A deviation inequality for Banach-valued random fields and some applications

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Plan

Definition of a martingale difference sequence

Definition

Let $(\Omega, \mathbb{P}, \mathcal{F})$ be a probability space. We say that the real-valued sequence $(D_i)_{i \geqslant 1}$ is a martingale difference sequence with respect to the filtration $(\mathcal{F}_i)_{i \geqslant 0}$ if

1. *for each $i \geqslant 0$, $\mathcal{F}_i \subset \mathcal{F}_{i+1}$,*
2. *for each $i \geqslant 1$, D_i is integrable and $\mathcal{F}_i$-measurable and*
3. *for each $i \geqslant 1$, $\mathbb{E}\left[D_i \mid \mathcal{F}_{i-1}\right] = 0$.*

Examples

1. If $(D_i)_{i \geqslant 1}$ is independent and centered and $\mathcal{F}_i = \sigma\left(D_j, j \leqslant i\right)$, then $(D_i)_{i \geqslant 1}$ is a martingale difference sequence with respect to the filtration $(\mathcal{F}_i)_{i \geqslant 0}$.
2. If $(\mathcal{F}_i)_{i \geqslant 0}$ is a filtration and X an integrable random variable, then $D_i := \mathbb{E}\left[X \mid \mathcal{F}_i\right] - \mathbb{E}\left[X \mid \mathcal{F}_{i-1}\right]$ is a martingale difference sequence.

Moment inequalities (1)

Notice that if $(D_i)_{i \geqslant 1}$ is a martingale difference sequence, then for each $i < j$,

$$\mathbb{E}\left[D_i D_j\right] = \mathbb{E}\left[\mathbb{E}\left[D_i D_j \mid \mathcal{F}_i\right]\right] = \mathbb{E}\left[D_i \mathbb{E}\left[D_j \mid \mathcal{F}_i\right]\right] = 0$$

hence denoting $S_n = \sum_{i=1}^n D_i$,

$$\mathbb{E}\left[S_n^2\right] = \sum_{i=1}^n \mathbb{E}\left[D_i^2\right].$$

Moreover, by Doob's inequality,

$$\mathbb{E}\left[\max_{1 \leqslant n \leqslant N} S_n^2\right] \leqslant 4 \sum_{i=1}^N \mathbb{E}\left[D_i^2\right].$$

Moment inequalities (2)

Burkholder's inequality:

- for $1 < p \leqslant 2$,

$$\mathbb{E}\left[\max_{1 \leqslant n \leqslant N} |S_n|^p\right] \leqslant C(p) \sum_{i=1}^{N} \mathbb{E}\left[|D_i|^p\right].$$

- For $p \geqslant 2$,

$$\mathbb{E}\left[\max_{1 \leqslant n \leqslant N} |S_n|^p\right] \leqslant C(p) \left(\sum_{i=1}^{N} \mathbb{E}\left[|D_i|^p\right]^{2/p}\right)^{p/2}.$$

Why deviation inequalities?

Suppose that for some sequence $(a_N)_{N\geqslant 1}$, one can get a bound of the form

$$\mathbb{P}\left(\frac{1}{a_N}\max_{1\leqslant n\leqslant N}|S_n|>x\right)\leqslant\int_0^\infty g(u)\,\mathbb{P}(Y>ux)\,du \qquad (*)$$

where Y is independent on N.

Then under integrability conditions on Y, we can deduce convergence of series of the form

$$\sum_N b_N\mathbb{P}\left(\frac{1}{c_N}\max_{1\leqslant n\leqslant N}|S_n|>x\right)$$

by a direct application of (*).

In particular, no truncation is needed.

A deviation inequality with conditional moments

Theorem (Nagaev, 2003)

If $(S_n)_{n\geqslant 1}$ is a martingale defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $X_i := S_i - S_{i-1}$, then

$$\mathbb{P}\left\{|S_n| > x\right\} \leqslant C(q) \int_0^1 \mathbb{P}\left\{\max_{1\leqslant i\leqslant n} |X_i| > xu\right\} u^{q-1} \mathrm{d}u + C(q) \int_0^1 \mathbb{P}\left\{\left(\sum_{i=1}^n \mathbb{E}\left[X_i^2 \mid \mathcal{F}_{i-1}\right]\right) > xu\right\} u^{q-1} \mathrm{d}u.$$

Definition of a smooth Banach space, examples

Definition

Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a separable Banach space. We say that $\mathbb{B}$ is *r*-smooth for $1 < r \leqslant 2$ if there exists an equivalent norm $\|\cdot\|'_{\mathbb{B}}$ on $\mathbb{B}$ such that

$$\sup_{t>0} \sup_{x,y\in B, \|x\|'_{\mathbb{B}}=\|y\|'_{\mathbb{B}}=1,} \frac{\|x+ty\|'_{\mathbb{B}} + \|x-ty\|'_{\mathbb{B}} - 2}{t^r} < \infty.$$

For example, if μ is σ-finite on the Borel σ-algebra of $\mathbb{R}$, then $\mathbb{L}^p(\mathbb{R}, \mu)$ is $\min\{p, 2\}$-smooth.

Moreover, a separable Hilbert space is 2-smooth.

Link between martingales and smooth Banach spaces

Proposition (Xu, Roasch, 1991)

Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be an r-smooth Banach space for some $r \in (1,2]$. For each $z \in \mathbb{B}$, there exists a unique continuous linear functional $J_z : \mathbb{B} \to \mathbb{R}$ such that the following conditions hold:

1. $\|J_z\|_{\mathbb{B}'} = \|z\|_{\mathbb{B}}^{r-1}$
2. $J_z(z) = \|z\|_{\mathbb{B}}^{r}$
3. *for each* $x, y \in \mathbb{B}$,

$$\|x+y\|_{\mathbb{B}}^r \leqslant \|x\|_{\mathbb{B}}^r + rJ_x(y) + C(\mathbb{B})\|y\|_{\mathbb{B}}^r .$$

We apply this to $x = \sum_{i=1}^{n-1} D_i$ and $y = D_n$, then integrate in order to get, by induction,

$$\mathbb{E}\left[\left\|\sum_{i=1}^{n} D_i\right\|_{\mathbb{B}}^r\right] \leqslant C(\mathbb{B}) \sum_{i=1}^{n} \mathbb{E}\left[\|D_i\|_{\mathbb{B}}^r\right] .$$

Important constants for a smooth Banach space

Noticing that an r-smooth Banach space is also p-smooth for each $1 < p \leqslant r$, we can define

$$C_{p,\mathbb{B}} := \sup_{n \geqslant 1} \sup_{(D_i)_{i=1}^n \in \Delta_n} \frac{\mathbb{E}\left[\left\|\sum_{i=1}^n D_i\right\|_{\mathbb{B}}^p\right]}{\sum_{i=1}^n \mathbb{E}\left[\left\|D_i\right\|_{\mathbb{B}}^p\right]},$$

where the Δ_n denotes the set of the martingale differences sequences $(D_i)_{i=1}^n$ such that $\sum_{i=1}^n \|D_i\|_{\mathbb{B}}^p$ is not identically 0.

Deviation inequality with conditional moments

Theorem (G., 2020)

Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a separable r-smooth Banach space where $1 < r \leqslant 2$. For each $1 < p \leqslant r$, $q > 0$ and for any $\mathbb{B}$-valued martingale differences sequence $(D_i, \mathcal{F}_i)_{i \geqslant 1}$, the following inequality holds for each $n \geqslant 1$ and $x > 0$:

$$\mathbb{P}\left(\max_{1\leqslant k\leqslant n} \|S_k\|_{\mathbb{B}} > t\right) \leqslant f(p,q,C_{p,\mathbb{B}}) \int_0^1 \mathbb{P}\left(\max_{1\leqslant i\leqslant n} \|D_i\|_{\mathbb{B}} > tu\right) u^{q-1}\mathrm{d}u$$
$$+ f(p,q,C_{p,\mathbb{B}}) \int_0^1 \mathbb{P}\left(\left(\sum_{i=1}^n \mathbb{E}\left[\|D_i\|_{\mathbb{B}}^p \mid \mathcal{F}_{i-1}\right]\right)^{1/p} > tu\right) u^{q-1}\mathrm{d}u,$$

where $S_k = \sum_{i=1}^k D_i$ and $C_{p,\mathbb{B}}$ is a constant satisfying

$$\mathbb{E}\left[\left\|\sum_{i=1}^n D_i\right\|_{\mathbb{B}}^p\right] \leqslant C_{p,\mathbb{B}} \sum_{i=1}^n \mathbb{E}\left[\|D_i\|_{\mathbb{B}}^p\right]$$

for any n and any martingale differences sequence.

Deviation inequality without conditional moments

Proposition (G., 2022+)

Let $1 < r \leqslant 2$ and let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a separable r-smooth Banach space. For each $p \in (1, r]$, and $q > p$, there exists a function $f_{p,q} \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that if $(D_i)_{i \geqslant 1}$ is a $\mathbb{B}$-valued martingale difference sequence with respect to the filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ then for each $1 < p \leqslant r$, $q > p$ and $x > 0$, the following inequality holds:

$$\mathbb{P}\left(\max_{1 \leqslant n \leqslant N} \left\| \sum_{i=1}^{n} D_i \right\|_{\mathbb{B}} > t\right) \leqslant f_{p,q}\left(C_{p,\mathbb{B}}\right) \int_0^{\infty} \min\left\{u^{q-1}, u^{p-1}\right\} \mathbb{P}\left(\left(\sum_{i=1}^{N} \|D_i\|_{\mathbb{B}}^p\right)^{1/p} > tu\right) \mathrm{d}u.$$

Plan

Commuting filtration

For $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{Z}^d$, $\boldsymbol{i} \succcurlyeq \boldsymbol{j}$ means that $i_\ell \leqslant j_\ell$ for each $\ell \in \{1, \dots, d\}$.

Definition

We say that a collection of σ-algebras $(\mathcal{F}_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{Z}^d}$ is a completely commuting filtration if

1. for each $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{Z}^d$ such that $\boldsymbol{i} \preccurlyeq \boldsymbol{j}$, $\mathcal{F}_{\boldsymbol{i}} \subset \mathcal{F}_{\boldsymbol{j}}$ and
2. for each $Y \in \mathbb{L}^1$ and each $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{Z}^d$,

$$\mathbb{E}\left[\mathbb{E}\left[Y \mid \mathcal{F}_{\boldsymbol{i}}\right] \mid \mathcal{F}_{\boldsymbol{j}}\right] = \mathbb{E}\left[Y \mid \mathcal{F}_{\min\{\boldsymbol{i}, \boldsymbol{j}\}}\right],$$

where $\min\{\boldsymbol{i}, \boldsymbol{j}\}$ is the element of $\mathbb{Z}^d$ defined as the coordinatewise minimum of $\boldsymbol{i}$ and $\boldsymbol{j}$, that is, $\min\{\boldsymbol{i}, \boldsymbol{j}\} = (\min\{i_\ell, j_\ell\})_{\ell=1}^d$.

Examples of commuting filtrations

Proposition

1. *If* $(\varepsilon_{\boldsymbol{u}})_{\boldsymbol{u}\in\mathbb{Z}^d}$ *is i.i.d., then defining* $\mathcal{F}_{\boldsymbol{i}} = \sigma\left(\varepsilon_{\boldsymbol{u}}, \boldsymbol{u}\in\mathbb{Z}^d, \boldsymbol{u}\preccurlyeq \boldsymbol{i}\right)$, *the filtration* $(\mathcal{F}_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ *is completely commuting.*
2. *Suppose that* $\left(\mathcal{F}^{(\ell)}_{\boldsymbol{i}^{(\ell)}}\right)_{\boldsymbol{i}^{(\ell)}\in\mathbb{Z}^{d_\ell}}$, $1\leqslant \ell\leqslant L$, *are completely commuting filtrations on a probability space* $(\Omega,\mathcal{F},\mathbb{P})$. *Suppose that for each* $\boldsymbol{i}^{(1)}\in\mathbb{Z}^{d_1},\dots,\boldsymbol{i}^{(L)}\in\mathbb{Z}$, *the* σ*-algebras* $\mathcal{F}^{(\ell)}_{\boldsymbol{i}^{(\ell)}}$, $1\leqslant\ell\leqslant L$, *are independent. Let* $d=\sum_{\ell=1}^L d_\ell$ *and for* $\boldsymbol{i}=\left(i^{(\ell)}\right)_{\ell=1}^L\in\mathbb{Z}^d$, *let* $\mathcal{F}_{\boldsymbol{i}}=\bigvee_{\ell=1}^L \mathcal{F}^{(\ell)}_{\boldsymbol{i}_{\boldsymbol{\ell}}}$. *Then* $(\mathcal{F}_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ *is completely commuting.*

Definition of orthomartingales

Denote $\boldsymbol{e}_{\boldsymbol{\ell}} = \Big(\underbrace{0, \dots, 0}_{\ell-1}, 1, \underbrace{0 \dots, 0}_{d-\ell}\Big) \in \mathbb{Z}^d$.

Definition

Let $(X_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{Z}^d}$ be a random field taking values in a separable Banach space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$. We say that $(X_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{Z}^d}$ is an orthomartingale martingale difference random field with respect to the completely commuting filtration $(\mathcal{F}_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{Z}^d}$ if for each $\boldsymbol{i} \in \mathbb{Z}^d$,

1. $\|X_i\|_{\mathbb{B}}$ is integrable,
2. $X_{\boldsymbol{i}}$ is $\mathcal{F}_{\boldsymbol{i}}$-measurable and
3. for each $\ell \in \{1, \dots, d\}$, $\mathbb{E}\left[X_{\boldsymbol{i}} \mid \mathcal{F}_{\boldsymbol{i}-\boldsymbol{e}_{\boldsymbol{\ell}}}\right] = 0$.

Properties of orthomartingales

Let $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ be an orthomartingale difference random field with respect to $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$. Then for each fixed j, $\left(X_{i_1,\dots,i_{d-1},j}\right)_{i_1 n\dots,i_{d-1}\in\mathbb{Z}}$ is an orthomartingale difference with respect to the commuting filtration $\left(\mathcal{F}_{i_1,\dots,i_{d-1},j}\right)_{i_1,\dots,i_{d-1}\in\mathbb{Z}}$.

This allows arguments by induction on the dimension.

For example in the real-valued case, for $p\geqslant 2$,

$$\mathbb{E}\left[\left|\sum_{\mathbf{1}\preccurlyeq\mathbf{i}\preccurlyeq\mathbf{n}} X_{\mathbf{i}}\right|^p\right]\leqslant C\left(p,d\right)\left(\sum_{\mathbf{1}\preccurlyeq\mathbf{i}\preccurlyeq\mathbf{n}}\mathbb{E}\left[\left|X_{\mathbf{i}}\right|^p\right]^{2/p}\right)^{p/2}.$$

Main result

Theorem (G., 2022+)

Let $1 < r \leqslant 2$ and let $(B, \|\cdot\|_{\mathbb{B}})$ be a separable r-smooth Banach space. For each $p \in (1, r]$, $q > p$ and $d \geqslant 1$, there exists a function $f_{p,q,d} \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that if $(X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ is a an orthomartingale martingale differences random field with respect to a completely commuting filtration $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$, and taking values in a $\mathbb{B}$, then for each $1 < p \leqslant r$, $q > p$ and $x > 0$, the following inequality holds:

$$\mathbb{P}\left(\max_{\mathbf{1} \preccurlyeq \mathbf{n} \preccurlyeq \mathbf{N}} \left\| \sum_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}} X_{\mathbf{i}} \right\|_{\mathbb{B}} > t \right) \leqslant f_{p,q,d}\left(C_{p,\mathbb{B}}\right) \mathbb{E}\left[\left(\frac{Y}{t}\right)^q \mathbf{1}_{Y \leqslant t}\right] + f_{p,q,d}\left(C_{p,\mathbb{B}}\right) \mathbb{E}\left[\left(\frac{Y}{t}\right)^p \left(1 + \log\left(\frac{Y}{t}\right)\right)^{d-1} \mathbf{1}_{Y > t}\right]$$

with $Y = Y_{N,p} = \left(\sum_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{N}} \|X_{\mathbf{i}}\|_{\mathbb{B}}^p\right)^{\frac{1}{p}}$.

Case of stochastic domination

Corollary (G., 2022+)

Let $1 < r \leqslant 2$ and let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a separable r-smooth Banach space. Let $(X_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ be a $\mathbb{B}$-valued orthomartingale martingale differences random field with respect to a completely commuting filtration $(\mathcal{F}_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$, and such that there exists a real-valued random variable V such that for each convex increasing function φ,

$$\mathbb{E}\left[\varphi\left(\sum_{\boldsymbol{1}\preccurlyeq\boldsymbol{i}\preccurlyeq\boldsymbol{N}} \|X_{\boldsymbol{i}}\|_{\mathbb{B}}^p\right)\right] \leqslant \mathbb{E}\left[\varphi\left(V^p\right)\right],$$

then for each $1 < p \leqslant r$, $q > 0$ and $x > 0$, the following inequality holds:

$$\mathbb{P}\left(\max_{\boldsymbol{1}\preccurlyeq\boldsymbol{n}\preccurlyeq\boldsymbol{N}}\left\|\sum_{\boldsymbol{1}\preccurlyeq\boldsymbol{i}\preccurlyeq\boldsymbol{n}} X_{\boldsymbol{i}}\right\|_{\mathbb{B}} > t\right) \leqslant f_{p,q,d}\left(C_{p,\mathbb{B}}\right)\int_0^1 u^{q-1}\mathbb{P}\left(V > tu\right)\mathrm{d}u$$
$$+ f_{p,q,d}\left(C_{p,\mathbb{B}}\right)\int_1^\infty u^{p-1}\left(1+\log u\right)^d\mathbb{P}\left(V > tu\right)\mathrm{d}u.$$

Idea of proof of the theorem

For $d = 2$, let $(X_{i,j})_{i,j\in\mathbb{Z}}$ the considered orthomartingale difference random field.
Applying an inequality for martingale difference sequence with $D_j = \sum_{i=1}^{n_1} X_{i,j}$ to get

$$\mathbb{P}\left(\left\|\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} X_{i,j}\right\|_{\mathbb{B}} > t\right)$$

$$\leqslant f_{p,q}\left(C_{p,\mathbb{B}}\right)\int_0^{\infty} \min\left\{u^{q-1}, u^{p-1}\right\}\mathbb{P}\left(\left(\sum_{j=1}^{n_2}\left\|\sum_{i=1}^{n_1} X_{i,j}\right\|_{\mathbb{B}}^p\right)^{1/p} > tu\right)\mathrm{d}u.$$

Then we view $\left\|\sum_{i=1}^{n_1} X_{i,j}\right\|_{\mathbb{B}}^p$ as $\left\|\sum_{i=1}^{n_1} d_i\right\|_{\widetilde{\mathbb{B}}}^p$, with $\widetilde{\mathbb{B}} = \mathbb{B}^{n_2}$, $\left\|(x_j)_{j=1}^{n_2}\right\|_{\widetilde{\mathbb{B}}}^p = \sum_{j=1}^{n_2} \|x_j\|_{\mathbb{B}}^p$ and $d_i = (X_{i,j})_{j=1}^{n_2}$.
Key point: $C_{p,\mathbb{B}} = C_{p,\widetilde{\mathbb{B}}}$.

Plan

Rates in the law of large numbers for random fields

Let $\varphi_{p,s} \colon t \mapsto t^p \left(1 + \mathbf{1}_{t>1} \log t\right)^s$ and $|\boldsymbol{n}| = \prod_{\ell=1}^d n_\ell$.

A sufficient condition for the almost sure convergence of $S_{\boldsymbol{n}} / |\boldsymbol{n}|^{1/p} \to 0$ as $\max \boldsymbol{n} \to \infty$ is $\mathbb{E}\left[\varphi_{p,d-1}\left(\|X_{\mathbf{1}}\|_{\mathbb{B}}\right)\right] < \infty$.

Theorem (G., 2022+)

Let $\mathbb{B}$ be a separable r-smooth Banach space and $s > r$. For each identically distributed $\mathbb{B}$-valued orthomartingale difference random field $(X_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$, for each positive ε and each $\alpha \in (1/r, 1]$, the following inequality takes place

$$\sum_{\boldsymbol{n}\in\mathbb{N}^d} |\boldsymbol{n}|^{s(\alpha-1/r)-1} \mathbb{P}\left(\max_{\mathbf{1}\preccurlyeq \boldsymbol{i}\preccurlyeq \boldsymbol{n}} \|S_{\boldsymbol{i}}\|_{\mathbb{B}} > \varepsilon |\boldsymbol{n}|^{\alpha}\right) \leqslant C(r, d, \mathbb{B})\, \mathbb{E}\left[\varphi_{s,d}\left(\frac{\|X_{\mathbf{1}}\|_{\mathbb{B}}}{\varepsilon}\right)\right].$$

Definition

We consider the following regression model:

$$Y_{\boldsymbol{i}} = g\left(\frac{\boldsymbol{i}}{n}\right) + X_{\boldsymbol{i}}, \quad \boldsymbol{i} \in \Lambda_n := \{1, \dots, n\}^d,$$

where $g\colon [0,1]^d \to \mathbb{R}$ is an unknown smooth function and $(X_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ is an orthomartingale difference random field. Let K be a probability kernel defined on $\mathbb{R}^d$ and let $(h_n)_{n\geqslant 1}$ be a sequence of positive numbers which converges to zero and which satisfies

$$\lim_{n\to+\infty} nh_n = +\infty \text{ and } \lim_{n\to+\infty} nh_n^{d+1} = 0.$$

We estimate the function g by the kernel estimator g_n defined by

$$g_n(\boldsymbol{x}) = \frac{\sum_{\boldsymbol{i}\in\Lambda_n} Y_{\boldsymbol{i}} K\left(\frac{\boldsymbol{x}-\boldsymbol{i}/n}{h_n}\right)}{\sum_{\boldsymbol{i}\in\Lambda_n} K\left(\frac{\boldsymbol{x}-\boldsymbol{i}/n}{h_n}\right)}, \quad \boldsymbol{x} \in [0,1]^d,$$

where

$$\Lambda_n = \{1, \dots, n\}^d.$$

Goal, assumptions

Goal : provide tail bounds for $\|g_n(\cdot) - \mathbb{E}[g_n(\cdot)]\|_{\mathbb{L}^p([0,1]^d)}$.

Assumptions on K:

1. The probability kernel K fulfills $\int_{\mathbb{R}^d} K(\boldsymbol{u}) \,\mathrm{d}\boldsymbol{u} = 1$, is symmetric, non-negative, supported by $[-1,1]^d$.
2. There exist positive constants c and C such that for any $\boldsymbol{x} \in [-1,1]^d$, $c \leqslant K(\boldsymbol{x}) \leqslant C$.

Convergence in $\mathbb{L}^p$, $1 < p \leqslant 2$

Theorem (G., 2022+)

Let $1 < p \leqslant 2$ and let $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ be an identically distributed real-valued orthomartingale difference random field and let $g_n\colon [0,1]^d \to \mathbb{R}$ be given by

$$g_n(\mathbf{x}) = \frac{\sum_{\mathbf{i}\in\Lambda_n} Y_{\mathbf{i}} K\left(\frac{\mathbf{x}-\mathbf{i}/n}{h_n}\right)}{\sum_{\mathbf{i}\in\Lambda_n} K\left(\frac{\mathbf{x}-\mathbf{i}/n}{h_n}\right)}, \quad \mathbf{x}\in[0,1]^d.$$

For each positive t, the following inequality takes place:

$$\mathbb{P}\left(\left\|g_n(\cdot) - \mathbb{E}\left[g_n(\cdot)\right]\right\|_{\mathbb{L}^p([0,1]^d)} > t\right) \leqslant \kappa_{p,q,d}\int_0^1 u^{q-1}\mathbb{P}\left(\left|X_{\mathbf{1}}\right| > t\left(nh_n\right)^{d(1-1/p)} u\right)\mathrm{d}u + \kappa_{p,q,d}\int_1^\infty u^{p-1}\left(1+\log u\right)^d \mathbb{P}\left(\left|X_{\mathbf{1}}\right| > t\left(nh_n\right)^{d(1-1/p)} u\right)\mathrm{d}u.$$

Convergence in $\mathbb{L}^p$, $p > 2$

Theorem (G., 2022+)

Let $p > 2$ and let $(X_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ be an identically distributed real-valued orthomartingale difference random field and let $g_n\colon [0,1]^d \to \mathbb{R}$ be given by

$$g_n(\boldsymbol{x}) = \frac{\sum_{\boldsymbol{i}\in\Lambda_n} Y_{\boldsymbol{i}} K\left(\frac{\boldsymbol{x}-\boldsymbol{i}/n}{h_n}\right)}{\sum_{\boldsymbol{i}\in\Lambda_n} K\left(\frac{\boldsymbol{x}-\boldsymbol{i}/n}{h_n}\right)}, \quad \boldsymbol{x}\in[0,1]^d.$$

For each positive t, the following inequality takes place:

$$\begin{aligned}\mathbb{P}\left(\left\|g_n(\cdot) - \mathbb{E}\left[g_n(\cdot)\right]\right\|_{\mathbb{L}^p([0,1]^d)} > t\right) &\\ \leqslant \kappa_{p,q,d}\int_0^1 u^{q-1}\mathbb{P}\left(\left|X_{\boldsymbol{1}}\right| > t\left(nh_n\right)^{d(-p)/2} n^{\frac{2-p}{2p}} u\right)\mathrm{d}u &\\ + \kappa_{p,q,d}\int_1^\infty u^{p-1}\left(1+\log u\right)^d \mathbb{P}\left(\left|X_{\boldsymbol{1}}\right| > t\left(nh_n\right)^{d(1-p)/2} n^{\frac{2-p}{2p}} u\right)\mathrm{d}u.\end{aligned}$$