$.

# Further questions

1. Generalization for nonabelian loop groups $\boldsymbol{B} \rightsquigarrow$ functions $S^1 \to G$, $\boldsymbol{B}/\boldsymbol{B}^+ \rightsquigarrow$ affine Grassmanian.
2. Number theory generalization $\boldsymbol{B} \rightsquigarrow \mathbb{Q}_p$ (or Witt vectors), $\boldsymbol{B}/\boldsymbol{B}^+ \rightsquigarrow$ characters of $\mathbb{Q}_p/\mathbb{Z}_p$, $\Sigma \rightsquigarrow$ number field $\mathbb{F}$, $p_1, \ldots, p_s \rightsquigarrow$ places of $\mathbb{F}$.
3. Possible analogy: $\tau$ is constructed out of three subrgoups $\boldsymbol{B}^+$, $\boldsymbol{B}^-$ and $\boldsymbol{B}_L$ like a quadratic form is constructed out of three Lagrangian subspaces of a symplectic space.