Relaxation methods and low complexity implicit schemes

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Outline

Introduction

Implicit methods and hyperbolic PDE

Hyperbolic PDE and CFL constrains

Context

We want solve geophysical, compressible or plasmas flows (Tokamaks or astrophysical applications).

- We consider here the following type of equations:

$$\partial_t \boldsymbol{U} + \nabla \cdot (\boldsymbol{F}(\boldsymbol{U})) = \nu \nabla \cdot (\boldsymbol{D}(\boldsymbol{U}) \nabla \boldsymbol{U})$$

  with $\nu << 1$ or $\nu = 0$

- In general this type of problem are solved with explicit time integrators due to first orde CFL condition:

$$\Delta t < min(\frac{\Delta x}{\lambda_{max}(\partial \boldsymbol{F}(\boldsymbol{U}))}, \frac{\Delta x^2}{\nu})$$

- However it can be interesting to consider CFL-less approach in some cases:
  - when we want compute stationary flows,
  - when some cells are really small without physical reason (due to geometry for example),
  - for multiscale problems.

Implicit scheme and hyperbolic PDE

The hyperbolic PDE are not well-adapted to implicit method due to: nonlinearity, the directional structure and the multiscale dynamics.

Euler equations and the low Mach regime

- **Euler equations**:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0 \\ \partial_t (\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u} + p I_d) = 0 \\ \partial_t E + \nabla \cdot (E \boldsymbol{u} + p \boldsymbol{u}) = 0 \end{cases}$$

  with $\rho(t, \mathbf{x}) > 0$ the density, $\boldsymbol{u}(t, \mathbf{x})$ the velocity and $E(t, \mathbf{x}) > 0$ the total energy.

- **Hyperbolic system** with nonlinear waves. **Waves speed**: three differents eigenvalues: $(\boldsymbol{u}, \mathbf{n})$ and $(\boldsymbol{u}, \mathbf{n}) \pm c$ with the sound speed $c^2 = \gamma \frac{p}{\rho}$.

Euler equations and the low Mach regime

- **Euler equations**:

$$\begin{cases}\partial_t\rho+\nabla\cdot(\rho\boldsymbol{u})=0\\ \partial_t(\rho\boldsymbol{u})+\nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u}+pI_d)=0\\ \partial_t E+\nabla\cdot(E\boldsymbol{u}+p\boldsymbol{u})=0\end{cases}\longrightarrow\begin{cases}\partial_t\rho+\nabla\cdot(\rho\boldsymbol{u})=0\\ \partial_t(\rho\boldsymbol{u})+\nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u})+\dfrac{1}{M^2}\nabla p=0\\ \partial_t E+\nabla\cdot(E\boldsymbol{u}+p\boldsymbol{u})=0\end{cases}$$

  with $\rho(t,\mathbf{x})>0$ the density, $\boldsymbol{u}(t,\mathbf{x})$ the velocity and $E(t,\mathbf{x})>0$ the total energy.

- **Hyperbolic system** with nonlinear waves. **Waves speed**: three differents eigenvalues: $(\boldsymbol{u},\mathbf{n})$ and $(\boldsymbol{u},\mathbf{n})\pm c$ with the sound speed $c^2=\gamma\frac{p}{\rho}$.

Physic interpretation:

- **Two important velocity scales**: $\boldsymbol{u}$ and c, and their ratio (the Mach number) $M=\frac{|\boldsymbol{u}|}{c}$.

- When M tends to zero, we obtain the incompressible Euler equations:

$$\begin{cases}\partial_t\rho+\boldsymbol{u}\cdot\nabla\rho=0\\ \rho\partial_t\boldsymbol{u}+\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla p_2=0\\ \nabla\cdot\boldsymbol{u}=0\end{cases}$$

  In 1D, we only have an advection of ρ.
- **Aim**: construct an scheme valid at the limit with a uniform cost compared to M.
- **Other related problems**: Euler with gravity, low-Mach and low-β MHD.

Explicit vs implicit schemes

- **Explicit scheme**: issues with the CFL condition for low Mach flow:
  - Fast perturbative phenomena: acoustic waves at velocity c
  - Important phenomena: transport at velocity u
  - Expected CFL condition $\Delta t < \frac{\Delta x}{|u|}$; in practice, we need $\Delta t < \frac{\Delta x}{c} = M\frac{\Delta x}{|u|}$
  - At the end, we need a Δt multiplied by M compared to the expected Δt

First solution

Implicit time scheme. No CFL condition. Taking a larger time step, it allows to "filter" the fast acoustic waves which are not import to capture the limit regime.

- Implicit scheme: Newton method (important additional cost) + GMRES
- Simpler example (linearized compressible NS equations around $\mathbf{u}_0 = 0$):

$$\begin{cases} \partial_t p + \frac{1}{M}\nabla\cdot\mathbf{u} = 0 \\ \partial_t \mathbf{u} + \frac{1}{M}\nabla p = \nu\Delta\mathbf{u} \end{cases} \rightarrow \begin{cases} p^{n+1} + \frac{\Delta t}{M}\nabla\cdot\mathbf{u}^{n+1} = p^n \\ \mathbf{u}^{n+1} + \frac{\Delta t}{M}\nabla p^{n+1} - \nu\Delta t\Delta t\mathbf{u}^{n+1} = \mathbf{u}^n \end{cases}$$

- Matrix to invert:

$$\begin{pmatrix} \frac{M}{\Delta t}(I_d - \nu\Delta t^2) & \nabla\cdot \\ \nabla & \frac{M}{\Delta t}I_d \end{pmatrix}$$

- For $\Delta t \gg M$, the limit problem is ill-posed, and the matrix is difficult to invert.

Aim

Design simpler implicit schemes

Relaxation methods for implicit schemes

Second idea: relaxation approach

Relaxation approach

Keep the idea to replace the original model by one that is simpler to solve, use it as a solver rather than a preconditioner.

- **Relaxation** [JX95] : Used to design new schemes.
- **Idea**: Approximate the model

$$\partial_t \boldsymbol{U} + \partial_x \mathbf{F}(\boldsymbol{U}) = 0, \qquad \text{by} \qquad \partial_t \mathbf{f} + \partial_x \mathbf{A}(\mathbf{f}) = \frac{1}{\varepsilon}(Q(\mathbf{f}) - \mathbf{f})$$

  At the limit (Hilbert expansion) and taking $P\mathbf{f} = \boldsymbol{U}$ ($P \in \mathbb{R}^{n,m}$ with $n < m$) we obtain

$$\partial_t \boldsymbol{U} + \partial_x \mathbf{F}(\boldsymbol{U}) = \varepsilon \partial_x (D(\boldsymbol{U}) \partial_x \boldsymbol{U}) + O(\varepsilon^2)$$

- **Time scheme**: Splitting
  - We first solve

$$\frac{\mathbf{f}^* - \mathbf{f}^n}{\Delta t} + \partial_x \mathbf{A}(\mathbf{f}^{*,n}) = 0,$$

  - We solve the stiff source term using an implicit scheme.

Advantages of this approach

- In general, we construct **A** with a simpler structure than **F**, to easily designed a Godunov numerical flux.
- Here we use it to construct some simpler implicit schemes.

Xin-Jin relaxation method

- We consider the following nonlinear hyperbolic system

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = 0,$$

with a function $\boldsymbol{U} \in \mathbb{R}^N$, $x \in \mathbb{R}$.

- **Aim**: Find a way to approximate this system with a sequence of simple systems.
- **Idea**: Xin-Jin relaxation method (very popular in the hyperbolic and Finite Volume community) [JX95]-[Nat96]-[ADN00].

$$\begin{cases} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{V} = 0 \\ \partial_t \boldsymbol{V} + \lambda^2 \partial_x \boldsymbol{U} = \dfrac{1}{\varepsilon}(\boldsymbol{F}(\boldsymbol{U}) - \boldsymbol{V}) \end{cases}$$

Limit scheme for the hyperbolic relaxation

The limit equation of the relaxation system is

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \varepsilon \partial_x((\lambda^2 Id - |A(\boldsymbol{U})|^2)\partial_x \boldsymbol{U}) + O(\varepsilon^2),$$

with $A(\boldsymbol{U})$ the Jacobian of $\boldsymbol{F}(\boldsymbol{U})$.

- **Conclusion**: the relaxation system is an approximation of the original hyperbolic system (with an error in ε).

Xin-Jin implicit scheme

Main property

- **Relaxation system**: "the nonlinearity is local and the non-locality is linear".
- **Main idea**: **splitting scheme** between implicit transport and **implicit** relaxation.
- **Key point**: we have $\partial_t \boldsymbol{U} = 0$ during the relaxation step. Therefore $\boldsymbol{F}(\boldsymbol{U})$ **is explicit**.

- Relaxation step: we use a θ scheme:

$$\begin{cases} \boldsymbol{U}^{n+1} = \boldsymbol{U}^n \\ \boldsymbol{V}^{n+1} = \theta \dfrac{\Delta t}{\varepsilon}(\boldsymbol{F}(\boldsymbol{U}^{n+1}) - \boldsymbol{V}^{n+1}) + (1-\theta)\dfrac{\Delta t}{\varepsilon}(\boldsymbol{F}(\boldsymbol{U}^n) - \boldsymbol{V}^n) \end{cases}$$

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- Relaxation step: we use a θ scheme:

$$\begin{cases} \boldsymbol{U}^{n+1} = \boldsymbol{U}^n \\ \left(I_d + \theta \dfrac{\Delta t}{\varepsilon}\right) \boldsymbol{V}^{n+1} = \theta \dfrac{\Delta t}{\varepsilon} \boldsymbol{F}(\boldsymbol{U}^n) + (1-\theta)\dfrac{\Delta t}{\varepsilon}(\boldsymbol{F}(\boldsymbol{U}^n) - \boldsymbol{V}^n) \end{cases}$$

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$$\begin{cases} \boldsymbol{U}^{n+1} = \boldsymbol{U}^n \\ \boldsymbol{V}^{n+1} = \boldsymbol{V}^n + \underbrace{\frac{\Delta t}{\varepsilon + \theta \Delta t}}_{\omega} (\boldsymbol{F}(\boldsymbol{U}^n) - \boldsymbol{V}^n) \end{cases}$$

Xin-Jin implicit scheme

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- **Main idea**: **splitting scheme** between implicit transport and **implicit** relaxation.
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- Relaxation step: we use a θ scheme:
- Transport step (order 1) :

$$\left(I_d + \Delta t \begin{pmatrix} 0 & 1 \\ \lambda^2 & 0 \end{pmatrix} \partial_x\right) \begin{pmatrix} \boldsymbol{U}^{n+1} \\ \boldsymbol{V}^{n+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{U}^{n} \\ \boldsymbol{V}^{n} \end{pmatrix}$$

We plug the equation on $\boldsymbol{V}$ in the equation on $\boldsymbol{U}$ and obtain

$$(I_d - \Delta t^2 \lambda^2 \partial_{xx}) \boldsymbol{U}^{n+1} = \boldsymbol{U}^n - \Delta t \partial_x \boldsymbol{V}^n, \quad \boldsymbol{V}^{n+1} = \boldsymbol{V}^n - \Delta t \lambda^2 \partial_x \boldsymbol{U}^{n+1}$$

Numerical error of first splitting scheme

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \Delta t \left(\frac{2-\omega}{\omega}\right) \partial_x((\lambda^2 I_d - |A(\boldsymbol{U})|^2)\partial_x \boldsymbol{U}) + O(\Delta t^2)$$

- Remarks:
  - Coupling with Crank-Nicolson scheme for wave equation we can go to second order
  - We solve n uncoupled constant Laplacian in place to one nonlinear ill-conditioned system with n variables.
  - A. Thomann [Th2023] propose a specific IMEX scheme less dispersive than our approach.

Generic kinetic relaxation schemes

Kinetic relaxation systems

- **Model under consideration**:
$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = 0$$
- **Lattice**: $W = \{\lambda_1, ..., \lambda_{n_v}\}$ a set of velocities.
- **Mapping matrix**: P a matrix $n_c \times n_v$ ($n_c < n_v$) such that $\boldsymbol{U} = P\boldsymbol{f}$, with $\boldsymbol{U} \in \mathbb{R}^{n_c}$.
- **Kinetic relaxation system**:
$$\partial_t \boldsymbol{f} + \Lambda \partial_x \boldsymbol{f} = \frac{1}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f})$$
- Consistency condition (Natalini - Aregba [96-98-02], Bouchut [99-03]) :
$$\begin{cases} P\boldsymbol{f}^{eq}(\boldsymbol{U}) & = & \boldsymbol{U} \\ P\Lambda \boldsymbol{f}^{eq}(\boldsymbol{U}) & = & \boldsymbol{F}(\boldsymbol{U}) \end{cases} \qquad (\mathcal{C})$$

Chapman-Enskog stability

- **Limit system**:
$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \varepsilon \partial_x \left(\left(P\Lambda^2 \partial_{\boldsymbol{U}} \boldsymbol{f}^{eq}(\boldsymbol{U}) - |\partial \boldsymbol{F}(\boldsymbol{U})|^2 \right) \partial_x \boldsymbol{U} \right) + O(\varepsilon^2)$$
- This limit system is stable if the second order operator is entropy-dissipative. We also have partial stability results for the kinetic systems.
- **Strong Stability**: entropy theory equivalent to the H-theorem. Other criteria for stability are given in Bouchut [04].

Example of kinetic relaxation systems

- Example:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t \rho u + \partial_x \left(\rho u^2 + c^2 \rho\right) = 0 \end{cases}.$$

- Vectorial approach:
  - Each physical equation is represented by q transport equations.
  - We use 2 variables by physical variable so $\mathbf{f} = \left(f_-^r, f_+^r, f_-^{ru}, f_+^{ru}\right)$ with:

$$V = [-\lambda, \lambda]^2 .$$

and

$$\rho = f_-^r + f_+^r, \quad \rho u = f_-^{ru} + f_+^{ru}$$

$$f_{r,\pm}^{eq} = \frac{\rho}{2} \pm \frac{\rho u}{2\lambda}, \quad f_{ru,\pm}^{eq} = \frac{\rho u}{2} \pm \frac{\rho u^2 + c^2 \rho}{2\lambda}$$

  - Stable in all the physical regimes if we satisfy the sub-characteristic condition.
  - **dissipation**: similar to Rusanov scheme.

- Boltzmann approach:
  - We discretize with a minimal set of velocities the Boltzmann equations:
  - Ex: isothermal Euler equation
  - We use three variables $\mathbf{f} = \left(f_-, f_0, f_+\right)$ for all the variables $(\rho, \rho u)$ with:

$$V = [-\lambda, 0, \lambda]$$

$$\rho = f_- + f_0 + f_+, \quad \rho u = \lambda(f_+ - f_-)$$

$$f_\pm^{eq} = \frac{\rho u}{2}(u \pm 1) + \frac{\rho c^2}{2\lambda^2}, \quad f_0^{eq} = \rho - \rho^2 u - \frac{c^2 \rho}{2\lambda^2}$$

  - Stable in the Mach regime < 0.6 if we satisfy the sub-characteristic condition.
  - **dissipation**: low dissipation scheme.

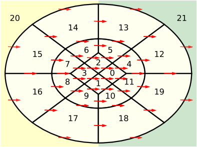
Implicit scheme based on kinetic relaxation I

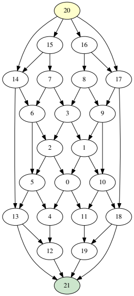
- **Advantage**: We replace independent wave equations by independent transport equations.
- We define the two operators for each step :

$$T(\Delta t) : e^{\Delta t \Lambda \partial_x} \boldsymbol{f}^{n+1} = \boldsymbol{f}^n$$

$$R(\Delta t) : \boldsymbol{f}^{n+1} = \boldsymbol{f}^n + \omega(\boldsymbol{f}^{eq}(\boldsymbol{U}^n) - \boldsymbol{f}^n)$$

- **First splitting scheme**: $T(\Delta t) \circ R(\Delta t)$ is consistent with
- How to deal with the transport step with constant velocity?
  - Exact transport (induce a CFL), is there the Lattice Boltzmann methods.
  - Semi-Lagrangian scheme,
  - CFL-less implicit DG scheme, with a downwind strategy: block triangular matrix using task graph numbering.





Implicit scheme based on kinetic relaxation II

High order scheme: composition method

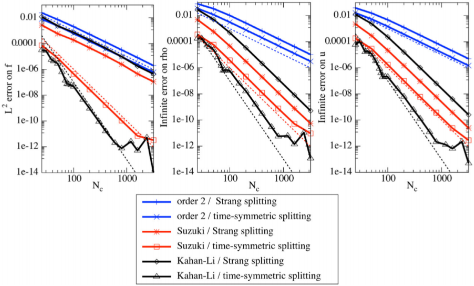
- If Ψ, a scheme that is second-order accurate in time, satisfies $\Psi(\Delta t)=\Psi^{-1}(-\Delta t)$ and $\Psi(0)=I_d$, then we can construct the high-order extension
$$M_p(\Delta t)=\Psi(\gamma_1\Delta t)\circ\Psi(\gamma_2\Delta t)\circ\cdots\circ\Psi(\gamma_s\Delta t),$$
with $\gamma_i\in[-1,1]$.
- Susuki scheme : $s=5$, $p=4$. Kahan-Li scheme: $s=9$, $p=6$.

New second-order scheme

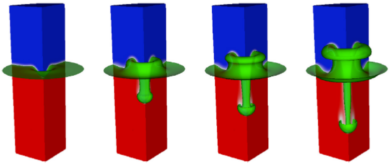
- The current second-order scheme is:
$$\Psi(\Delta t)=T\left(\frac{\Delta t}{2}\right)\circ R(\Delta t,\omega=2)\circ T\left(\frac{\Delta t}{2}\right).$$
- It satisfies the time symmetry, but not $\Psi(0)=I_d$ for $\varepsilon\approx 0$. Indeed,
$$R(\Delta t=0,\omega=2)\iff \boldsymbol{f}^n=2\boldsymbol{f}^{eq}-\boldsymbol{f}^n\neq\boldsymbol{f}^n$$
- However, $R(0,\omega=2)\circ R(0,\omega=2)=I_d$, and so we propose the following second-order scheme:
$$\Psi_{ap}(\Delta t)=T\left(\frac{\Delta t}{4}\right)\circ R(\Delta t,\omega=2)\circ T\left(\frac{\Delta t}{2}\right)\circ R(\Delta t,\omega=2)\circ T\left(\frac{\Delta t}{4}\right)$$

Implicit scheme based on kinetic relaxation III

- Error lines for the isothermal Euler equations. We have taken a CFL condition equal to 5 times the explicit one.



- Rayleigh-Taylor instability



- Theory and parallelization: Coulette and al [17-18-19].

Implicit scheme based on kinetic relaxation IV

- We have applied this strategy to the Guiding center for plasma physics, Helie [22]:

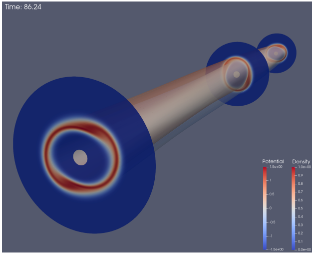
$$\begin{cases} \partial_t \rho - \nabla \cdot (((\nabla \phi)^{\perp} + \mathbf{B})\rho) = 0, \\ -\Delta \phi = \rho - \int \rho dx, \end{cases}$$

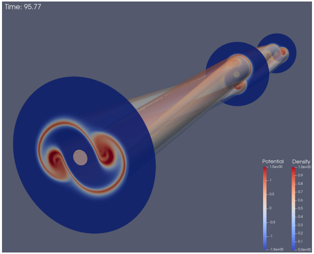
with $\mathbf{B} = (-b_\theta \sin(\theta), b_\theta \sin(\theta), b_\phi)^t$. We choose $b_\theta = 0.1$ and $b_\phi = 200$.

- **Scheme**: Exact transport in the toroidal direction, implicit DG kinetic scheme in the poloidal plane.
- **CFL conditions** for the classical and new schemes:

$$\Delta t_{exp} < \frac{\min(v_{pol}, v_\phi)}{\max(\Delta x_{pol}, \Delta x_\phi)} \qquad \rightarrow \qquad \Delta t_{new} < \frac{v_\phi}{\Delta x_\phi}$$

- **Test case**: 3D Diocotron instability. CFL condition equal to 33 times the explicit one.





Bc conditions

How impose the physical BC since we solve non physical equation ?

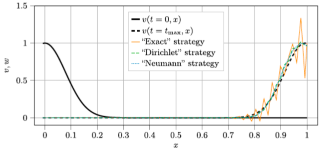
- Equivalence equation to kinetic scheme (R. Helie an al [2022-2023]).
- We consider the model $\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = 0$ and the kinetic relaxation model given by

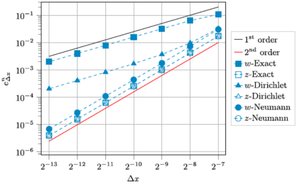
$$\begin{cases} \partial_t \mathbf{f}^+ + \lambda \partial_x \mathbf{f}^+ = \dfrac{1}{\varepsilon}\left(\mathbf{f}^{eq,+} - \mathbf{f}^+\right) \\ \partial_t \mathbf{f}^- - \lambda \partial_x \mathbf{f}^- = \dfrac{1}{\varepsilon}\left(\mathbf{f}^{eq,-} - \mathbf{f}^-\right) \end{cases}$$

- The solution of the kinetic scheme is solution of the next model with an error of $O(\Delta t^2)$

$$\partial_t \begin{pmatrix} \mathbf{U} \\ \mathbf{z} \end{pmatrix} + \begin{pmatrix} \mathbf{F}'(\mathbf{U}) & 0 \\ 0 & -\mathbf{F}'(\mathbf{U}) \end{pmatrix} \partial_x \begin{pmatrix} \mathbf{U} \\ \mathbf{z} \end{pmatrix} = 0$$

with $\mathbf{z} := \mathbf{V} - \mathbf{F}(\mathbf{U})$
- This analysis gives a tools to find good BC.





Relaxation methods and viscosity

How treat low viscosity models

Objective

Discretize hyperbolic systems with small viscosity term $\nu\partial_x(D(\boldsymbol{U})\partial_x\boldsymbol{U})$ keeping the good structure of the algorithm with transport step and local relaxation step.

- We recall that the first order transport-relaxation is consistant with the following equivalent equation:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \Delta t \partial_x \left(\left(P\Lambda^2 \partial_{\boldsymbol{U}} \boldsymbol{f}^{eq}(\boldsymbol{U}) - |\partial \boldsymbol{F}(\boldsymbol{U})|^2 \right) \partial_x \boldsymbol{U} \right) + O(\Delta t^2)$$

- If we take the new relaxation model

$$\partial_t \boldsymbol{f} + \Lambda \partial_x \boldsymbol{f} = \frac{R(\boldsymbol{U})}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f})$$

- the equivalent equation becomes:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \Delta t \partial_x \left(R_u(\boldsymbol{U})^{-1} \left(P\Lambda^2 \partial_{\boldsymbol{U}} \boldsymbol{f}^{eq}(\boldsymbol{U}) - |\partial \boldsymbol{F}(\boldsymbol{U})|^2 \right) \partial_x \boldsymbol{U} \right) + O(\Delta t^2)$$

  with $R_u(\boldsymbol{U})$ a subpart of $R(\boldsymbol{U})$

Solution

Choosing $R_u(\boldsymbol{U})$ correctly we wil obtain:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \nu \partial_x \left(D(\boldsymbol{U}) \partial_x \boldsymbol{U} \right) + O(\nu^2)$$

- For high-viscosity flow there exist relaxation scheme but the situation is more complex.

Artificial viscosity and stabilization I

Idea

Use this modified relaxation model to apply artificial viscosity to stabilize the method.

- We consider here the scalar case:

$$\partial\rho + \nabla\cdot(\boldsymbol{f}(\rho)) = 0.$$

- $[D2Q4]$ scheme :
  1. 4 new variables $f_1(t,x), \dots, f_4(t,x)$
  2. At each time step transport step:

  $$f_i^*(t_n, \boldsymbol{x}_j) = f_i(t_n, \boldsymbol{x}_j - \Delta t \boldsymbol{v}_i), \quad \forall 1 \leq i \leq 4$$

  3. At each time step collisional step:

  $$f_i(t_{n+1}, \boldsymbol{x}_j) = f_i^*(\boldsymbol{x}_j) + W(\rho(\boldsymbol{x}_j))(f_i^{eq}(\rho(\boldsymbol{x}_j)) - f_i^*(\boldsymbol{x}_j))$$

  with $\rho(\boldsymbol{x}_j) = \sum_{i=1}^4 f_i^*(\boldsymbol{x}_j)$, M a change of basis matrix and

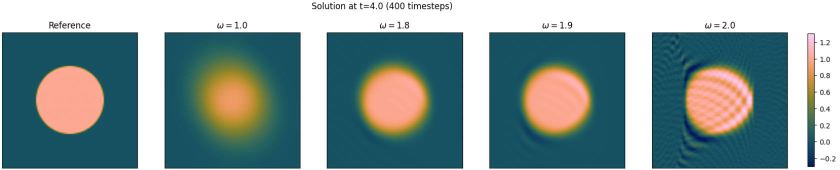
  $$W(\rho) = M^{-1}\left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & & & 0 \\ 0 & \multicolumn{2}{c|}{\Omega(\rho)} & 0 \\ \hline 0 & 0 & 0 & {\color{red}\tau} \end{array}\right) M.$$

- Error :

$$\partial\rho + \nabla\cdot(\boldsymbol{f}(\rho)) = \Delta t \nabla\cdot\left(\underbrace{\left(\Omega^{-1}(\rho) - \frac{1}{2}I\right)\left(\frac{\lambda^2}{2}I - \boldsymbol{f}'(\rho)\otimes\boldsymbol{f}'(\rho)\right)}_{D(\rho)}\nabla\rho\right) + O(\Delta t^2)$$

Artificial viscosity and stabilization II

- Choice of Ω for scalar advection with 400 time step.



Aim

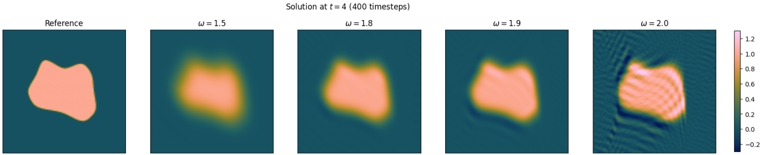
Construct the viscosity using neural network

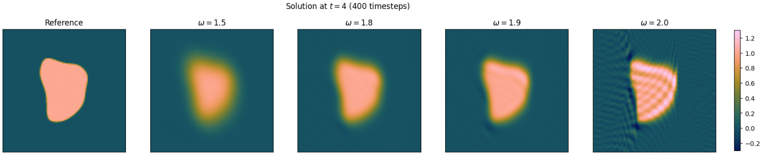
Algorithm 1: Training algorithm

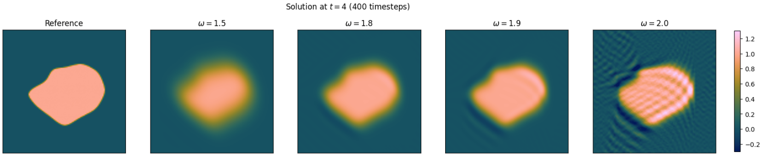
Set the total number of timesteps desired N
Compute reference solutions $\{\mathbf{u}_{\text{ref},i}\}_{i=1}^{m}$ on all timesteps t^0, ..., t^N
Build the neural network π_θ and initialize its parameters θ
Set the number of timesteps for the training n
while *True* **do**
    for $k \in \{1, ..., \textit{batch size}\}$ **do**
        for $i \in \{1, ..., m\}$ **do**
            Draw a random starting time $t^i \in \{t^0, ..., t^{N-n}\}$
            Compute the numerical solutions $\mathbf{u}_{\theta,i}^{[t^i,t^{i+n}]}$
            Compute the error $E_i(\theta) = \|\mathbf{u}_{\theta,i}^{[t^i,t^{i+n}]} - \mathbf{u}_{\text{ref},i}^{[t^i,t^{i+n}]}\|$
        end
        Compute the approximated loss for this sample $\mathcal{L}_k(\theta) = \sum_{i=1}^{m} E_i(\theta)$
    end
    Compute the approximated loss for this batch $\mathcal{L}(\theta) = \sum_{k=1}^{\text{batch size}} \mathcal{L}_k(\theta)$
    Update the parameters θ with $\nabla\mathcal{L}(\theta)$
end

Artificial viscosity and stabilization III

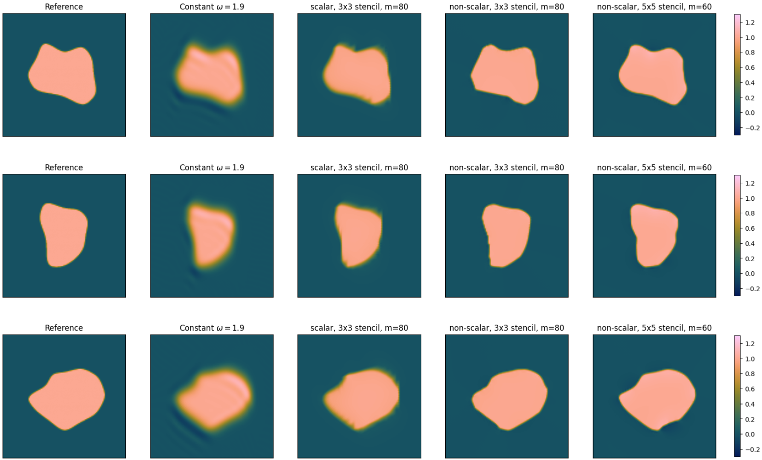
- Examples with the classical scheme $\Omega = 1.9 I_d$:







- Examples with learned viscosities:



- Low dissipation but low deformation of the shape. How avoid this ?
- Unfinished work. Interesting to treat system. How avoid instability in the training ?

Semi implicit scheme and relaxation

Low Mach regime

➜ We consider the isothermal Euler equation in the low Mach regime:

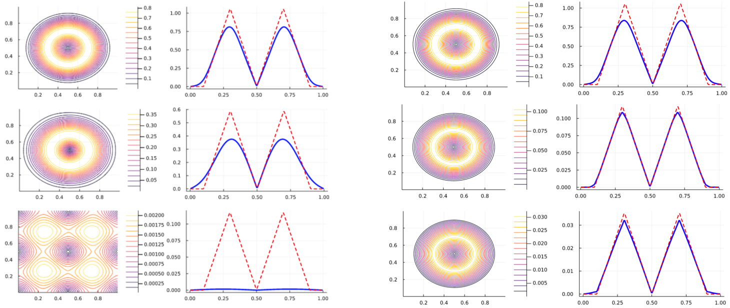
$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\ \partial_t(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + \frac{1}{M^2}\rho) = 0 \end{cases}$$

Numerical error

➜ Asymptotic limit:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p \approx \left(\frac{2-\omega}{\omega}\right) \frac{\Delta t}{2M^2} |\mathbf{u}|^2 \Delta \mathbf{u} + O(\Delta t^2)$$

➜ **Conclusion**: The scheme is asymptotic preserving for $\omega = 2 - M^2$.



Low Mach regime II

- We consider the isothermal Euler equation in the low Mach regime:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \frac{1}{M^2} p) = 0 \\ \partial_t(E) + \partial_x(Eu + pu) = 0 \end{cases}$$

- **Limit in 1D**: a transport equation on the density with $p(t,x) = p_0$ and $u(t,x) = u_0$.
- A uniformly AP scheme: the error on the transport depend only on u_0.
- **Excepted behavior**: accuracy must be the same for the different u_0 by taking $\Delta t = 0.5 \frac{\Delta x}{u_0}$ and $T_f = \frac{0.15}{u_0}$

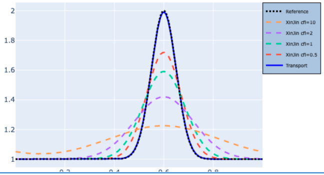
Numerical error

- Equivalent equation:

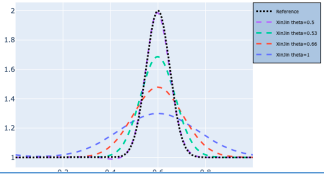
$$\partial_t \rho + u \partial_x \rho = \Delta t \left(\frac{2-\omega}{\omega} \right) \partial_x \left(\lambda^2 - u^2 \right) \partial \rho + O(\Delta t^2)$$

- For stability reason we must choose $\lambda > \frac{c}{M} + u$. **Conclusion**: The scheme is too dissipative.

Varying CFL



Varying w



Low Mach regime II

- We consider the isothermal Euler equation in the low Mach regime:

$$\left\{\begin{array}{l}\partial_t\rho+\partial_x(\rho u)=0\\ \partial_t(\rho u)+\partial_x(\rho u^2+\frac{1}{M^2}p)=0\\ \partial_t(E)+\partial_x(Eu+pu)=0\end{array}\right.$$

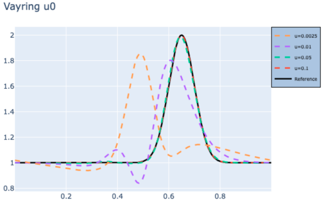
- **Limit in 1D**: a transport equation on the density with $p(t,x)=p_0$ and $u(t,x)=u_0$.
- A uniformly AP scheme: the error on the transport depend only on u_0.
- **Excepted behavior**: accuracy must be the same for the different u_0 by taking $\Delta t=0.5\frac{\Delta x}{u_0}$ and $T_f=\frac{0.15}{u_0}$

Numerical error for $\omega=2$

- Equivalent equation:

$$\partial_t\rho+u\partial_x\rho=\Delta t^2 O(\lambda^4)+O(\Delta t^2)$$

- For stability reason we must choose $\lambda>\frac{c}{M}+u$. **Conclusion**: The scheme is too dispersive.



Suliciu relaxation for the Low-Mach regime I

Suliciu type relaxation

Originally the relaxation methods have been used to construct Godunov solver. The Suliciu approach consist to linearize only the genuinely nonlinear waves compared to the Xin-Jin/Kinetic relaxation methods which linearize all the waves.

- **Idea**: Linearize only the fast wave with relaxation.
- Non-conservative form and acoustic terms:

$$\left\{\begin{array}{l}\partial_t\rho+\partial_x(\rho u)=0\\ \partial_t p+u\partial_x p+\rho c^2\partial_x u=0\\ \partial_t u+u\partial_x u+\frac{1}{\rho}\partial_x p=0\end{array}\right.$$

Suliciu type relaxation

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- Non-conservative form and acoustic terms:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t p + u\partial_x p + \rho c^2 \partial_x u = 0 \\ \partial_t u + u\partial_x u + \frac{1}{\rho}\partial_x p = 0 \end{cases}$$

- **Idea**: Relax only the acoustic part ([BCG18]) to linearize the implicit part.

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u v + \Pi) = 0 \\ \partial_t E + \partial_x(E v + \Pi v) = 0 \\ \partial_t \Pi + v\partial_x \Pi + \phi\lambda^2 \partial_x v = \frac{1}{\varepsilon}(p - \Pi) \\ \partial_t v + v\partial_x v + \frac{1}{\phi}\partial_x \Pi = \frac{1}{\varepsilon}(u - v) \end{cases}$$

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- **Idea**: Linearize only the fast wave with relaxation.
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$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t p + u\partial_x p + \rho c^2 \partial_x u = 0 \\ \partial_t u + u\partial_x u + \frac{1}{\rho}\partial_x p = 0 \end{cases}$$

- **Idea**: Relax only the acoustic part ([BCG18]) to linearize the implicit part.

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u v + \Pi) = 0 \\ \partial_t E + \partial_x(E v + \Pi v) = 0 \\ \partial_t \Pi + v\partial_x \Pi + \phi\lambda^2 \partial_x v = \frac{1}{\varepsilon}(p - \Pi) \\ \partial_t v + v\partial_x v + \frac{1}{\phi}\partial_x \Pi = \frac{1}{\varepsilon}(u - v) \end{cases}$$

Advantage

We keep the conservative form for the original variables and obtain fully linear acoustics.

Suliciu relaxation for the Low-Mach regime II

- **Limit**:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = \varepsilon \partial_x \left[A \partial_x p \right] \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = \varepsilon \partial_x \left[(A u \partial_x p) + B \partial_x u \right] \\ \partial_t E + \partial_x(Eu + pu) = \varepsilon \partial_x \left[A E \partial_x p + A \partial_x \frac{p^2}{2} + B \partial_x \frac{u^2}{2} \right], \end{cases}$$

with $A = \frac{1}{\rho}\left(\frac{\rho}{\phi} - 1\right)$ and $B = \left(\rho\phi\lambda^2 - \rho^2 c^2\right)$.

- **Stability**: The dissipation is entropically stable if $\phi\lambda > \rho c^2$ and $\rho > \phi$.

Suliciu relaxation for the Low-Mach regime II

- **Limit**:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = \varepsilon \partial_x \left[A \partial_x p \right] \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = \varepsilon \partial_x \left[(A u \partial_x p) + B \partial_x u \right] \\ \partial_t E + \partial_x(E u + p u) = \varepsilon \partial_x \left[A E \partial_x p + A \partial_x \frac{p^2}{2} + B \partial_x \frac{u^2}{2} \right], \end{cases}$$

with $A = \frac{1}{\rho}\left(\frac{\rho}{\phi} - 1\right)$ and $B = \left(\rho\phi\lambda^2 - \rho^2 c^2\right)$.

- **Stability**: The dissipation is entropically stable if $\phi\lambda > \rho c^2$ and $\rho > \phi$.

Results

The relaxation system is hyperbolic and all the characteristic fields are linearly degenerate. The characteristic speeds are given by:

$$\Sigma = \left\{ v, v, v, v, v, v - \frac{a}{\rho}, v + \frac{a}{\rho} \right\}$$

Advantage

We keep the conservative form for the original variables and obtain fully linear acoustics.

Suliciu relaxation for Low-Mach III

- Splitting: Convective part treated explicitly / Acoustic part treated implicitly.

$$
\left\{
\begin{array}{l}
\partial_t\rho + \partial_x(\rho v) = 0\\
\partial_t(\rho u) + \partial_x(\rho u v + \mathcal{M}^2(t)\Pi) = 0\\
\partial_t E + \partial_x(E v + \mathcal{M}^2(t)\Pi v) = 0\\
\partial_t\Pi + v\partial_x\Pi + \phi\lambda_c^2\partial_x v = 0\\
\partial_t v + v\partial_x v + \frac{\mathcal{M}^2(t)}{\phi}\partial_x\Pi = 0
\end{array}
\right.
\quad \text{and} \quad
\left\{
\begin{array}{l}
\partial_t\rho = 0\\
\partial_t(\rho u) + (1-\mathcal{M}^2(t))\partial_x\Pi = 0\\
\partial_t E + (1-\mathcal{M}^2(t))\partial_x(\Pi v) = 0\\
\partial_t\Pi + \phi(1-\mathcal{M}^2(t))\lambda_a^2\partial_x v = 0\\
\partial_t v + (1-\mathcal{M}^2(t))\frac{1}{\phi}\partial_x\Pi = 0
\end{array}
\right.
$$

with $\mathcal{M}(t) \approx \max\left(\mathcal{M}_{min}, \min\left(\max\frac{|u|}{c}, 1\right)\right)$.

Suliciu relaxation for Low-Mach III

- Splitting: Convective part treated explicitly / Acoustic part treated implicitly.

$$\left\{\begin{array}{l}\partial_t\rho+\partial_x(\rho v)=0\\ \partial_t(\rho u)+\partial_x(\rho u v+\mathcal{M}^2(t)\Pi)=0\\ \partial_t E+\partial_x(Ev+\mathcal{M}^2(t)\Pi v)=0\\ \partial_t\Pi+v\partial_x\Pi+\phi\lambda_c^2\partial_x v=0\\ \partial_t v+v\partial_x v+\frac{\mathcal{M}^2(t)}{\phi}\partial_x\Pi=0\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l}\partial_t\rho=0\\ \partial_t(\rho u)+(1-\mathcal{M}^2(t))\partial_x\Pi=0\\ \partial_t E+(1-\mathcal{M}^2(t))\partial_x(\Pi v)=0\\ \partial_t\Pi+\phi(1-\mathcal{M}^2(t))\lambda_a^2\partial_x v=0\\ \partial_t v+(1-\mathcal{M}^2(t))\frac{1}{\phi}\partial_x\Pi=0\end{array}\right.$$

with $\mathcal{M}(t)\approx \max\left(\mathcal{M}_{min}, \min\left(\max\frac{|u|}{c},1\right)\right)$.

- Eigenvalues: explicit part: v, $v\pm\mathcal{M}(t)\underbrace{\lambda_c}_{\approx c}$; implicit part: 0, $\pm(1-\mathcal{M}^2(t))\underbrace{\lambda_a}_{\approx c}$.
  - Step 1: we solve

$$(I_d-(1-\mathcal{M}^2(t_n))^2\Delta t^2\lambda_a^2\partial_{xx})\Pi^{n+1}=\Pi^n-\Delta t(1-\mathcal{M}^2(t_n))\phi\lambda_a^2\partial_x v^n$$

  - Step 2: we compute v^{n+1} and ρu^{n+1} using Π^{n+1}, and E^{v+1} using $\Pi^{n+1}v^{n+1}$.

Suliciu relaxation for Low-Mach III

- Splitting: Convective part treated explicitly / Acoustic part treated implicitly.

$$\left\{\begin{array}{l}\partial_t\rho+\partial_x(\rho v)=0\\ \partial_t(\rho u)+\partial_x(\rho u v+\mathcal{M}^2(t)\Pi)=0\\ \partial_t E+\partial_x(Ev+\mathcal{M}^2(t)\Pi v)=0\\ \partial_t\Pi+v\partial_x\Pi+\phi\lambda_c^2\partial_x v=0\\ \partial_t v+v\partial_x v+\frac{\mathcal{M}^2(t)}{\phi}\partial_x\Pi=0\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l}\partial_t\rho=0\\ \partial_t(\rho u)+(1-\mathcal{M}^2(t))\partial_x\Pi=0\\ \partial_t E+(1-\mathcal{M}^2(t))\partial_x(\Pi v)=0\\ \partial_t\Pi+\phi(1-\mathcal{M}^2(t))\lambda_a^2\partial_x v=0\\ \partial_t v+(1-\mathcal{M}^2(t))\frac{1}{\phi}\partial_x\Pi=0\end{array}\right.$$

with $\mathcal{M}(t)\approx\max\left(\mathcal{M}_{min},\min\left(\max\frac{|u|}{c},1\right)\right)$.

- Eigenvalues: explicit part: v, $v\pm\mathcal{M}(t)\underbrace{\lambda_c}_{\approx c}$; implicit part: 0, $\pm(1-\mathcal{M}^2(t))\underbrace{\lambda_a}_{\approx c}$.
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  - Step 2: we compute v^{n+1} and ρu^{n+1} using Π^{n+1}, and E^{v+1} using $\Pi^{n+1}v^{n+1}$.

Advantages

- We construct an efficient Godunov relaxation scheme for the explicit part.
- We solve only a linear and constant Laplacian. The matrix is only assembled once.
- No conditioning issues coming from large gradients of ρ.

Results in 1D

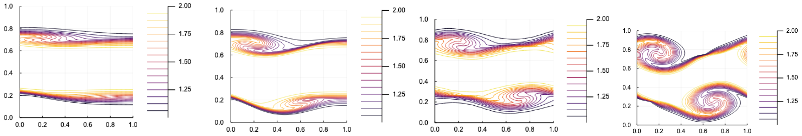
- We consider the same low problem as before with density transport.
- Convergence of different schemes:

		$N=250$	$N=500$	$N=1000$	$N=2000$
Explicit (Rusanov)	error	0.77	0.67	0.53	0.38
	order	-	0.2	0.34	0.48
Explicit (FVS)	error	$1.63E^{-2}$	$8.3E^{-3}$	$4.1E^{-3}$	$2.0E^{-3}$
	order	-	0.96	1.02	1.03
SI Suliciu (Rusanov)	error	$5.0E^{-2}$	$2.54E^{-2}$	$1.3E^{-2}$	$6.55E^{-3}$
	order	-	0.97	0.98	0.99
SI two-speed (Rusanov)	error	$1.1E^{-1}$	$6.5E^{-2}$	$3.4E^{-2}$	$1.7E^{-2}$
	order	-	0.76	0.93	1.0
SI two-speed (FVS)	error	$1.55E^{-2}$	$7.8E^{-3}$	$3.9E^{-3}$	$2.0E^{-3}$
	order	-	0.99	1.0	1.0
SI two-speed (Godunov)	error	$1.54E^{-2}$	$7.8E^{-3}$	$3.9E^{-3}$	$2.0E^{-3}$
	order	-	1.0	1.0	1.0

- CFL comparison

Scheme	$\lambda_{\max}$	Δt (PG law)	Δt (SG law)
Explicit	$\max(\lvert u-c\rvert, \lvert u+c\rvert)$	$1.3E^{-4}$	$2.7E^{-5}$
SI Suliciu	$\max\left(\lvert u-\mathcal{E}(t)\lambda_c/\rho\rvert, \lvert u+\mathcal{E}(t)\lambda_c/\rho\rvert\right)$	0.0038	0.004
SI two-speed	$\max\left(\lvert v-\mathcal{E}(t)\lambda_c I, \lvert v+\mathcal{E}(t)\lambda_c\rvert\right)$	0.029	0.03

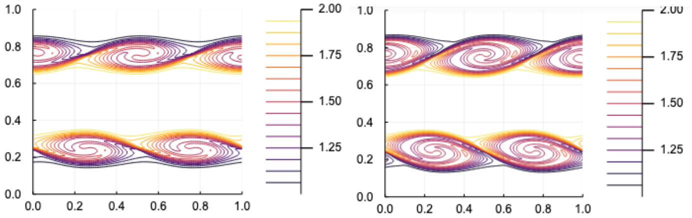
- Kelvin-Helmholtz instability. Density:



- Density at time $T_f = 3$, $k = 1$, $M_0 = 0.1$. Explicit Lagrange-Remap scheme with 120×120 (left) and 360×360 cells (middle left), SI two-speed relaxation scheme ($\lambda_c = 18$, $\lambda_a = 15$, $\phi = 0.98$) with 42×42 (middle right) and 120×120 cells (right).

Results in 2D: Kelvin-Helmholtz instability

- Kelvin-Helmholtz instability. Density:



- Density at time $T_f = 3$, $k = 2$, $M_0 = 0.01$ with SI two-speed relaxation scheme ($\lambda_c = 180$, $\lambda_a = 150$, $\phi = 0.98$). Left: 120×120 cells. Right: 240×240 cells.

Well balanced extension I

- We consider the Ripa model:

$$\begin{cases} \partial_t h + \partial_x(hu) = 0 \\ \partial_t(hu) + \partial_x\left(hu^2 + \frac{p(h,\Theta)}{F_r^2}\right) = -\frac{gh\Theta\partial_x z}{F_r^2} \\ \partial_t(h\Theta) + \partial_x(h\Theta u) = 0 \end{cases}$$

with $p = g\Theta\frac{1}{2}h^2$ the pressure.

- Steady states:

$$\begin{cases} u = 0 \\ \Theta = cst \\ h + z = cst, \end{cases}, \quad \begin{cases} u = 0 \\ z = cst \\ \Theta\frac{h^2}{2} = Cts, \end{cases}, \quad \begin{cases} u = 0 \\ h = cst \\ z + \frac{h}{2}\ln(\Theta) = Cts \end{cases}$$

- We want solve around equilibrium:

$$\begin{cases} u = O\left(F_r\right) \\ \Theta = cst + O\left(F_r\right) \\ h + z = cst + O\left(F_r\right) \end{cases}$$

Idea

Propose the same semi implicit relaxation scheme as before coupled with a procedure to plug the source into the fluxes. This procedure is call the Jin-Levermore method. It allows to obtain WB scheme.

Well balanced extension II

- Relaxation model:

$$\left\{\begin{array}{l}\partial_t h + \partial_x(hv) = 0,\\ \partial_t(hu) + \partial_x(huv + \Pi) = -gh\partial_x z,\\ \partial_t(h\Theta) + \partial_x(h\Theta v) = 0,\\ \partial_t\Pi + v\partial_x\Pi + \frac{ab}{h}\lambda^2\partial_x v = \frac{1}{\varepsilon}\left(\Pi - g\Theta\frac{1}{2}h^2\right)\\ \partial_t v + v\partial_x v + \frac{a}{bh}\partial_x\Pi = -\frac{a}{b}g\Theta\partial_x z + \frac{1}{\varepsilon}(v-u)\end{array}\right.$$

- Limit of the relaxation model:

$$\left\{\begin{array}{l}\partial_t h + \partial_x(hu) = \varepsilon\partial_x\left(\beta\left(\partial_x p + gh\partial_x z\right)\right)\\ \partial_t(hu) + \partial_x\left(hu^2 + \Theta\frac{g}{2}h^2\right) = -hg\Theta\partial_x z + \varepsilon\partial_x\left(u\beta\left(\partial_x p + gh\partial_x z\right)\right) + \varepsilon\partial_x\left(\gamma\partial_x u\right)\\ \partial_t(h\Theta) + \partial_x(h\Theta u) = \varepsilon\partial_x\left(\Theta\gamma\left(\partial_x p + gh\partial_x z\right)\right)\end{array}\right.$$

Results

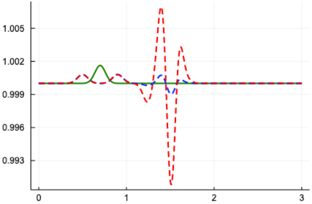
In the relaxation limite the model preserve the equilibriums of Ripa model.

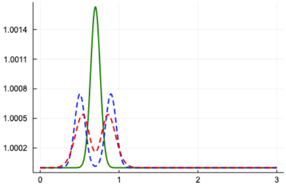
Results for Well balanced extension

- Test case: equilibrium preservation:

$\Delta t/$ Error	Tests	Rusanov Ex	Relaxation Exp	Relaxation SI
ST1	Error h	$1.5E^{-2}$	$1.5E^{-17}$	$3.6E^{-13}$
	Error u	$5.9E^{-3}$	$1.5E^{-15}$	$6.7E^{-13}$
	Error Θ	0.0	0.0	0.0
	Δt	$8.1E^{-4}$	$7.1E^{-4}$	$1.42E^{-1}$
ST2	Error h	$9.3E^{-2}$	0.0	$8.4E^{-12}$
	Error u	$7.3E^{-9}$	0.0	$1.3E^{-13}$
	Error Θ	0.13	$1.8E^{-17}$	$6.0E^{-12}$
	Δt	$2.5E^{-3}$	$2.3E^{-3}$	$4.7E^{-1}$
ST3	Error h	0.59	0.0	$1.38E^{-12}$
	Error u	0.65	$1.6E^{-15}$	$4.4E^{-14}$
	Error Θ	0.19	0.0	$1.4E^{-12}$
	Δt	$2.4E^{-3}$	$1.8E^{-3}$	0.49

- Test case: equilibrium perturbation:





Conclusion

Conclusion

Relaxation methods

The global idea of relaxation methods is to replace a PDE complex to discretize by a larger PDE but simpler to discretize.

Relaxation methods and Implicit schemes

With this idea we have develop different implicit schemes very simple, cheaper than classical approaches for hyperbolic PDEs

Low Mach regimes

In the low Mach regime we obtain a very simple scheme which is uniformly accurate.

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