Hölderian invariance principle for stationary sequences

Davide Giraudo

Laboratoire de Mathématiques Raphaël Salem

November, 20th 2015

Context

- Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $T\colon \Omega \to \Omega$ be a bijective bi-measurable *measure-preserving* function.

- The sequence $\left(f \circ T^j\right)_{j \geqslant 0}$ is a *strictly stationary sequence*, that is, the sequences $\left(f \circ T^j\right)_{j \geqslant 0}$ and $\left(f \circ T^{j+1}\right)_{j \geqslant 0}$ have the same distribution. In this talk, we shall only deal with the case of *ergodic* dynamical systems, that is, if $A \in \mathcal{F}$ is such that $T^{-1}A = A$, then $\mu(A) = 0$ or 1.

- We define $S_N(f) := \sum_{j=0}^{N-1} f \circ T^j$. In probability theory, an important problem is the understanding of the asymptotic behaviour of the sequence $(S_N(f))_{N \geqslant 1}$.

- To this aim, we consider the *polygonal line process* $t \mapsto S_n^{\mathrm{pl}}(f, t)$, that is, a piecewise linear function on $[0, 1]$ whose value at k/n is $S_k(f)$, $1 \leqslant k \leqslant n$.

- We study the asymptotic behaviour of the normalised partial sum process in some function spaces.

Invariance principle on the space of continuous functions

Let $C[0,1]$ denote the *space of continuous functions* on the unit interval endowed with the uniform norm. The random function $t \mapsto S_n^{\mathrm{pl}}(f,t)$ belongs to this space.

Donsker (1952) showed that if $\left(f \circ T^j\right)_{j \geqslant 0}$ is independent, centered and $\mathbb{E}\left[f^2\right] = \sigma^2$, then for each $F \colon C[0,1] \to \mathbb{R}$ continuous and bounded,

$$\lim_{n \to +\infty} \mathbb{E}\left[F\left(n^{-1/2} S_n^{\mathrm{pl}}(f,\cdot)\right)\right] = \mathbb{E}\left[F(\sigma W)\right],$$

where W a standard Brownian motion. When this convergence holds, we say that f satisfies the invariance principle in $C[0,1]$.

Invariance principle on the space of continuous functions

Let $C[0,1]$ denote the *space of continuous functions* on the unit interval endowed with the uniform norm. The random function $t \mapsto S_n^{\mathrm{pl}}(f,t)$ belongs to this space.

Donsker (1952) showed that if $\left(f \circ T^j\right)_{j \geqslant 0}$ is independent, centered and $\mathbb{E}\left[f^2\right] = \sigma^2$, then for each $F \colon C[0,1] \to \mathbb{R}$ continuous and bounded,

$$\lim_{n \to +\infty} \mathbb{E}\left[F\left(n^{-1/2} S_n^{\mathrm{pl}}(f,\cdot)\right)\right] = \mathbb{E}\left[F(\sigma W)\right],$$

where W a standard Brownian motion. When this convergence holds, we say that f satisfies the invariance principle in $C[0,1]$.

Let $\mathcal{M}$ be a sub-σ-algebra of $\mathcal{F}$ such that $T\mathcal{M} \subset \mathcal{M}$ (in this way, $(T^{-i}\mathcal{M})_{i \geqslant 0}$ is a filtration). We say that $\left(m \circ T^j\right)_{j \geqslant 0}$ is a *martingale differences sequence* if the function m is $\mathcal{M}$-measurable, integrable and $\mathbb{E}\left[m \mid T\mathcal{M}\right] = 0$.

If $\left(m \circ T^j\right)_{j \geqslant 0}$ is a martingale differences sequence such that $\mathbb{E}\left[m^2\right] = \sigma^2$, then m satisfies the invariance principle in $C[0,1]$ (see Billingsley (1968)).

The Hölder spaces

Space	Definition	Separable
$\mathcal{H}_\alpha[0,1]$	$\left\|x\right\|_\alpha := \sup_{s\neq t}\frac{\lvert x(s)-x(t)\rvert}{\lvert s-t\rvert^\alpha} + \lvert x(0)\rvert < +\infty$	No

Space	Definition	Separable
$\mathcal{H}_\alpha[0,1]$	$\left\lVert x\right\rVert_\alpha := \sup_{s\neq t}\frac{\lvert x(s)-x(t)\rvert}{\lvert s-t\rvert^\alpha}+\lvert x(0)\rvert<+\infty$	No
$\mathcal{H}^o_\alpha[0,1]$		

The Hölder spaces

Space	Definition	Separable
$\mathcal{H}_\alpha[0,1]$	$\left\lVert x\right\rVert_\alpha := \sup_{s\neq t}\frac{\lvert x(s)-x(t)\rvert}{\lvert s-t\rvert^\alpha} + \lvert x(0)\rvert < +\infty$	No
$\mathcal{H}_\alpha^o[0,1]$	$\lim_{\delta\to 0}\sup\left\{\frac{\lvert x(t)-x(s)\rvert}{\lvert t-s\rvert^\alpha} : \lvert t-s\rvert<\delta, s,t\in[0,1]\right\}=0$	

The Hölder spaces

Space	Definition	Separable
$\mathcal{H}_\alpha[0,1]$	$\left\Vert x\right\Vert_\alpha := \sup_{s\neq t} \frac{\lvert x(s)-x(t)\rvert}{\lvert s-t\rvert^\alpha} + \lvert x(0)\rvert < +\infty$	No
$\mathcal{H}^o_\alpha[0,1]$	$\lim_{\delta\to 0} \sup \left\{ \frac{\lvert x(t)-x(s)\rvert}{\lvert t-s\rvert^\alpha} : \lvert t-s\rvert < \delta, s,t\in[0,1]\right\} = 0$	Yes

The Hölder spaces

Space	Definition	Separable
$\mathcal{H}_\alpha[0,1]$	$\|x\|_\alpha := \sup_{s\neq t} \frac{\lvert x(s)-x(t)\rvert}{\lvert s-t\rvert^\alpha} + \lvert x(0)\rvert < +\infty$	No
$\mathcal{H}^o_\alpha[0,1]$	$\lim_{\delta\to 0} \sup \left\{ \frac{\lvert x(t)-x(s)\rvert}{\lvert t-s\rvert^\alpha} : \lvert t-s\rvert < \delta, s,t \in [0,1] \right\} = 0$	Yes

The convergence in distribution in $\mathcal{H}_\alpha[0,1]$ is implied by that on $\mathcal{H}^o_\alpha[0,1]$. We shall work on $\mathcal{H}^o_\alpha[0,1]$.

The paths of a standard Brownian motion belong to $\mathcal{H}_\alpha[0,1]$ for each $\alpha \in (0,1/2)$.

Since the embedding $\iota\colon \left(\mathcal{H}^o_\alpha[0,1], \|\cdot\|_\alpha\right) \to \left(C[0,1], \|\cdot\|_\infty\right)$ is continuous, any continuous functional on $C[0,1]$ is continuous on $\mathcal{H}^o_\alpha[0,1]$. Therefore, the invariance principle on $\mathcal{H}^o_\alpha[0,1]$ for $0<\alpha<1/2$ is stronger than the corresponding result on $C[0,1]$. It has potential statistical applications like change point detection.

The first work was done by Lamperti, (1962): his goal was to prove the convergence $\mathbb{E}\left[F\left(S_n^{\mathrm{pl}}(f,\cdot)/\sqrt{n}\right)\right] \to \mathbb{E}\left[F(W)\right]$ for the largest possible class of functionals F.

The i.i.d. and martingale case

A (centered) function $f\colon \Omega \to \mathbb{R}$ satisfies the Hölderian invariance principle (HIP) in $\mathcal{H}_\alpha[0,1]$ if the convergence

$$\lim_{n\to\infty} \mathbb{E}\left[F\left(S_n^{\mathrm{pl}}(f,\cdot)/\sqrt{n}\right)\right] = \mathbb{E}\left[F(W)\right]$$

holds for each continuous and bounded function $F\colon \mathcal{H}_\alpha^o[0,1] \to \mathbb{R}$, where W is a scalar multiple of a standard Brownian motion. Let $p := (1/2-\alpha)^{-1}$ where $0 < \alpha < 1/2$.

Dependence of $(f \circ T^i)_{i\geqslant 0}$	**Integrability**	**Does f satisfy the HIP?**

The i.i.d. and martingale case

A (centered) function $f\colon \Omega \to \mathbb{R}$ satisfies the Hölderian invariance principle (HIP) in $\mathcal{H}_\alpha[0,1]$ if the convergence

$$\lim_{n\to\infty} \mathbb{E}\left[F\left(S_n^{\mathrm{pl}}(f,\cdot)/\sqrt{n}\right)\right] = \mathbb{E}\left[F(W)\right]$$

holds for each continuous and bounded function $F\colon \mathcal{H}_\alpha^o[0,1] \to \mathbb{R}$, where W is a scalar multiple of a standard Brownian motion. Let $p := (1/2-\alpha)^{-1}$ where $0<\alpha<1/2$.

Dependence of $(f\circ T^i)_{i\geqslant 0}$	**Integrability**	**Does f satisfy the HIP?**
Independent		

The i.i.d. and martingale case

A (centered) function $f\colon \Omega \to \mathbb{R}$ satisfies the Hölderian invariance principle (HIP) in $\mathcal{H}_\alpha[0,1]$ if the convergence

$$\lim_{n\to\infty} \mathbb{E}\left[F\left(S_n^{\mathrm{pl}}(f,\cdot)/\sqrt{n}\right)\right] = \mathbb{E}\left[F(W)\right]$$

holds for each continuous and bounded function $F\colon \mathcal{H}_\alpha^o[0,1] \to \mathbb{R}$, where W is a scalar multiple of a standard Brownian motion. Let $p := (1/2-\alpha)^{-1}$ where $0 < \alpha < 1/2$.

Dependence of $(f \circ T^i)_{i \geqslant 0}$	**Integrability**	**Does f satisfy the HIP?**
Independent	For each t, $0 < c_1 \leqslant t^p \mu\{\lvert f\rvert > t\} \leqslant c_2$	

The i.i.d. and martingale case

A (centered) function $f\colon \Omega \to \mathbb{R}$ satisfies the Hölderian invariance principle (HIP) in $\mathcal{H}_\alpha[0,1]$ if the convergence

$$\lim_{n\to\infty} \mathbb{E}\left[F\left(S_n^{\mathrm{pl}}(f,\cdot)/\sqrt{n}\right)\right] = \mathbb{E}\left[F(W)\right]$$

holds for each continuous and bounded function $F\colon \mathcal{H}_\alpha^o[0,1] \to \mathbb{R}$, where W is a scalar multiple of a standard Brownian motion. Let $p := (1/2-\alpha)^{-1}$ where $0 < \alpha < 1/2$.

Dependence of $(f\circ T^i)_{i\geqslant 0}$	**Integrability**	**Does f satisfy the HIP?**
Independent	For each t, $0 < c_1 \leqslant t^p \mu\left\{\lvert f\rvert > t\right\} \leqslant c_2$	No (Lamperti, 1962)

A (centered) function $f\colon \Omega \to \mathbb{R}$ satisfies the Hölderian invariance principle (HIP) in $\mathcal{H}_\alpha[0,1]$ if the convergence

$$\lim_{n\to\infty} \mathbb{E}\left[F\left(S_n^{\mathrm{pl}}(f,\cdot)/\sqrt{n}\right)\right] = \mathbb{E}\left[F(W)\right]$$

holds for each continuous and bounded function $F\colon \mathcal{H}_\alpha^o[0,1] \to \mathbb{R}$, where W is a scalar multiple of a standard Brownian motion. Let $p := (1/2-\alpha)^{-1}$ where $0 < \alpha < 1/2$.

Dependence of $(f\circ T^i)_{i\geqslant 0}$	Integrability	Does f satisfy the **HIP**?
Independent	For each t, $0 < c_1 \leqslant t^p \mu\{\lvert f\rvert > t\} \leqslant c_2$	No (Lamperti, 1962)
Independent		

The i.i.d. and martingale case

A (centered) function $f\colon \Omega \to \mathbb{R}$ satisfies the Hölderian invariance principle (HIP) in $\mathcal{H}_\alpha[0,1]$ if the convergence

$$\lim_{n\to\infty} \mathbb{E}\left[F\left(S_n^{\mathrm{pl}}(f,\cdot)/\sqrt{n}\right)\right] = \mathbb{E}\left[F(W)\right]$$

holds for each continuous and bounded function $F\colon \mathcal{H}_\alpha^o[0,1] \to \mathbb{R}$, where W is a scalar multiple of a standard Brownian motion. Let $p := (1/2-\alpha)^{-1}$ where $0 < \alpha < 1/2$.

Dependence of $(f\circ T^i)_{i\geqslant 0}$	**Integrability**	**Does f satisfy the HIP?**
Independent	For each t, $0 < c_1 \leqslant t^p \mu\{\lvert f\rvert > t\} \leqslant c_2$	No (Lamperti, 1962)
Independent	$t^p \mu\{\lvert f\rvert > t\} \to 0$	

The i.i.d. and martingale case

A (centered) function $f\colon \Omega \to \mathbb{R}$ satisfies the Hölderian invariance principle (HIP) in $\mathcal{H}_\alpha[0,1]$ if the convergence

$$\lim_{n\to\infty} \mathbb{E}\left[F\left(S_n^{\mathrm{pl}}(f,\cdot)/\sqrt{n}\right)\right] = \mathbb{E}\left[F(W)\right]$$

holds for each continuous and bounded function $F\colon \mathcal{H}_\alpha^o[0,1] \to \mathbb{R}$, where W is a scalar multiple of a standard Brownian motion. Let $p := (1/2-\alpha)^{-1}$ where $0 < \alpha < 1/2$.

Dependence of $(f \circ T^i)_{i \geqslant 0}$	**Integrability**	**Does f satisfy the HIP?**
Independent	For each t, $0 < c_1 \leqslant t^p \mu\{\lvert f\rvert > t\} \leqslant c_2$	No (Lamperti, 1962)
Independent	$t^p \mu\{\lvert f\rvert > t\} \to 0$	Yes (Račkauskas, Suquet, 2003)

The i.i.d. and martingale case

A (centered) function $f\colon \Omega \to \mathbb{R}$ satisfies the Hölderian invariance principle (HIP) in $\mathcal{H}_\alpha[0,1]$ if the convergence

$$\lim_{n\to\infty} \mathbb{E}\left[F\left(S_n^{\mathrm{pl}}(f,\cdot)/\sqrt{n}\right)\right] = \mathbb{E}\left[F(W)\right]$$

holds for each continuous and bounded function $F\colon \mathcal{H}_\alpha^o[0,1] \to \mathbb{R}$, where W is a scalar multiple of a standard Brownian motion. Let $p := (1/2-\alpha)^{-1}$ where $0 < \alpha < 1/2$.

Dependence of $(f\circ T^i)_{i\geqslant 0}$	**Integrability**	**Does f satisfy the HIP?**
Independent	For each t, $0 < c_1 \leqslant t^p \mu\{\lvert f\rvert > t\} \leqslant c_2$	No (Lamperti, 1962)
Independent	$t^p \mu\{\lvert f\rvert > t\} \to 0$	Yes (Račkauskas, Suquet, 2003)
Martingale differences		

A (centered) function $f\colon \Omega \to \mathbb{R}$ satisfies the Hölderian invariance principle (HIP) in $\mathcal{H}_\alpha[0,1]$ if the convergence

$$\lim_{n\to\infty} \mathbb{E}\left[F\left(S_n^{\mathrm{pl}}(f,\cdot)/\sqrt{n}\right)\right] = \mathbb{E}\left[F(W)\right]$$

holds for each continuous and bounded function $F\colon \mathcal{H}_\alpha^o[0,1] \to \mathbb{R}$, where W is a scalar multiple of a standard Brownian motion. Let $p := (1/2-\alpha)^{-1}$ where $0 < \alpha < 1/2$.

Dependence of $(f \circ T^i)_{i \geqslant 0}$	**Integrability**	**Does f satisfy the HIP?**
Independent	For each t, $0 < c_1 \leqslant t^p \mu\{\lvert f\rvert > t\} \leqslant c_2$	No (Lamperti, 1962)
Independent	$t^p \mu\{\lvert f\rvert > t\} \to 0$	Yes (Račkauskas, Suquet, 2003)
Martingale differences	$t^p \mu\{\lvert f\rvert > t\} \to 0$	

A (centered) function $f\colon \Omega \to \mathbb{R}$ satisfies the Hölderian invariance principle (HIP) in $\mathcal{H}_\alpha[0,1]$ if the convergence

$$\lim_{n\to\infty} \mathbb{E}\left[F\left(S_n^{\mathrm{pl}}(f,\cdot)/\sqrt{n}\right)\right] = \mathbb{E}\left[F(W)\right]$$

holds for each continuous and bounded function $F\colon \mathcal{H}_\alpha^o[0,1] \to \mathbb{R}$, where W is a scalar multiple of a standard Brownian motion. Let $p := (1/2-\alpha)^{-1}$ where $0 < \alpha < 1/2$.

Dependence of $(f\circ T^i)_{i\geqslant 0}$	**Integrability**	**Does f satisfy the HIP?**
Independent	For each t, $0 < c_1 \leqslant t^p\mu\{\lvert f\rvert > t\} \leqslant c_2$	No (Lamperti, 1962)
Independent	$t^p\mu\{\lvert f\rvert > t\} \to 0$	Yes (Račkauskas, Suquet, 2003)
Martingale differences	$t^p\mu\{\lvert f\rvert > t\} \to 0$	Not necessarily (G., 2015)

The i.i.d. and martingale case

A (centered) function $f\colon \Omega \to \mathbb{R}$ satisfies the Hölderian invariance principle (HIP) in $\mathcal{H}_\alpha[0,1]$ if the convergence

$$\lim_{n\to\infty} \mathbb{E}\left[F\left(S_n^{\mathrm{pl}}(f,\cdot)/\sqrt{n}\right)\right] = \mathbb{E}\left[F(W)\right]$$

holds for each continuous and bounded function $F\colon \mathcal{H}_\alpha^o[0,1] \to \mathbb{R}$, where W is a scalar multiple of a standard Brownian motion. Let $p := (1/2-\alpha)^{-1}$ where $0 < \alpha < 1/2$.

Dependence of $(f \circ T^i)_{i\geqslant 0}$	**Integrability**	**Does f satisfy the HIP?**
Independent	For each t, $0 < c_1 \leqslant t^p\mu\left\{\lvert f\rvert > t\right\} \leqslant c_2$	No (Lamperti, 1962)
Independent	$t^p\mu\left\{\lvert f\rvert > t\right\} \to 0$	Yes (Račkauskas, Suquet, 2003)
Martingale differences	$t^p\mu\left\{\lvert f\rvert > t\right\} \to 0$	Not necessarily (G., 2015)
Martingale differences		

The i.i.d. and martingale case

A (centered) function $f \colon \Omega \to \mathbb{R}$ satisfies the Hölderian invariance principle (HIP) in $\mathcal{H}_\alpha[0,1]$ if the convergence

$$\lim_{n\to\infty} \mathbb{E}\left[F\left(S_n^{\mathrm{pl}}(f,\cdot)/\sqrt{n}\right)\right] = \mathbb{E}\left[F(W)\right]$$

holds for each continuous and bounded function $F \colon \mathcal{H}_\alpha^o[0,1] \to \mathbb{R}$, where W is a scalar multiple of a standard Brownian motion. Let $p := (1/2-\alpha)^{-1}$ where $0 < \alpha < 1/2$.

Dependence of $(f \circ T^i)_{i \geqslant 0}$	**Integrability**	**Does f satisfy the HIP?**
Independent	For each t, $0 < c_1 \leqslant t^p \mu\{\lvert f\rvert > t\} \leqslant c_2$	No (Lamperti, 1962)
Independent	$t^p \mu\{\lvert f\rvert > t\} \to 0$	Yes (Račkauskas, Suquet, 2003)
Martingale differences	$t^p \mu\{\lvert f\rvert > t\} \to 0$	Not necessarily (G., 2015)
Martingale differences	$t^p \mu\{\lvert f\rvert > t\} \to 0$ and $\mathbb{E}\left[f^2 \mid T\mathcal{M}\right] \in \mathbb{L}^{p/2}$	

The i.i.d. and martingale case

A (centered) function $f\colon \Omega \to \mathbb{R}$ satisfies the Hölderian invariance principle (HIP) in $\mathcal{H}_\alpha[0,1]$ if the convergence

$$\lim_{n\to\infty} \mathbb{E}\left[F\left(S_n^{\mathrm{pl}}(f,\cdot)/\sqrt{n}\right)\right] = \mathbb{E}\left[F(W)\right]$$

holds for each continuous and bounded function $F\colon \mathcal{H}_\alpha^o[0,1] \to \mathbb{R}$, where W is a scalar multiple of a standard Brownian motion. Let $p := (1/2-\alpha)^{-1}$ where $0 < \alpha < 1/2$.

Dependence of $(f\circ T^i)_{i\geqslant 0}$	**Integrability**	**Does f satisfy the HIP?**
Independent	For each t, $0 < c_1 \leqslant t^p\mu\{\lvert f\rvert > t\} \leqslant c_2$	No (Lamperti, 1962)
Independent	$t^p\mu\{\lvert f\rvert > t\} \to 0$	Yes (Račkauskas, Suquet, 2003)
Martingale differences	$t^p\mu\{\lvert f\rvert > t\} \to 0$	Not necessarily (G., 2015)
Martingale differences	$t^p\mu\{\lvert f\rvert > t\} \to 0$ and $\mathbb{E}\left[f^2 \mid T\mathcal{M}\right] \in \mathbb{L}^{p/2}$	Yes (G., 2015)

We want to find some conditions on $f\colon \Omega \to \mathbb{R}$ which guarantee the convergence

$$\lim_{n\to\infty} \mathbb{E}\left[F\left(S_n^{\mathrm{pl}}(f,\cdot)/\sqrt{n}\right)\right] = \mathbb{E}\left[F(W)\right]$$

for each continuous and bounded function $F\colon \mathcal{H}_\alpha^o[0,1] \to \mathbb{R}$ where W is (a scalar multiple of) a standard Brownian motion and $\alpha \in (0,1/2)$.

- The *finite dimensional distributions* characterize probability measures on $\mathcal{H}_\alpha^o[0,1]$.
- It thus suffices to show *tightness* of the sequence $\left(S_n^{\mathrm{pl}}(f,\cdot)/\sqrt{n}\right)_{n\geqslant 1}$ in $\mathcal{H}_\alpha^o[0,1]$. In this context, this means that

$$\forall \varepsilon > 0, \quad \lim_{\delta\to 0} \limsup_{n\to\infty} \mu\left\{\sup_{s,t:|t-s|<\delta} \frac{\left|S_n^{\mathrm{pl}}(f,t) - S_n^{\mathrm{pl}}(f,s)\right|}{|t-s|^\alpha} > \varepsilon\sqrt{n}\right\} = 0.$$

- Using a dyadic splitting of the unit interval, we derive a *tightness criterion*:

$$\forall \varepsilon > 0, \quad \lim_{\delta\to 0} \limsup_{n\to\infty} n \sum_{k=1}^{\log[n\delta]} 2^{-k} \mu\left\{\max_{1\leqslant i\leqslant 2^k} |S_i(f)| > \varepsilon 2^{k\alpha} n^{1/2-\alpha}\right\} = 0.$$

Some remaining questions

- Give an estimate for the speed of convergence (no available results in the i.i.d. case).
- What happens for more general modulus of regularity, that is, when the role of the function $t \mapsto t^\alpha$ is played by $t \mapsto t^\alpha L(1/t)$, where L is a slowly varying function (e.g. $L(t) = \log(t)^\beta, \beta > 1/2$ and $0 < \alpha \leqslant 1/2$)?

  The i.i.d. case was considered by Račkauskas, Suquet, (2004).

Some remaining questions

- Give an estimate for the speed of convergence (no available results in the i.i.d. case).
- What happens for more general modulus of regularity, that is, when the role of the function $t \mapsto t^{\alpha}$ is played by $t \mapsto t^{\alpha} L(1/t)$, where L is a slowly varying function (e.g. $L(t) = \log(t)^{\beta}, \beta > 1/2$ and $0 < \alpha \leqslant 1/2$)?

  The i.i.d. case was considered by Račkauskas, Suquet, (2004).

Thanks for your attention!