Optimization Problems in Risk theory

Stefan Thonhauser

HEC, University of Lausanne

Strasbourg, February 21, 2012

based on joint works with H. Albrecher and V. Troncale
Outline

Popular risk models

Optimization problems - Control Possibilities
  Ruin probability
  Dividends

A Concrete Problem
  Optimal investment with transaction costs
  Statement of optimization problem
  Characterization of solution

Outlook
Classical risk process

Classical risk process $R = (R_t)_{t \geq 0}$ (collective model, Cramér-Lundberg)

$$R_t = x + ct - \sum_{i=0}^{N_t} Y_i, \quad t \geq 0$$

Components:

- $x \geq 0$ ... deterministic initial capital
- $c > 0$ ... constant premium intensity
- $\{Y_i\} \overset{iid}{\sim} F_Y$ ... claim amounts with $F_Y(0) = 0$
- $N = (N_t)_{t \geq 0}$ ... claim counting process, $N_t \sim \text{Poi}(\lambda t)$
- $\{T_i\}$ ... sequence of claim occurrence times

One usually assumes that $N$ and $\{Y_i\}$ are independent

As a consequence we have $T_i - T_{i-1} \overset{iid}{\sim} \text{Exp}(\lambda)$
Some associated quantities of interest

Time of ruin $\tau = \inf \{ t > 0 \mid R_t < 0 \}$
The probability of ruin $\psi$:

$$\psi(x) = P(\tau < \infty \mid R_0 = x)$$

Some properties of $\psi(x)$:

- $\psi$ fulfills the integro-differential equation
  $$0 = c\psi'(x) + \lambda \left( 1 - \psi(x) - F_Y(x) + \int_0^x \psi(x - y)dF_Y(y) \right)$$
  $$0 = \lim_{x \to \infty} \psi(x)$$

- If $c \leq \lambda \mathbb{E}(Y_1) \Rightarrow \psi(x) = 1$
- $c > \lambda \mathbb{E}(Y_1)$ is called net profit condition

- If $\mathbb{E}(e^{rY_i}) < \infty$ for some $r > 0 \Rightarrow \psi(x) \sim C e^{-Rx}$
- $R > 0$ solves $\lambda(\mathbb{E}(e^{rY_i}) - 1) = cr$, $R$... adjustment coefficient
Alternative models

- Diffusion risk model
  Classical risk reserve \((R_t)_{t \geq 0}\) approximated by diffusion process

\[
R_t = x + \mu t + \sigma W_t, \ t \geq 0 \quad (\mu, \sigma > 0)
\]

\((W_t)_{t \geq 0}\) is Brownian motion
→ lack of jumps sometimes leads to easier expressions

\[
\varphi(x) = 1 - \psi(x) = 1 - e^{-\frac{2\mu x}{\sigma^2}} \ldots \text{survival probability}
\]

- Piecewise-deterministic-Markov-processes (Davis (1993))
  → allows for incorporation of different economic environments
  → includes Sparre-Andersen risk model (renewal process)

- Spectrally negative Lévy processes
  (Lévy measure support in \((−\infty, 0]\))
Control possibilities

Once basic model is fixed there are no further interventions possible

→ need for extensions to control evolution of risk reserve

For example one can use:

- Introduce reinsurance opportunity
- Allow for investment in financial market
  ruin probability: exponential decay can turn into power decay!
  Kalashnikov & Norberg (2002)
  → control of investment crucial
- Consider shareholders via dividend payments
- Additional capital injections if reserve is negative
- Adapt premium rate (recently considered by Norberg (2012))
  idea: use CL-approximation for determining $c$
Minimization of ruin probability by reinsurance

Reinsurance for classical model
Let \( b : [0, \infty) \to [0, \infty) \) be a measurable function, \( 0 \leq b(u) \leq u \), the modified process is

\[
R^b_t = x + c(b, t) - \sum_{i=1}^{N_t} b(Y_i)
\]

Usual reinsurance schemes:
- \( b(x) = b \times \) for some \( 0 \leq b \leq 1 \), proportional reinsurance
- \( b(x) = \min\{x, M\} \) for some \( M > 0 \), XL reinsurance

Problem of minimizing ruin probability,

\[
\psi^*(x) = \inf_{b \in B} \psi(x, b)
\]

Optimal reinsurance example

Schmidli (2001) with $Y_i \sim \text{Exp}(1)$

Minimal ruin probability and optimal reinsurance strategy
Minimization of ruin probability by investment

Investment for classical model
Let $W = (W_t)_{t \geq 0}$ be BM independent to $N$ and $\{Y_i\}$
Asset price $S = (S_t)_{t \geq 0}$ modeled by geometric Brownian motion

$$dS_t = S_t(adt + bdW_t), \quad S_0 = s, \ a, \ b > 0$$

For some $\theta \geq 0$ the modified process behaves like

$$dT_t^\theta = dR_t + \theta dS_t, \quad T_0^\theta = x.$$ 

The related optimization problem is: $\psi^*(x) = \inf_{\theta \in \Theta} \psi(x, \theta)$


Small claims - $A(x)$ constant ($A(x)$ invested amount reserve level $x$)
Large claims - $A(x)/x$ constant
Optimal investment example

Hipp & Plum (2000) with $Y_i \sim Erl(3, 2)$

Minimal ruin probability and optimal investment strategy
Dividend payments

Observations: when taking care of ruin probability one has

either ruin occurs or \( R_t \to \infty \)

\( \rightarrow \) leads to unrealistic behaviour in practice

De Finetti (1957) proposed to measure performance of insurance portfolio by expected discounted dividend payments

\[
V_L(x) = \mathbb{E}_x \left( \int_0^{\tau_L} e^{-\delta t} dL_t \right)
\]

- \( L = (L_t)_{t \geq 0} \ldots \) evolution of accumulated dividends
  \( \delta > 0 \ldots \) discount/preference rate
- for comparing portfolios, maximal payments are important

Admissible dividend strategy - formal

Process $L = (L_t)_{t \geq 0}$ is admissible strategy, if
- predictable (càglàd $L_{t-} = L_t$), non-decreasing
- no payments after ruin, $L_t = L_{\tau^L}$ for $t > \tau^L$
- paying dividends can not cause ruin, $L_{t^+} - L_t \leq R_t^L$

$L_t$ denotes accumulated dividends up to time $t$

Controlled reserve process $R^L = (R_t^L)_{t \geq 0}$ given by

$$R_t^L = R_t - L_t$$

If $\mathcal{L}$ is some class of strategies, optimization problem is:

$$V(x) = \sup_{L \in \mathcal{L}} V_L(x)$$

$\tau^L = \inf\{ t > 0 \mid R_t^L < 0 \}$ time of ruin of $R^L$
Dividend strategies - examples

Barrier and threshold strategies

Band and impulse type strategy
Different ways to measure performance

Add *time value of ruin*

\[ V_L(x) = \mathbb{E}_x \left( \int_0^{\tau} e^{-\delta s} dL_s + \Lambda \int_0^{\tau} e^{-\delta s} ds \right) \]

T. & Albrecher (2007)

Transaction costs and utility \( u \) of dividends

\[ V_L(x) = \mathbb{E}_x \left( \sum_{i=1}^{\infty} e^{-\delta \theta_i} u(k L_i - K) \mathbb{I}_{\{\theta_i < \tau\}} \right) \]

T. & Albrecher (2011)

Measure utility \( U \) of dividends

\[ V_L(x) = \mathbb{E}_x \left( \int_0^{\tau} e^{-\delta s} U(I_s) ds \right), \quad V_L(x) = \mathbb{E}_x \left( U \left( \int_0^{\tau} e^{-\delta s} dL_s \right) \right) \]

Hubalek & Schachermayer (2004), Grandits et al. (2007)
Problem with dividends - remarks

- Insurance business includes safety aspect
  → optimal strategies lead to ruin with probability 1

- Add ruin probability constraint to dividend optimization
  → difficult, up to now hardly any results!
  Hipp (2007)

- Alternative, reinvestments to avoid ruin and maximize:
  \[
  V_{L,Z}(x) = \mathbb{E}_x \left( \int_0^\infty e^{-\delta t} \, dL_t - \theta \int_0^\infty e^{-\delta t} \, dZ_t \right)
  \]
  \[
  R_{t}^{L,Z} = R_t - L_t + Z_t
  \]
  Dickson & Waters (2004), Kulenko & Schmidli (2008)
  Eisenberg & Schmidli (2009, 2010)

- Difference to consumption problems in finance:
  dividends change underlying process
Risk model with investment

Goal: maximization of survival probability using investment

For simplicity we use a diffusion model:

\[ dR_t = \mu \, dt + \sigma \, dW^{(1)}_t, \quad \text{for} \ t \geq 0 \ldots \text{surplus} \]

\[ dS_t = S_t (a \, dt + b \, dW^{(2)}_t), \quad \text{for} \ t \geq 0 \ldots \text{asset price} \]

- Insurer invests amount \( A_0 = A \geq 0 \), and leaves it unchanged
- Surplus \( T^A = (T^A_t)_{t \geq 0} \) follows:

\[ dT^A_t = dR_t + dA_t, \quad \text{for} \ t \geq 0, \quad T^A_0 = x > 0 \]

\[ dA_t = A_t (a \, dt + b \, dW^{(2)}_t), \quad \text{for} \ t \geq 0 \]

Associated time of ruin and survival probability:

\[ \tau^A = \inf \{ t \geq 0 \mid T^A_t \leq 0 \} \quad \varphi^0(x, A) = P_{x,A}(\tau^A = \infty) \]

\[ W^{(1)} = (W^{(1)}_t)_{t \geq 0}, \ W^{(2)} = (W^{(2)}_t)_{t \geq 0} \quad \ldots \text{independent BMs,} \ (\mu, \sigma, a, b > 0) \]
Investment can be dangerous

Survival probability as function of initial surplus and initial investment

\[ \varphi^0(x, A) \text{ for } \mu = 0.3, \sigma = 0.8, a = 0.2, b = 0.4(0.6) \; A^* = 1(0.507) \]

Usage of investment needs to be controlled!
Review of classical optimal solution

Suppose continuous adoptions of investment position are feasible

→ admissible investment policy $A = (A_t)_{t \geq 0}$ is adapted, càdlàg process

Using policy $(A_t)$ surplus follows (1-dimensional problem):

$$dT_t^A = (\mu + aA_t)dt + \sigma dW_t^{(1)} + bA_t dW_t^{(2)}, \quad T_0^A = x > 0$$

Using dynamic programming approach one gets:

$$A^* = \frac{\mu \sigma^2}{b^2 (\mu + \sqrt{\mu^2 + \sigma^2 (a/b)^2})} > 0 \quad \text{is optimal policy}$$

$A^*$ constant, implies continuously buying and selling fractions of asset

see Browne (1995), Schmidli (2008) for complete solution
Introducing transaction costs I

Now insurer can modify investment position, but changes are subject to transaction costs:

\[
\text{a change } A \rightarrow A + \Delta A \text{ leads to costs } K + k|\Delta A|
\]

\(K > 0\) ... fixed cost, \(k > 0\) ... proportional cost factor

Observations:

- continuous adaptations lead to unbounded transaction costs
- admissible investment policies are impulse controls

\[
\pi = \{(\theta_n, A_n)\}_{n \in \mathbb{N}}
\]

\(\theta_i\) ... intervention time, \(A_i\) ... new investment position
- one needs to know actual investment \(\rightarrow\) bivariate modeling needed
Introducing transaction costs II

Controlled surplus process \( T^{\pi} = (T^{\pi}_t)_{t \geq 0} \) is:

\[
T^{\pi}_t = x + \int_0^t dR_s + \sum_{n=1}^{\infty} \int_{\theta_{n-1} \wedge t}^{\theta_n \wedge t} A^{\pi}_s (a \, ds + b \, dW^{(2)}_s) - \sum_{n=1}^{\infty} (K + |\Delta A^{\pi}_{\theta_n}|) I_{\{\theta_n \leq t\}}
\]

\[
dA^{\pi}_s = A^{\pi}_s (a \, ds + b \, dW^{(2)}_s), \text{ for } \theta_{n-1} \leq s < \theta_n
\]

\[
A^{\pi}_{\theta_n} = A^{\pi}_{\theta_{n-1}} + \Delta A^{\pi}_{\theta_n} = A_n
\]

\[
A^{\pi}_{0} = A
\]

Value function is maximal survival probability:

\[
\varphi(x, A) = \sup_{\pi \in \Pi} \varphi^{\pi}(x, A)
\]

Two types of characterization:

- \( \varphi \) is fixed point of an operator
- \( \varphi \) is solution to quasi-variational-inequalities
- finally both are useful
Characterization through iterated optimal stopping

If immediate intervention is optimal it should be equal to:

$$\sup_{\{\Delta A \in D(x,A)\}} \{ f(x - K - k|\Delta A|, A + \Delta A) \} =: Mf(x, A)$$

Define *implicit* optimal stopping operator and sequence \( \{\varphi_n\}_{n \in \mathbb{N}} \)

\[ Gf(x, A) := \max\{\varphi^0(x, A), \sup_{\theta < \tau} \mathbb{E}_{x,A}(Mf(T^A_{\theta}, A_{\theta}))\} \]

\[ \varphi_n(x, A) := G\varphi_{n-1}(x, A) \]

One obtains (\( \prod_n \) policies with at most \( n \) interventions):

\[ \varphi_n(x, A) = \sup_{\pi \in \prod_n} \varphi^\pi(x, A) \quad \text{and} \quad \lim_{n \to \infty} \varphi_n = \varphi \]

Fixed point property is equal to *dynamic programming principle*

\[ \varphi(x, A) = G\varphi(x, A) \]

Crucial: approximation procedure through finitely many interventions

How to determine \( \varphi_n \)?

\[ \rightarrow \text{QVI characterization} \]
Characterization through QVI

Heuristically $\varphi$ is associated to

$$\max\{L\varphi(x, A), M\varphi(x, A) - \varphi(x, A)\} = 0.$$ 

Motivation:

- around $(x, A)$ not optimal to intervene
  $\rightarrow L\varphi(x, A) = 0$
  $L$ generator of $T^A$
- intervention optimal in $(x, A)$
  $\rightarrow \varphi(x, A) = M\varphi(x, A)$
- $\varphi$ is viscosity solution to QVI
- every solution dominates $\varphi$
  (verification)
Numerical examples

$\varphi(x, A)$ without and 5 interventions

Optimal policy 5 iterations and stylized behaviour

Calculations based on suitable approximation and policy improvement procedure
Outlook

Various directions for future research

- Inclusion of model uncertainties, changes in economic environment → Hidden Markov models, stochastic filtering, time dependent parameters

- Up to now, investment problem/dividend problem, assumption of constant interest rate → inclusion of stochastic interest rate

- Up to now hardly any results for general PDMPs in risk theory or for problems on a finite time horizon

...
Some references

- S. Asmussen & H. Albrecher. *Ruin probabilities*  

  Optimality results for dividend problems in insurance  
  *RACSAM*, 103(2), 2009.

- C. Hipp & M. Plum.  
  Optimal investment for insurers  

- H. Schmidli. *Stochastic Control in Insurance*  

- S. Thonhauser & V. Troncale.  
  Optimal investment under transaction costs for an insurer  
  *Preprint*, 2012.