Lamperti invariance principle for strictly stationary sequences

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The invariance principle

Let $(X_j)_{j\geqslant 0}$ be a strictly stationary sequence.

We define S_n^{pl} as a random function, whose graph is piecewise linear and $S_n^{\mathrm{pl}}(k/n) = S_k := \sum_{i=1}^{k} X_i$.

If $(X_j)_{j\geqslant 0}$ is i.i.d., $\mathbb{E}X_0^2 = 1$, and $\mathbb{E}X_0 = 0$, then

$$\text{(IP)} \qquad \frac{1}{\sqrt{n}} S_n^{\mathrm{pl}} \to W \text{ in distribution in } C[0,1],$$

where W is a standard Brownian motion (Donsker, 1952).

When the convergence (IP) takes places for a strictly stationary sequence $(X_j)_{j\geqslant 0}$ in a function space E, we say that $(X_j)_{j\geqslant 0}$ satisfies the invariance principle in E.

Two possible generalisations:

- weakly dependent random variables instead of independent ones;
- consider other functional spaces than $C[0,1]$.

Holderian framework

For $\alpha \in (0,1)$ the Hölder space $H_\alpha[0,1]$ is the space of functions $x\colon [0,1] \to \mathbb{R}$ such that $\sup_{s \neq t \leqslant 1} |x(s) - x(t)| / |s-t|^\alpha$ is finite. The analoguous of the continuity modulus in $C[0,1]$ is w_α, defined by

$$w_\alpha(f,\delta) = \sup_{0<|t-s|<\delta} \frac{|f(t)-f(s)|}{|t-s|^\alpha}.$$

We then define $H_\alpha^0[0,1]$ by $H_\alpha^0[0,1] := \{h \in H_\alpha[0,1], \lim_{\delta\to 0} w_\alpha(h,\delta) = 0\}$ and $\|h\|_\alpha := |h(0)| + w_\alpha(h,1)$.

Denote by D_j the set of dyadic numbers in $[0,1]$ of level j:

$$D_0 := \{0,1\}, \quad D_j := \left\{(2l-1)2^{-j}; 1 \leqslant l \leqslant 2^{j-1}\right\}, j \geqslant 1.$$

If $r \in D_j$ for some $j \geqslant 0$, we define $r^+ := r + 2^{-j}$ and $r^- := r - 2^{-j}$. For $r \in D_j$, $j \geqslant 1$, let Λ_r be the function whose graph is the polygonal path joining the points $(0,0)$, $(r^-,0)$, $(r,1)$, $(r^+,0)$ and $(1,0)$.

Sequential norm and tightness criterion

If $x \in C[0,1]$, then

$$x = \sum_{r \in D} \lambda_r(x) \Lambda_r,$$

with uniform convergence on $[0,1]$.

In order to prove an invariance principle in $H_\alpha^0[0,1]$, we need to handle quantities like $\mu \left\{ n^{-1/p} \max_{1 \leqslant i < j \leqslant n} |S_j - S_i| / (j-i)^\alpha > \varepsilon \right\}$, which is not an easy task.

The sequential norm is defined by

$$\|x\|_\alpha^{\text{seq}} := \sup_{j \geqslant 0} 2^{j\alpha} \max_{r \in D_j} |\lambda_r(x)| .$$

It is equivalent to $\|\cdot\|_\alpha$.

Proposition

A sequence $(\xi_n)_{n \geqslant 1}$ of random elements of H_α^0 for which $\xi_n(0) = 0$ is tight if and only if for each positive ε,

$$\lim_{J \to \infty} \limsup_{n \to \infty} \mu \left\{ \sup_{j \geqslant J} 2^{j\alpha} \max_{r \in D_j} |\lambda_r(\xi_n)| > \varepsilon \right\} = 0.$$

Outline

Using the continuity mapping theorem, we can see that if $(X_j)_j$ satisfies the invariance principle in H_α, then

$$\mu \left\{ \max_{1 \leqslant j \leqslant n} |X_j| > n^{1/2-\alpha} \right\} \to 0.$$

In the independent case, this is equivalent to

$$n \cdot \mu \left\{ |X_0| > n^{1/p} \right\} \to 0, \quad \frac{1}{2} - \frac{1}{p} = \alpha.$$

We denote by $\mathbb{L}_0^{p,\infty}$ the space of function $f \colon \Omega \to \mathbb{R}$ such that $t^p \mu \{|f| > t\} \to 0$.

Račkauskas and Suquet showed that for an i.i.d. zero mean sequence $(X_j)_{j \geqslant 0}$, a necessary and sufficient condition to obtain the invariance principle in $H^0_{1/2-1/p}[0,1]$ is $X_0 \in \mathbb{L}_0^{p,\infty}$.

We want to investigate the invariance principle in $H^0_{1/2-1/p}[0,1]$ for weakly dependent stationary sequences with $X_0 \in \mathbb{L}_0^{p,\infty}$.

We use the strategy "convergence of the finite dimensional distributions and tightness".

Method of proof (1)

We have to prove that for each positive ε,

$$\lim_{J\to\infty} \limsup_{n\to\infty} \mu \left\{ \sup_{j\geqslant J} 2^{j\alpha} \max_{r\in D_j} \left| \lambda_r(S_n^{\mathrm{pl}}) \right| > n^{1/2}\varepsilon \right\} = 0.$$

1. We can prove that

$$\mu \left\{ \sup_{\log n\leqslant j} 2^{j\alpha} \max_{r\in D_j} \left| \lambda_r(S_n^{\mathrm{pl}}) \right| > n^{1/2}\varepsilon \right\} \leqslant C(\alpha,\varepsilon) n \cdot \mu \left\{ |X_0| > n^{1/p}\varepsilon \right\}.$$

2. Define $X_i' := X_i \chi\left(\left\{|X_i| > n^{1/p}\delta\right\}\right) - \mathbb{E}\left[X_i \chi\left(\left\{|X_i| > n^{1/p}\delta\right\}\right)\right]$ for a fixed n and δ, and $\widetilde{S}_N^{\mathrm{pl}}$ the partial sum process associated to these X_j'. We can show that

$$\mu \left\{ \sup_{J\leqslant j\leqslant \log n} 2^{j\alpha} \max_{r\in D_j} \left| \lambda_r(\widetilde{S}_n^{\mathrm{pl}}) \right| > n^{1/2}\varepsilon \right\} \leqslant C(\varepsilon,\alpha,\delta) n \mu \left\{ |X_0| > n^{1/p}\varepsilon \right\}.$$

Method of proof (2)

3. It remains to control

$$P(J,n,\delta) := \sum_{j=J}^{\log n} 2^j \mu \left\{ \left| \sum_{i=1}^{2n2^{-j}} (X_i - X_i') \right| > \varepsilon n^{1/2} 2^{-\alpha j} \right\}.$$

4. By Markov's inequality, for $\eta > 0$,

$$P(J,n,\delta) \leqslant \varepsilon^{-p-\eta} n^{-p/2-\eta/2} \sum_{j=J}^{\log n} 2^{pj/2} 2^{j\alpha\eta} \mathbb{E} \left| \sum_{i=1}^{2n2^{-j}} (X_i - X_i') \right|^{p+\eta}.$$

5. Then we use Rosenthal's inequality

$$\mathbb{E} \left| \sum_{j=1}^{N} \xi_j \right|^q \leqslant C(q) \left(\sum_{j=1}^{N} \mathbb{E} |\xi_j|^q + \left(\sum_{j=1}^{N} \mathbb{E} \left[\xi_j^2 \right] \right)^{q/2} \right), (\xi_j) \text{ independent and centered}$$

and the bound

$$\mathbb{E} \left| X_i - X_i' \right|^{p+\eta} \leqslant C(p) n^{\eta/p} \delta^\eta \cdot \sup_t t^p \mu \left\{ |X_0| > t \right\}.$$

Mixing conditions (1)

Let $\mathcal{A}$ and $\mathcal{B}$ be two sub-σ-algebras of $\mathcal{F}$. We define

- the α-mixing coefficients

$$\alpha(\mathcal{A},\mathcal{B}) := \sup\left\{\left|\mu(A\cap B)-\mu(A)\mu(B)\right|, A\in\mathcal{A}, B\in\mathcal{B}\right\},$$

- the ρ-mixing coefficients

$$\rho(\mathcal{A},\mathcal{B}) := \sup\left\{\operatorname{Corr}(f,g), f\in\mathbb{L}^2(\mathcal{A}), g\in\mathbb{L}^2(\mathcal{B})\right\}.$$

The coefficients are related by the inequalities

$$\alpha(\mathcal{A},\mathcal{B}) \leqslant \rho(\mathcal{A},\mathcal{B}).$$

Mixing conditions (2)

For a sequence $X = (X_k, k \in \mathbb{Z})$ and $n \geqslant 0$ we define

$$\alpha_X(n) = \alpha(n) = \sup_{m \in \mathbb{Z}} \alpha(\mathcal{F}_{-\infty}^{m}, \mathcal{F}_{n+m}^{\infty})$$

where $\mathcal{F}_u^v$ is the σ-algebra generated by X_k with $u \leqslant k \leqslant v$ (if $u = -\infty$ or $v = \infty$, the corresponding inequality is strict). In the same way we define coefficients $\rho_X(n)$. Denote $\mathcal{F}_I := \sigma(X_i, i \in I)$. Then we define

$$\rho^*(n) := \sup \left\{ \rho(\mathcal{F}_I, \mathcal{F}_J), \inf |i - j| \geqslant n \right\}.$$

If $(X_j)_{j \geqslant 0}$ is a one-sided sequence, then $\alpha_X(n) := \alpha_{X'}(n)$ where $X' = (\dots, 0, X_0, X_1, \dots)$.

We have the inequalities

$$\alpha_X(n) \leqslant \rho_X(n) \leqslant \rho_X^*(n).$$

X is α-mixing $\Leftrightarrow \alpha_X(n) \to 0$.

Doukhan, Massart and Rio's result

Let $X = (X_j)_{j \geqslant 0}$ be a strictly stationary sequence.

- $\alpha^{-1}(u) := \inf \{k, \alpha_X(k) \leqslant u\}$;
- Q is the right-continuous inverse of the quantile function $t \mapsto \mu \{X_0 > t\}$, i.e. $Q_X(u) = \inf \{t \geqslant 0, \mu \{|X| > t\} \leqslant u\}$.

Theorem (Doukhan et al. (1995))

Let $(X_j)_{j \geqslant 1}$ be a strictly stationary zero-mean sequence such that

$$\int_0^1 \alpha^{-1}(u) Q_{X_0}^2(u) du < \infty.$$

Then $\lim_{n \to +\infty} \sigma_n^2 / n =: \sigma$ exists and

$$\frac{1}{\sqrt{n}} S_n^{\mathrm{pl}}(f) \to \sigma W \text{ in distribution in } D[0,1].$$

A Doukhan et al. like condition

Theorem

Let $p > 2$ and let $(X_j)_{j\geqslant 0}$ be a zero-mean strictly stationary sequence. Assume that one of the following two conditions holds

(C) *for some positive η,*

$$\int_0^1 [\alpha^{-1}(u)]^{p+\eta-1} Q_{X_0}(u)^p \mathrm{d}u < +\infty.$$

(C') $\lim_{t\to+\infty} t^p \lambda \left\{ u \mid \alpha^{-1}(u) Q(u) > t \right\} = 0.$

Then $(X_j)_{j\geqslant 0}$ satisfies the weak invariance principle in $H^0_{1/2-1/p}[0,1]$, .i.e.,

$$\frac{1}{\sqrt{n}} S_n^{\mathrm{pl}} \to \sigma W,$$

where $\sigma^2 = \lim_{n\to\infty} \sigma_n^2/n$.

Comparison of (C) and (C')

1. If $(X_j)_{j\geqslant 0}$ is an m-dependent sequence (identically distributed), we have

$$(C) \Leftrightarrow X_0 \in \mathbb{L}^p \text{ and } (C') \Leftrightarrow X_0 \in \mathbb{L}_0^{p,\infty}.$$

2. If $\alpha^{-1}(u) \overset{u\to 0}{\sim} \log u$ and $Q(u)\alpha^{-1}(u) \geqslant u^{-1/p}$, then condition (C) holds (with $\eta = 1/2$) while condition (C') is not satisfied.

Conditions (C) and (C') are thus independent.

Some comments

1. If $(X_j)_{j\geqslant 0}$ is an m-dependent sequence (identically distributed), Račkauskas and Suquet's result can be deduced from condition (C').
2. Hamadouche (2000) required moments of order $> p$ to deduce the invariance principle in $H_\gamma[0,1]$ for each $\gamma < 1/2 - 1/p$. Neither condition (C) nor condition (C') need such a moment condition.
3. Let $(X_j)_{j\geqslant 1}$ be a strictly stationary zero-mean sequence such that $t^p\mu\left\{|X_0| > t\right\} \leqslant \varepsilon(t)$ with $\varepsilon(t) \to 0$ as $t \to +\infty$. Defining $\delta(u) := \inf\left\{s > 0 \mid \varepsilon(su^{-1/p}) \leqslant s^p\right\}$, we obtain $Q(u) \leqslant u^{-1/p}\delta(u)$, hence

$$\int_0^1 \frac{\alpha^{-1}(u)\delta(u)^p}{u}\mathrm{d}u < \infty \Rightarrow (C').$$

Main tool: a Fuk-Nagaev type inequality

- $(X_i)_{i\geqslant 1}$ sequence of random variables;
- $Q := \sup_{i\geqslant 1} Q_{X_i}$; $R(u) = \alpha^{-1}(u)Q(u)$ and $H(u) = R^{-1}(u)$;
- $s_N^2 := \sum_{i,j=1}^N |\mathrm{Cov}(X_i, X_j)|$.

Theorem (Rio (1995))

Let $(X_i)_{i\geqslant 1}$ be a sequence of real-valued and centered random variables with finite variance. Then, for any positive λ, any integer $N \geqslant 1$ and any $r \geqslant 1$,

$$\mu\left\{\max_{1\leqslant k\leqslant N} |S_k| \geqslant 4\lambda\right\} \leqslant 4\left(1 + \frac{\lambda^2}{r s_N^2}\right)^{-r/2} + 4N\lambda^{-1}\int_0^{H(\lambda/r)} Q(u)\mathrm{d}u.$$

Proposition

Let $p \geqslant 2$ and $X = (X_j)_{j\in\mathbb{N}}$ a sequence of centered random variables. For each integer $N \geqslant 1$, the following inequality holds:

$$\mathbb{E}\left[\max_{1\leqslant k\leqslant N} |S_k|^p\right] \leqslant a_p s_N^p + N b_p \int_0^1 \left[\alpha_X^{-1}(u)\right]^{p-1} Q^p(u)\mathrm{d}u,$$

where a_p and b_p depend only on p and $Q(u) = \max_{1\leqslant j\leqslant N} Q_{X_j}(u)$.

Moment inequality for ρ-mixing sequences

Proposition (Shao, 1995)

Let $(X_j)_{j\geqslant 0}$ be a strictly stationary sequence of centered random variables and $q \geqslant 2$. Then there exists a constant K depending only on q and the sequence $(\rho_X(n))_{n\geqslant 1}$ such that for each integer n,

$$\mathbb{E}\,|S_n|^q \leqslant K \cdot n^{q/2} \exp\left(K \sum_{i=0}^{[\log n]} \rho(2^i)\right) \|X_0\|_2^q + K \cdot n \exp\left(K \sum_{i=0}^{[\log n]} \rho^{2/q}(2^i)\right) \|X_0\|_q^q .$$

Proposition (Peligrad, Gut (1999))

Let $X := (X_j)_{j\geqslant 1}$ be a sequence of centered random variables satisfying $\mathbb{E}\,|X_i|^q < \infty$ for all $i \geqslant 1$ and some $q \geqslant 2$. Suppose that $\rho_X^*(N) < 1$ for some $N \geqslant 1$. Then there exists a positive constant $C = C(N, \rho_X^*(N), q)$ for each integer $n \geqslant 1$,

$$\mathbb{E}\,|S_n|^q \leqslant C \left(\sum_{i=1}^{n} \mathbb{E}\,|X_i|^q + \left(\sum_{i=1}^{n} \mathbb{E} X_i^2\right)^{q/2}\right) .$$

The case of ρ-mixing sequences

Theorem

Let $p > 2$ and let $(X_j)_{j \geqslant 0}$ be a zero-mean strictly stationary sequence with $X_0 \in \mathbb{L}_0^{p,\infty}$. Suppose that one of the following two conditions is satisfied:

(C1) *the series $\sum_{i=1}^{+\infty} \rho(2^i)$ is convergent;*

(C2) $\lim_{n\to\infty} \rho_X^*(n) = 0$.

Then

$$\frac{1}{\sqrt{n}} S_n^{\mathrm{pl}} \to \sigma W \text{ in distribution in } H^0_{1/2-1/p}[0,1],$$

with $\sigma^2 = \lim_{n\to\infty} \sigma_n^2 / n$.

If we merely assume $X_0 \in \mathbb{L}_0^{p,\infty}$, $\rho_X(n) \to 0$ and $\sigma_n \to \infty$, then for each $\gamma < \frac{1}{2} - \frac{1}{p}$,

$$\frac{1}{\sigma_n} S_n^{\mathrm{pl}} \to W \text{ in distribution in } H^0_\gamma[0,1].$$

Martingale differences

We say that $(X_j)_{j \geqslant 0}$ is a martingale difference sequence if there exists a a filtration $(\mathcal{F}_j)_{j \geqslant 0}$ such that X_j is $\mathcal{F}_j$ measurable and $\mathbb{E}[X_j \mid \mathcal{F}_{j-1}] = 0$.

Theorem

Let $\alpha \in (0, 1/2)$. There exists a strictly stationary ergodic martingale difference sequence $(X_j)_{j \geqslant 0}$ such that

- $\lim_{t \to \infty} t^{\frac{1}{1/2-\alpha}} \mu \{|X_0| > t\} = 0$;
- *the sequence $(X_j)_{j \geqslant 0}$ does not satisfy the invariance principle in $H_\alpha^0[0, 1]$.*

Theorem

Let $\alpha \in (0, 1/2)$ and $(X_j)_{j \geqslant 0}$ be a strictly stationary ergodic martingale difference sequence such that

$$\sum_{j=1}^{+\infty} \sup_{k \geqslant j} 2^{\frac{k}{1/2-\alpha}} \mu \left\{ |X_0| > 2^k \right\} < +\infty.$$

Then the sequence $(X_j)_{j \geqslant 0}$ satisfies the invariance principle in $H_\alpha^0[0, 1]$.