Implicit kinetic relaxation schemes
Application to the fluid and plasma physics

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Outline

Physical and mathematical context

Models

Model for the center: Gyro-kinetic model

$$\begin{cases} \partial_t(B_{\parallel} f) + \nabla \cdot \left(\frac{d\mathbf{x}_g}{dt} f\right) + \partial_{v_{\parallel}} \left(B_{\parallel} \frac{dv_{\parallel}}{dt} f\right) = 0 \\ -\nabla \cdot_{\perp} (\rho_e(\mathbf{x}) \nabla_{\perp} \phi) = \rho(\mathbf{x}) - 1 + S(\phi) \end{cases}$$

- The guiding center motion $\frac{d\mathbf{x}_g}{dt}$ and $\frac{dv_{\parallel}}{dt}$ depend of $\boldsymbol{B}_{\parallel}$ and $\nabla\phi$ and ρ is the density of the gyro-distribution f.
- **Other models**: Vlasov-Maxwell or Vlasov-Poisson.
- **Kinetic models** coupled with **elliptic model**.

Model for the edge: Resistive MHD

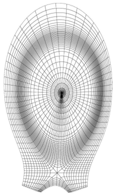
$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \nu \nabla \cdot \boldsymbol{\Pi} \\ \partial_t p + \nabla \cdot (p \boldsymbol{u}) + (\gamma - 1) p \nabla \cdot \boldsymbol{u} = \nabla \cdot ((\kappa \boldsymbol{B} \otimes \boldsymbol{B} + \varepsilon I_d) \nabla T) + \eta \mid \nabla \times \boldsymbol{B} \mid^2 + \nu \boldsymbol{\Pi} : \nabla \boldsymbol{u} \\ \partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = \eta \nabla \times (\nabla \times \boldsymbol{B}) \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

- Scaling: $\nu, \eta << 1$ and $\kappa \mid \boldsymbol{B} \mid^2 >> 1 >> \varepsilon$.
- **Other models**: Reduced incompressible MHD model or Extended MHD.
- **Hyperbolic model** coupled with **parabolic model**.

Geometries and times schemes

Geometry

- **3D geometry**: Torus with a non circular section.
- **Poloidal geometry**: aligned with the magnetic surfaces of the equilibrium.
- Non structured grids and singularities.



Time schemes for kinetic model

- **Vlasov**: large kinetic velocities.
- **Gyrokinetic**: large poloïdal velocities due to the electric field variation.
- Characteristic time larger that time associated to fast velocities. We need CFL-free schemes.
- **Turbulence**: We need high-order scheme and fine grids.

Time schemes for MHD model

- **Anisotropic diffusion**: We need CFL-free schemes.
- **Perp magneto-acoustic waves**: larger than characteristic velocity. We need CFL-free schemes.
- **Usual schemes**: Implicit high-order schemes. Very hard to invert the nonlinear problem.

Semi Lagrangian scheme

- One of the main scheme to treat transport and kinetic equations.
- **Idea**: use the characteristic method.

- Example: Backward SL

$$\partial_t f + a\partial_x f = 0$$

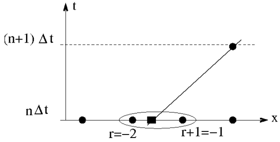
  - **Aim**: compute at the mesh point x_j:

$$f(t+\Delta t, x_j)$$

  - Solution:

$$f(t+\Delta t, x_j) = f(t, x_j - a\Delta t)$$

  - $x_n = x_j - a\Delta t$ is not a mesh point.
  - Using $f(t, x_i)$ we interpolate the function at x_n.



- **BSL/FSL**: follow the backward characteristic and interpolate/follow the forward characteristic and distribute on the mesh.

Advantages/drawbacks

- **Advantages**: infinite/high order in time/space. CFL-less and no matrix inversion.
- **Drawbacks**: BC and Gibbs oscillations due to high-order methods.
- **Interesting works**: Positive SL (B. Després), Artificial diffusion for SL, limiting.

Aim

Aim:

Construct High-Order Solver like SL-Solver (no matrix inversion, no CFL) for the different type of PDE.

Approximate BGK method

BGK and approximate BGK theory

- Distribution $f(t, \mathbf{x}, \boldsymbol{v})$ with $\boldsymbol{v} \in \mathbb{R}^d$.
- BGK equation:

$$\partial_t f + \boldsymbol{v} \cdot \nabla f = \frac{1}{K_n}(M_{\rho,\mathbf{u},T}(\boldsymbol{v}) - f)$$

with the moment:

$$\rho(\mathbf{x}) = \int M(\boldsymbol{v}) d\boldsymbol{v}, \quad \rho\boldsymbol{u}(\mathbf{x}) = M(\boldsymbol{v})\boldsymbol{v}d\boldsymbol{v}$$

and the T link to the third moment of $M(\boldsymbol{v})$.

- The equilibrium Maxwellian is:

$$M(\boldsymbol{v}) = \frac{\rho(\mathbf{x})}{(2\pi k T(\mathbf{x}))^{\frac{d}{2}}} e^{-\frac{-|\boldsymbol{u}-\boldsymbol{v}|^2}{2T}}$$

- **Limit**: $K_n \longrightarrow 0$
- Hilbert expansion: $f = f_0 + K_n f_1 + O(K_n^2)$

$$f_0 = M_0(\boldsymbol{v})$$

So

$$\partial_t f_0 + \boldsymbol{v} \cdot \nabla f_0 = (M_1(\boldsymbol{v}) - f_1)$$

- Distribution $f(t, \mathbf{x}, \boldsymbol{v})$ with $\boldsymbol{v} \in \{\boldsymbol{v}_1..\boldsymbol{v}_m\}$.
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- Taking $P\boldsymbol{f}^{eq} = \boldsymbol{U}$ and $P\Lambda\boldsymbol{f}^{eq} = \boldsymbol{F}(\boldsymbol{U})$. We obtain the limit

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = 0$$

Generic kinetic relaxation scheme

Kinetic relaxation system

- **Considered model**:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = 0$$

- **Lattice**: $W = \{\lambda_1, ..., \lambda_{n_v}\}$ a set of velocities.
- **Mapping matrix**: P a matrix $n_c \times n_v$ $(n_c < n_v)$ such that $\boldsymbol{U} = P\boldsymbol{f}$, with $\boldsymbol{U} \in \mathbb{R}^{n_c}$.
- **Kinetic relaxation system**:

$$\partial_t \boldsymbol{f} + \Lambda \partial_x \boldsymbol{f} = \frac{1}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f})$$

- Consistence condition (Natalini - Aregba [96-98-20], Bouchut [99-03]) :

$$\mathcal{C} \left\{ \begin{array}{ll} P\boldsymbol{f}^{eq}(\boldsymbol{U}) & = \boldsymbol{U} \\ P\Lambda\boldsymbol{f}^{eq}(\boldsymbol{U}) & = \boldsymbol{F}(\boldsymbol{U}) \end{array} \right.$$

Chapman-Enskog stability

- **Limit of the system**:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \varepsilon \partial_x \left(\left(P\Lambda^2 \partial_{\boldsymbol{U}} \boldsymbol{f}^{eq}(\boldsymbol{U}) - \mid \partial \boldsymbol{F}(\boldsymbol{U}) \mid^2 \right) \partial_x \boldsymbol{U} \right) + O(\varepsilon^2)$$

- This limit system is stable if the second order operator is dissipative for the entropy. Partial stability result for the kinetic system.
- **Strong-Stability**: entropy theory equivalent to the H-theorem. Other criteria for stability Bouchut [04].

Example of Approximate BGK model I

- We consider the classical Xin-Jin [95] relaxation for a scalar system $\partial_t u + \partial_x F(u) = 0$:

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \lambda^2 \partial_x u = \frac{1}{\varepsilon}(F(u) - v) \end{cases}$$

Limit

- The limit scheme of the relaxation system is

$$\partial_t u + \partial_x F(u) = \varepsilon \partial_x((\lambda^2 - \mid \partial F(u) \mid^2)\partial_x u) + O(\varepsilon^2)$$

- **Stability**: the limit system is dissipative if $(\lambda^2 - \mid \partial F(u) \mid^2) > 0$.

- We diagonalize the hyperbolic matrix $\begin{pmatrix} 0 & 1 \\ \lambda^2 & 0 \end{pmatrix}$ to obtain

$$\begin{cases} \partial_t f_- - \lambda \partial_x f_- = \frac{1}{\varepsilon}(f_{eq}^- - f_-) \\ \partial_t f_+ + \lambda \partial_x f_+ = \frac{1}{\varepsilon}(f_{eq}^+ - f_+) \end{cases}$$

- with $u = f_- + f_+$ and $f_{eq}^{\pm} = \frac{u}{2} \pm \frac{F(u)}{2\lambda}$.
- This system is called the D1Q2 model.

Example of Approximate BGK model II

Vectorial $[D1Q2]^n$ model

- The idea is simple: use one D1Q2 par macroscopic equation.
- Consider $\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = 0$. We consider two velocities $\pm\lambda$. For each U_i we have:

$$P\boldsymbol{f}^i = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} f_-^i \\ f_+^i \end{pmatrix} = U_i, \quad P\Lambda \boldsymbol{f}^i = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix} \begin{pmatrix} f_-^i \\ f_+^i \end{pmatrix} = F_i(\boldsymbol{U})$$

- The unique solution is: $f_\pm^{eq,i} = \frac{U_i}{2} \pm \frac{F_i(\boldsymbol{U})}{2\lambda}$. The limit:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \varepsilon \partial_x \left(\left(\lambda^2 I_n - \mid \partial \boldsymbol{F}(\boldsymbol{U}) \mid^2 \right) \partial_x \boldsymbol{U} \right) + O(\varepsilon^2)$$

Vectorial $[D1Q3]^n$ model

- Use one D1Q3 par macroscopic equation (generalization to D1Qq).
- Consider $\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = 0$. We consider two velocities $\pm\lambda$ and $\lambda_0 = 0$. For each U_i we have:

$$P\boldsymbol{f}^i = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} f_-^i \\ f_0^i \\ f_+^i \end{pmatrix} = U_i, \quad P\Lambda \boldsymbol{f} = F_i(\boldsymbol{U}), \quad P\Lambda^2 \boldsymbol{f} = G_i(\boldsymbol{U})$$

- We obtain

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \varepsilon \partial_x \left(\left(\partial \boldsymbol{G}(\boldsymbol{U}) - \mid \partial \boldsymbol{F}(\boldsymbol{U}) \mid^2 \right) \partial_x \boldsymbol{U} \right) + O(\varepsilon^2)$$

- with $\partial \boldsymbol{G}(\boldsymbol{U}) \approx \mid \partial \boldsymbol{F}(\boldsymbol{U}) \mid^2$. **Difficulty**: construct $\boldsymbol{G}(\boldsymbol{U})$.
- **Classic choice**: $\boldsymbol{G}(\boldsymbol{U}) = \boldsymbol{F}^+(\boldsymbol{U}) - \boldsymbol{F}^-(\boldsymbol{U})$ with $\boldsymbol{F} = \boldsymbol{F}^+ + \boldsymbol{F}^-$ a flux vector splitting.

Time schemes

Time discretization: First order

Space scheme

- Semi-Lagrangian method for advection: high order/exact in space/time.

Main property

- **Relaxation system**: "the nonlinearity is local and the non locality is linear".
- **Many schemes**: Jin-Filbet [10], Dimarco-Pareschi [11-14-17], Lafitte-Samaey [17] etc.
- **Main idea**: **splitting scheme** between transport and **implicit** relaxation (Dellar [13]).
- **Key point**: the $\partial_t \boldsymbol{U} = 0$ during the relaxation step. Therefore $\boldsymbol{f}^{eq}(\boldsymbol{U})$ **is explicit**.

First order scheme (exact transport)

- We define the two operators for each step :

$$T_{\Delta t} : e^{\Delta t \Lambda \partial_x} \boldsymbol{f}^{n+1} = \boldsymbol{f}^n$$

$$R_{\Delta t} : \boldsymbol{f}^{n+1} + \frac{\Delta t}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}^n) - \boldsymbol{f}^{n+1}) = \boldsymbol{f}^n$$

- **Final scheme**: $T_{\Delta t} \circ R_{\Delta t}$ is consistent with

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \left(\frac{\Delta t}{2}\right) \partial_x \left(D(\boldsymbol{U}) \partial_x \boldsymbol{U}\right) + O(\Delta t^2)$$

- with $\omega = \frac{\Delta t}{\varepsilon + \theta \Delta t}$ and $D(\boldsymbol{U}) = \left(P\Lambda^2 \partial_{\boldsymbol{U}} \boldsymbol{f}^{eq}(\boldsymbol{U}) - \partial \boldsymbol{F}(\boldsymbol{U})^2\right)$.

Time scheme: second order

- **Classical result**: Strang Splitting + exact scheme for relaxation converge at first order for $\varepsilon \approx 0$ [Jin96].
- Comparison between different scheme for relaxation (+ Strang splitting):

	Exact		SSP RK2		CN	
	Error	Order	Error	Order	Error	Order
$\Delta t = 4.10^{-3}$	$2.0E^{-2}$	-	$2.0E^{-2}$	-	$4.8E^{-4}$	-
$\Delta t = 2.10^{-3}$	$1.1E^{-2}$	0.86	$1.1E^{-2}$	0.86	$1.2E^{-4}$	2.0
$\Delta t = 1.10^{-3}$	$5.7E^{-3}$	0.95	$5.5E^{-3}$	1.0	$2.9E^{-5}$	2.05
$\Delta t = 5.10^{-4}$	$2.9E^{-3}$	0.97	$2.8E^{-3}$	0.98	$7.4E^{-6}$	1.95

- **Remark**: we lose one order of cv with exact and SPP-RK solver not for CN.

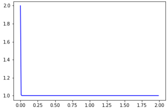
- **Schemes** for $\varepsilon \approx 0$:
- For Euler implicit, exact and SSP-RK2 schemes.

$$\boldsymbol{f}^{n+1} \approx \boldsymbol{f}^{eq}(\boldsymbol{U}^n) + O(\varepsilon)$$

- For Crank-Nicolson.

$$\boldsymbol{f}^{n+1} \approx 2\boldsymbol{f}^{eq}(\boldsymbol{U}^n) - \boldsymbol{f}^n + O(\varepsilon)$$

- We solve the EDO $\partial_t u = \frac{1}{\varepsilon}(u_{eq} - u)$.



- Implicit Euler scheme. $\Delta t = 100\varepsilon$

Conclusion:

- Order two only with CN scheme but large dispersive effect far to the equilibrium.

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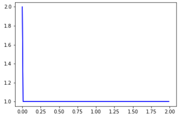

- Exact time scheme. $\Delta t = 100\varepsilon$

Conclusion:

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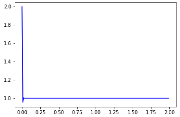

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Conclusion:

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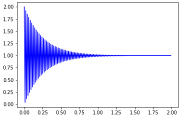


- Crank-Nicolson scheme. $\Delta t = 100\varepsilon$

Conclusion:

- Order two only with CN scheme but large dispersive effect far to the equilibrium.

Analysis of the second order scheme

Consistance analysis I

- Making Taylor expansion we can prove that Ψ_{ap} for the $[D1Q2]^n$ is consistant with

$$\begin{cases} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = O(\Delta t^2) \\ \partial_t \boldsymbol{W} - \partial \boldsymbol{F}(\boldsymbol{U}) \partial_x \boldsymbol{W} = O(\Delta t^2) \end{cases}$$

with $U_i = f_-^i + f_+^i$, $V_i = \lambda(f_+^i - f_-^i)$ and $\boldsymbol{W} = \boldsymbol{V} - \boldsymbol{F}(\boldsymbol{U})$.

Consistance analysis II

- Particular case: Ψ_{ap} and D1Q2 for $\partial_t u + \partial_x(cu) = 0$:

$$\partial_t \begin{pmatrix} u \\ w \end{pmatrix} + \begin{pmatrix} c\partial_x u \\ -c\partial_x w \end{pmatrix} + \underbrace{\begin{pmatrix} (\lambda^2 - c^2) & 3c \\ 3c(\lambda^2 - c^2) & -(\lambda^2 - c^2) \end{pmatrix} \partial_{xxx} \begin{pmatrix} u \\ w \end{pmatrix}}_{A} = O(\Delta t^3)$$

with $u = f_-^i + f_+^i$, $v = \lambda(f_+^i - f_-^i)$ and $w = u - cu$.

- **Chapman-Enskog stability**: The previous third order approximation is stable since the following energy is preserved:

$$E(t) = \int ((\lambda^2 - c^2)u^2 + w^2)$$

- We recover the sub-characteristic condition $\lambda > c$.

High order scheme: composition method

- If Ψ second order time scheme satisfy $\Psi(\Delta t) = \Psi^{-1}(-\Delta t)$ and $\Psi(0) = I_d$ we can construct high order extension with

$$M_p(\Delta t) = \Psi(\gamma_1 \Delta t) \circ \Psi(\gamma_2 \Delta t) \circ ..... \circ \Psi(\gamma_s \Delta t)$$

- with $\gamma_i \in [-1, 1]$.
- Susuki scheme : $s = 5$, $p = 4$. Kahan-Li scheme: $s = 9$, $p = 6$.

Second-order scheme

- For now we have

$$\Psi(\Delta t) = T\left(\frac{\Delta t}{2}\right) \circ R(\Delta t, \theta = 0.5) \circ T\left(\frac{\Delta t}{2}\right).$$

- We have symmetry in time but not $\Psi(0) = I_d$ for $\varepsilon \approx 0$. Indeed

$$R(\Delta t = 0, \theta = 0.5)\boldsymbol{f}^n = 2\boldsymbol{f}^{eq} - \boldsymbol{f}^n \neq \boldsymbol{f}^n$$

- However $R(0, \theta = 0.5) \circ R(0, \theta = 0.5) = I_d$ consequently we can propose a new second order scheme:

$$\Psi_{ap}(\Delta t) = T\left(\frac{\Delta t}{4}\right) \circ R(\Delta t, \theta = 0.5) \circ T\left(\frac{\Delta t}{2}\right) \circ R(\Delta t, \theta = 0.5) \circ T\left(\frac{\Delta t}{4}\right)$$

Numerical results

Burgers: convergence results

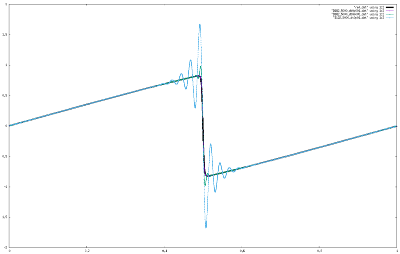
- **Model**: Burgers equation

$$\partial_t \rho + \partial_x \left(\frac{\rho^2}{2} \right) = 0$$

- Spatial discretization: SL-scheme, 2000 cells, degree 11.
- **Test**: $\rho(t=0,x) = sin(2\pi x)$. $T_f = 0.14$ (before the shock) and no viscosity.
- Scheme: splitting schemes and Suzuki composition + splitting.

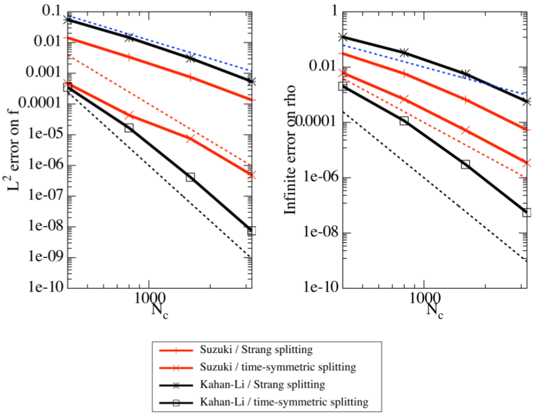
	SPL 1, $\theta=1$		SPL 1, $\theta=0.5$		SPL 2, $\theta=0.5$		Suzuki	
Δt	Error	order	Error	order	Error	order	Error	order
0.005	$2.6E^{-2}$	-	$1.3E^{-3}$	-	$7.6E^{-4}$	-	$4.0E^{-4}$	-
0.0025	$1.4E^{-2}$	0.91	$3.4E^{-4}$	1.90	$1.9E^{-4}$	2.0	$3.3E^{-5}$	3.61
0.00125	$7.1E^{-3}$	0.93	$8.7E^{-5}$	1.96	$4.7E^{-5}$	2.0	$2.4E^{-6}$	3.77
0.000625	$3.7E^{-3}$	0.95	$2.2E^{-5}$	1.99	$1.2E^{-5}$	2.0	$1.6E^{-7}$	3.89

- Scheme: second order splitting scheme.
- Same test after the shock:



Convergence

- **Equation**: Euler isothermal
- Model $[D1Q2]^2$ High-order space scheme. Comparison of the time scheme.
- **Test case**: smooth solution. $\Delta t = \frac{\beta \Delta x}{\lambda}$ with $\beta = 50$



- With Strang splitting: only order 2 for f.
- Loss of convergence for macroscopic variables for Kahan-li + Strang splitting.

Limitation

- Drawback of $[D1Q2]^n$ model: diffusion/dispersion homogenous to the larger speed.

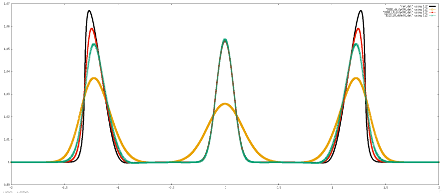
$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \sigma \Delta t \partial_x((\lambda^2 I_d - \mid \partial \boldsymbol{F}(\boldsymbol{U}) \mid^2)\partial_x \boldsymbol{U}) + O(\Delta t^2 \lambda^3)$$

- Low-mach limit: contact wave admit an error homogeneous to acoustic speed.

- **Euler equation**: Flux splitting for low-mach flow.

$$\begin{cases} \boldsymbol{f}_-^{eq}(\boldsymbol{U}) = \frac{1}{\lambda_-}\boldsymbol{F}^-(\boldsymbol{U}) \\ \boldsymbol{f}_0^{eq}(\boldsymbol{U}) = \left(\boldsymbol{U} - \left(\frac{\boldsymbol{F}^+(\boldsymbol{U})}{\lambda_+} + \frac{\boldsymbol{F}^-(\boldsymbol{U})}{\lambda_-}\right)\right) \\ \boldsymbol{f}_+^{eq}(\boldsymbol{U}) = \frac{1}{\lambda_+}\boldsymbol{F}^+(\boldsymbol{U}) \end{cases}$$

- **Test case**: Acoustic wave. SL order 11, 4000 cells.



- D1Q2 $\Delta t = 0.005$ (yellow), D1Q3 $\Delta t = 0.005/0.01$ (red, green). Contact captured.

Limitation

- Drawback of $[D1Q2]^n$ model: diffusion/dispersion homogenous to the larger speed.

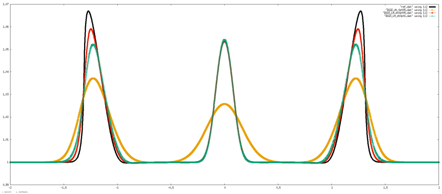
$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \sigma \Delta t \partial_x ((\lambda^2 I_d - \mid \partial \boldsymbol{F}(\boldsymbol{U}) \mid^2) \partial_x \boldsymbol{U}) + O(\Delta t^2 \lambda^3)$$

- Low-mach limit: contact wave admit an error homogeneous to acoustic speed.

- **Euler equation**: Flux splitting for low-mach flow.

$$\boldsymbol{F}(\boldsymbol{U}) = \begin{pmatrix} (\rho)u \\ (\rho u)u + p \\ (E)u + pu \end{pmatrix}, \quad \boldsymbol{F}^{\pm}(\boldsymbol{U}) = \frac{1}{2} \begin{pmatrix} (\rho u \pm \alpha \frac{u^2}{\lambda} \rho) + p \\ (\rho u^2 \pm \alpha \frac{u^2}{\lambda} q) + p(1 \pm \gamma \frac{u}{\lambda}) \\ (Eu \pm \alpha \frac{u^2}{\lambda} E) + (pu \pm \frac{\gamma}{\lambda}(u^2 + \lambda^2) p) \end{pmatrix}$$

- **Test case**: Acoustic wave. SL order 11, 4000 cells.



- D1Q2 $\Delta t = 0.005$ (yellow), D1Q3 $\Delta t = 0.005/0.01$ (red, green). Contact captured.

BC : preliminary results

- **Question**: What BC for the kinetic variables. How keep the order ?

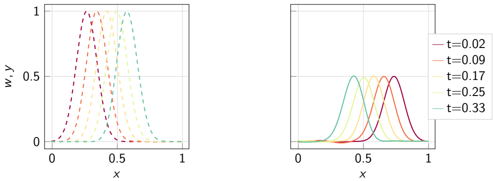
Consistency result (recall)

- The second order symmetric scheme (Ψ_{ap}) for the following equation (equivalent to $[D1Q2]^n$ kinetic model):

$$\begin{cases} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = O(\Delta t^2) \\ \partial_t \boldsymbol{W} - \partial \boldsymbol{F}(\boldsymbol{U}) \partial_x \boldsymbol{W} = O(\Delta t^2) \end{cases}$$

with $U_i = f_-^i + f_+^i$, $V_i = \lambda(f_+^i - f_-^i)$ and $\boldsymbol{W} = \boldsymbol{F}(\boldsymbol{U}) - \boldsymbol{V}$.

- **Natural BC**: entering condition for $\boldsymbol{U}$ and $\boldsymbol{W} = 0$ or $\partial_x \boldsymbol{W} = 0$.
- Example I: $F(u) = cu$ (transport):



- Transport of the u (dashed lines) and $w = v - f(u)$ (plain lines) quantities.

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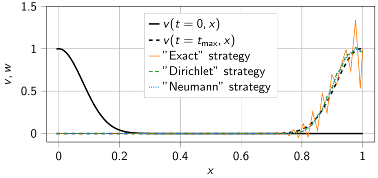

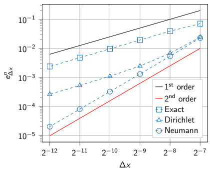

- Initial state and comparison of the final states. Gaussian initial profile, $\Delta x = 2^{-7}$.

BC : preliminary results

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Consistency result (recall)

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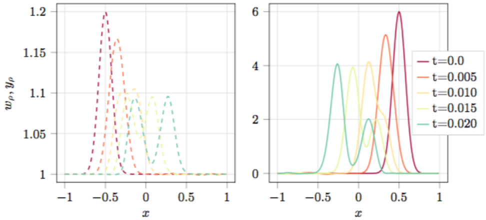
with $U_i = f_-^i + f_+^i$, $V_i = \lambda(f_+^i - f_-^i)$ and $\boldsymbol{W} = \boldsymbol{F}(\boldsymbol{U}) - \boldsymbol{V}$.

- **Natural BC**: entering condition for $\boldsymbol{U}$ and $\boldsymbol{W} = 0$ or $\partial_x \boldsymbol{W} = 0$.
- Example II: Barotropic Euler equation (supersonic):



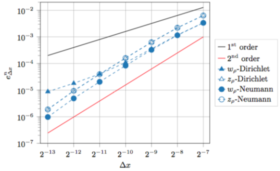


Figure 5: Convergence study for the Euler equations with Gaussian initial profile. Comparison of Dirichlet and Neumann strategies for the variables w_ρ and z_ρ.

- Left: ρ and w_ρ, Right: convergence result.

Parabolic systems

Relaxation scheme for diffusion

- We consider the classical Xin-Jin relaxation for a scalar system $\partial_t u - \nu \partial_{xx} u = 0$:

$$\left\{ \begin{array}{l} \partial_t u + \partial_x v = 0 \\ \partial_t v + \dfrac{\lambda^2}{\varepsilon^2} \partial_x u = -\dfrac{1}{\varepsilon^2} v \end{array} \right.$$

Limit

- The limit scheme of the relaxation system is

$$\partial_t u - \partial_x(\lambda^2 \partial_x u) = \varepsilon^2 \partial_{xxxx} u + O(\varepsilon^4)$$

- **Consistency**: Choosing $\lambda^2 = \nu$ we obtain the initial solution.

- We diagonalize the hyperbolic matrix $\begin{pmatrix} 0 & 1 \\ \lambda^2 & 0 \end{pmatrix}$ to obtain

$$\left\{ \begin{array}{l} \partial_t f_- - \dfrac{\lambda}{\varepsilon} \partial_x f_- = \dfrac{1}{\varepsilon^2}(f_{eq}^-(u) - f_-) \\ \partial_t f_+ + \dfrac{\lambda}{\varepsilon} \partial_x f_+ = \dfrac{1}{\varepsilon^2}(f_{eq}^+(u) - f_+) \end{array} \right.$$

- with $u = f_- + f_+$ and $f_{eq}(u)^{\pm} = \frac{u}{2}$.
- **Many schemes for this limit**. Hyperbolic case: Jin-Levermore [96], Gosse-Toscani [00] etc. Kinetic case: Lemou-Cresetto and al [09-14-17], Pareschi-Dimarco [07-10-14] etc.

Discretization

Consistency analysis

- We consider $\partial_t \rho - \nu \partial_{xx} \rho = 0$.
- We define the two operators for each step:

$$T_{\Delta t} : e^{\Delta t \frac{\Lambda}{\varepsilon} \partial_x} \boldsymbol{f}^{n+1} = \boldsymbol{f}^n$$

$$R_{\Delta t} : \boldsymbol{f}^{n+1} + \theta \frac{\Delta t}{\varepsilon^2} (\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^{n+1}) = \boldsymbol{f}^n - (1-\theta) \frac{\Delta t}{\varepsilon^2} (\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^n)$$

- **Final scheme**: $T_{\Delta t} \circ R_{\Delta t}$ is consistent with

$$\partial_t \rho = \Delta t \partial_x \left(\left(\frac{1-\omega}{\omega} + \frac{1}{2} \right) \frac{\lambda^2}{\varepsilon^2} \partial_x \rho \right) + O(\Delta t^2)$$

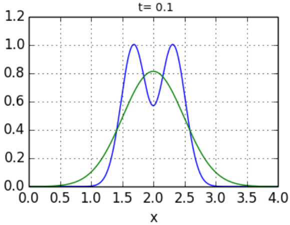
- We don't have convergence for all ε. The splitting scheme is not AP
- Taking $\nu = \lambda^2$, $\theta = 0.5$ and $\varepsilon = \sqrt{\Delta t}$ we obtain the diffusion equation.
- **Question**: When you choose like this. Consistence or not ?
- **First results** (for these choices of parameters):
  - Second order at the numerical level.
  - At the minimum the first order theoretically.

Numerical results for diffusion equation

- Heat equation. Scheme with $\varepsilon = \Delta t^{\gamma}$ and very high order SL + fine grid.

	$\gamma = \frac{1}{2}$		$\gamma = 1$		$\gamma = 2$	
	Error	order	Error	order	Error	order
$\Delta t = 0.04$	$1.87E^{-2}$	-	1.43	-	1.43	-
$\Delta t = 0.02$	$6.57E^{-3}$	1.50	0.2	0	0.23	0
$\Delta t = 0.01$	$1.85E^{-3}$	1.82	0.2	0	0.23	0
$\Delta t = 0.005$	$3.6E^{-4}$	2.36	0.2	0	0.23	0
$\Delta t = 0.0025$	$7.3E^{-5}$	2.30	0.2	0	0.23	0

- $\Delta t = 0.1$. The scheme oscillate. We cannot take very large time step.



Generalization

- With the general model

$$\partial_t \boldsymbol{f} + \Lambda \partial_x \boldsymbol{f} = \frac{R(x,\rho)}{\varepsilon} (\boldsymbol{f}^{eq}(\rho) - \boldsymbol{f})$$

- we can approximate the equation:

$$\partial_t \rho - \partial_x (D(x,\rho)\partial_x \rho) = 0$$

- **Convergence**: CV can be at the order 1.

Conclusion

Time scheme for BGK

- **High order Method**: Composition + Strang Splitting (or modified version) + Crank-Nicolson scheme for relaxation.
- **Default**: scheme not accurate far from the equilibrium and dispersive.
- **Advantage**: independent transport equation. Useful with an implicit transport solver.

Implicit Kinetic relaxation schemes

- We can approximate hyperbolic/parabolic PDE by small BGK models.
- Using this, we propose high-order scheme with large time step algorithm (SL method).
- This algorithm is very competitive against implicit schemes (no matrices, no solvers).

Future works

- Improve and study stability of low-Mach scheme in 1D
- Extension in 2D/3D for Low-Mach Euler/NS/MHD equations.
- Different approach/similar idea: Semi implicit Relaxation for NS and MHD.
- Study the scheme for elliptic problems and anisotropic diffusion.
- Propose artificial viscosity method for the total scheme (relaxation and SL steps) to avoid the oscillations.