3-dimensional spacetimes and crooked planes

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based on joint work with Jeff Danciger and Fanny Kassel

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Overview

1. Introduction & main result
2. Sketch of proof
3. Applications to Lorentzian geometry: crooked planes
4. A surface group acting properly discontinuously on $\mathbb{R}^6$

1. Introduction & main result

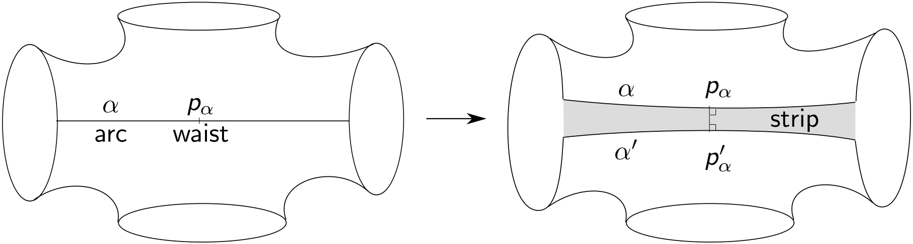
A problem in hyperbolic surfaces

Let (S, g) be a hyperbolic surface.

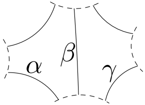
Question

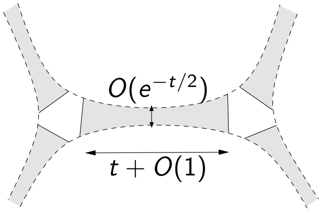
Can we find another hyperbolic metric g' for which all geodesics become uniformly longer "$(g' > g)$"?

- Not if S is compact.
- If S has boundary, Thurston's **strips** work:



Such a *strip deformation* lengthens all curves that cross the arc α. We can repeat with other arcs $\beta, \gamma, \ldots$ (disjoint) to lengthen all curves.





Question

Do we get *all* metrics longer than g?

Theorem (Danciger, G., Kassel)

Yes! and **uniquely**.

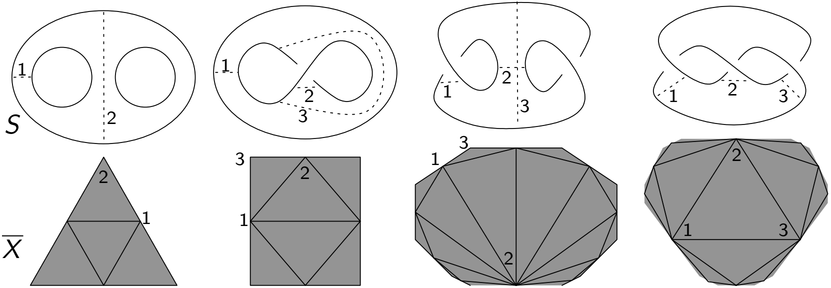
Also holds for *infinitesimal* deformations of g and *infinitesimal* strips.
But what does "uniquely" mean?

Uniqueness: the arc complex

Let $\overline{X}$ be the **arc complex** of S:

- $\overline{X}^{(0)} := \{\delta_\alpha \text{ isotopy class of properly embedded arcs } \alpha\}$
- $\overline{X}^{(1)} := \{(\delta_{\alpha_0}, \delta_{\alpha_1}) \mid \alpha_0, \alpha_1 \text{ distinct and isotopically disjoint}\}$
- $\overline{X}^{(2)} := \{(\delta_{\alpha_0}, \delta_{\alpha_1}, \delta_{\alpha_2}) \mid \alpha_0, \alpha_1, \alpha_2 \text{ distinct and isotopically disjoint}\}$
- $\ldots \overline{X}^{(N)} := \{(\delta_{\alpha_0}, \ldots, \delta_{\alpha_N}) \mid \alpha_0, \ldots, \alpha_N$ triangulate $S\}$

where $N + 1 = \dim(\mathrm{Teich}(S))$. Examples with $N = 2$:



For each arc $\alpha \in \overline{X}^{(0)}$, pick

- a geodesic representative
- a waist $p_\alpha \in \alpha$
- a width $w_\alpha > 0$.

This defines a "**strip map**"

$$\overline{\Phi} : \mathbb{R}_{\geq 0}\overline{X} \to \{\text{metrics} \geq g \text{ on } S\},$$

taking a formal linear combination of arcs

$$\boldsymbol{t\alpha} = t_0\alpha_0 + \cdots + t_k\alpha_k$$

to the deformation of (S, g) obtained by inserting (disjoint) strips

- at the arcs α_i
- of width $t_i w_{\alpha_i}$
- with waist p_{α_i}, for all $0 \leq i \leq k$.

Main result (made precise)

Definition

A simplex $\delta_{\alpha_0,\dots,\alpha_k}$ of $\overline{X}^{(k)}$ is **small** if $\alpha_0, \dots, \alpha_k$ fail to cut up S into disks (e.g. $k = 0$).

$$X := \overline{X} \smallsetminus \bigcup \{\text{small simplices}\} \qquad \simeq \mathbb{B}^N \text{ [Penner].}$$

Theorem (DGK)

The map $\overline{\Phi} : \mathbb{R}_{\geq 0}\overline{X} \to \{\textit{metrics} \geq g \textit{ on } S\}$ *induces a* **homeomorphism**

$$\Phi : \mathbb{R}_{>0}X \to \{\textit{metrics} > g \textit{ on } S\}.$$

2. Sketch of proof

Sketch of proof:

Let us focus on the infinitesimal version:

$$D\Phi : \mathbb{R}_{>0}X \underset{?}{\xrightarrow{\sim}} \{\text{lengthening infinitesimal deformations of } g\} \subset T_g\mathrm{Teich}(S).$$

Source $\simeq \mathbb{R}^{N+1} \simeq$ Range: so, show $D\Phi$ is $\begin{cases} (1) \text{ proper,} \\ (2) \text{ locally homeo.} \end{cases}$

For (1), note: $t\alpha \to \infty$ in $\mathbb{R}_{>0}X$ iff

Either the triangulation α goes to infinity, i.e. its arcs become very long: then they spin along a *geodesic lamination* (simple closed curve or Hausdorff limit thereof) which strips along α cannot lengthen much.

Or α stabilizes, but too many coordinates of t go to 0, so the remaining arcs fail to cut up S into disks (def. of X): the remaining strips fail to lengthen any curve in a non-disk component.

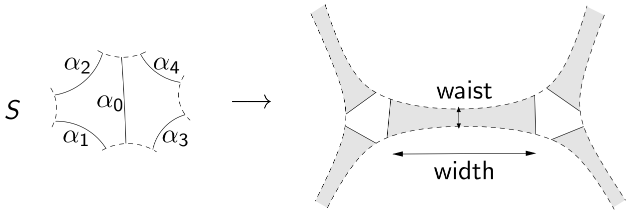
For (2): the projectivized strip map

$$\mathbb{P}D\Phi : X \to \mathbb{P}T_g\mathrm{Teich}(S) \simeq \mathbb{P}^N\mathbb{R}$$

is projective on each cell of the arc complex X.
Why is it locally a homeomorphism near any $x \in X$?

• **Fundamental case**: x belongs to the interior of a top-dimensional cell $(\alpha_0, \ldots, \alpha_N)$. Then we are asking:

Why are the $N+1$ infinitesimal strip deformations along arcs of a **triangulation** $(\alpha_0, \ldots, \alpha_N)$ **linearly independent** ($=$ a basis of $T_g\mathrm{Teich}$)?



Linear independence of strips from a triangulation

By contradiction, suppose $\sum_{i=0}^{N} t_i D\overline{\Phi}(\alpha_i) = 0 \;\in T_g\mathrm{Teich}(S)$.

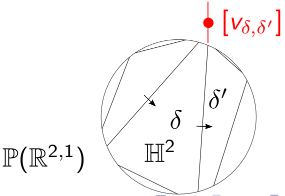
Lifting to $\mathbb{H}^2$ the *tiles* (triangles) of the triangulation α of S, this induces an **equivariant assignment of infinitesimal motions**

$$\mu : \left\{ \text{Tiles of } (\widetilde{S, g, \alpha}) \right\} \longrightarrow \mathrm{Kill}(\mathbb{H}^2) = \mathrm{Lie}(\mathrm{Isom}(\mathbb{H}^2)) = \mathfrak{psl}_2 = \mathbb{R}^{2,1}_{\langle\cdot|\cdot\rangle}$$

such that

$$v_{\delta,\delta'} := \mu(\delta) - \mu(\delta') \;\in\; \{\langle\cdot|\cdot\rangle > 0\} \;\cap\; \mathrm{Span}(\delta \cap \delta') \qquad (1)$$

for any adjacent tiles $\delta, \delta' \subset \mathbb{H}^2$.
Indeed, (1) says the *relative motion* of δ w.r.t. δ' is $(\pm)$ a strip deformation.
Note $\mathbb{H}^2 = \mathbb{P}\{\langle\cdot|\cdot\rangle < 0\}$.



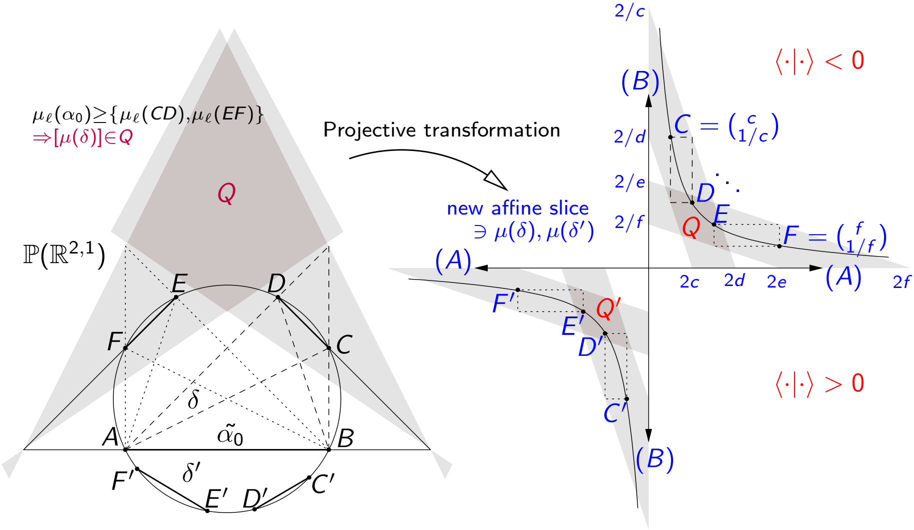
$\mu(\delta) - \mu(\delta') \in \mathrm{Span}(\delta \cap \delta')$ means the infinitesimal motions

$$\mu(\delta) \ \text{ and } \ \mu(\delta')$$

have the same **longitudinal component** $\mu_\ell(\tilde{\alpha})$ along the arc $\tilde{\alpha} = \delta \cap \delta'$.
By equivariance, μ_ℓ descends to $\{\alpha_0, \dots, \alpha_N\}$, inducing

$$\mu_\ell : \{\alpha_0, \dots, \alpha_N\} \longrightarrow \mathbb{R}_{\geq 0}.$$

Up to relabelling, $\mu_\ell(\alpha_0) = \max\limits_{\{\alpha_0, \dots, \alpha_N\}} \mu_\ell$. Then, picture!...

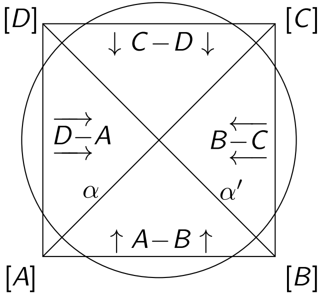


$\mu(\delta) - \mu(\delta') \in Q - Q' \subset \{\langle\cdot|\cdot\rangle < 0\} \Rightarrow\Leftarrow \square$

Codimension 1

Thus, Φ is homeomorphic at top-dimensional cells of the arc complex X.

• At a **codim-1 cell** (diagonal exchange $\alpha \leftrightarrow \alpha'$), the following choice of μ : {Tiles} $\to \mathbb{R}^{2,1}$



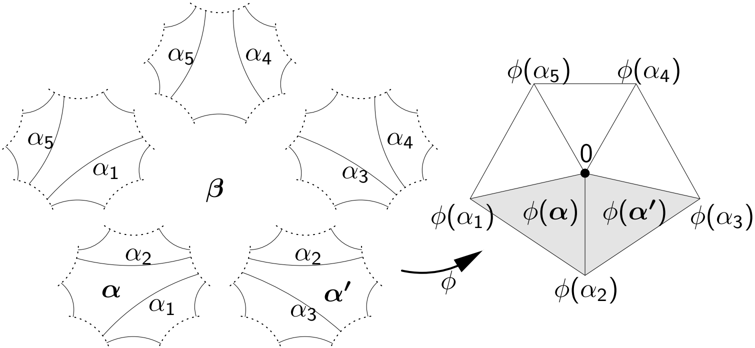
$\mathbb{PR}^{2,1}$

$A + C = B + D \in \mathbb{R}^{2,1}$

shows we can **cancel** positive strip deformations on $\alpha = AC$ and $\alpha' = BD$ by positive strip deformations on AB, BC, CD, DA: i.e., $\phi := \mathbb{P}D\Phi$ does not fold adjacent top-dimensional simplices $\boldsymbol{\alpha}, \boldsymbol{\alpha}'$ of X over.

Codimension ≥ 2

• At a **codim-2 cell** β (pentagon move), the map $\phi := \mathbb{P}D\Phi$ cannot wrap twice, by convexity of $\phi(\boldsymbol{\alpha} \cup \boldsymbol{\alpha}')$: modulo $\mathrm{Span}(\phi(\boldsymbol{\beta}))$,



• At **codimension-k cells** ($k \geq 3$), homeomorphicity of ϕ follows by induction on the link map $\mathbb{S}^{k-1} \to \mathbb{S}^{k-1}$ because $\pi_1(\mathbb{S}^{k-1}) = 1$. $\square$

The **infinitesimal** version of Theorem 2 is proved: every infinitesimal deformation of the hyperbolic metric (S, g) that lengthens all the curves (uniformly) is given by a unique positive linear combination of strip deformations on disjoint arcs, with the chosen waists.

The **macroscopic** version follows by a similar argument and a form of integration.

3. Applications to Lorentzian geometry: crooked planes

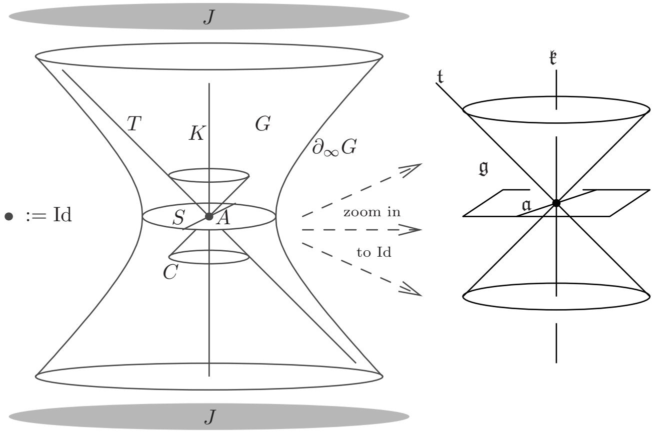
History

Let Γ (discrete) act properly discontinuously on $\mathbb{R}^n$ by **affine** transformations.

- 1900s, Bieberbach: Γ preserves $|\cdot|_{\text{Eucl}} \Rightarrow$ Γ virtually Abelian.
- '60s, Auslander conjecture: $\Gamma \backslash \mathbb{R}^n$ compact $\Rightarrow$ Γ virtually solvable?
- '70s, Milnor's question: can we drop cocompactness?
- '80s, Margulis's (counter)examples: **No!** ($\Gamma = \mathbb{F}_k$ acting on $\mathbb{R}^{2,1}$).
- (Goldman, Fried: if $n = 3$, these are virtually the only examples.)
- '90s, Drumm's **crooked planes**: understand the topology of *some* Margulis spacetimes via fundamental domains.
- '00s, Goldman-Labourie-Margulis: characterize dynamically the actions that yield Margulis spacetimes.
- '10s, DGK: understand the topology of *all* Margulis spacetimes / describe their geometry (fibrations) / parameterize their deformation space by the arc complex X / deform them to (< 0)-curvature, . . .

What is a crooked plane?

Here is a view of $G = \mathrm{PSL}_2\mathbb{R}$ and its Lie algebra $\mathfrak{g} := \mathbb{R}^{2,1}$.



A: diagonal S: symmetric C: parabolic
J: traceless K: rotations T: upper-triangular

Identify $G \simeq \mathrm{Isom}^0(\mathbb{H}^2)$, and $\mathfrak{g} \simeq \mathrm{Kill}(\mathbb{H}^2)$. As a homogeneous $(G \times G)$-space, G is also called **anti-de Sitter space** (AdS): $(g, g') \cdot h = g' h g^{-1}$. Let ℓ be a line of $\mathbb{H}^2$.

Definition

(i) An **AdS crooked plane** of G is any $(G \times G)$-translate of

$$\{g \in G \mid g \text{ has a nonrepelling fixed point in } \ell\}.$$

(ii) A **crooked plane** of $\mathfrak{g}$ is any $(\mathfrak{g} \rtimes_{\mathrm{Ad}} G)$-translate of

$$\{X \in \mathfrak{g} \mid X \text{ has a nonrepelling fixed point in } \ell\}.$$

Note that

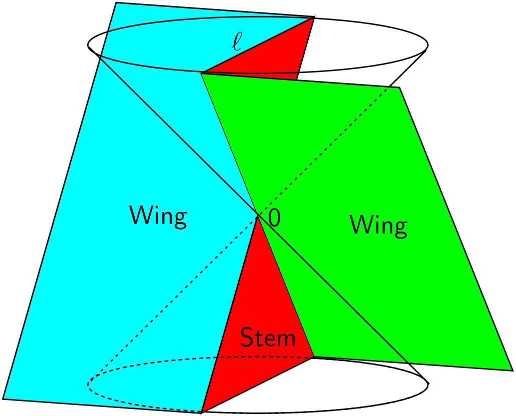
$$\begin{array}{ccc} G \rtimes_{\mathrm{Conj}} G & \xrightarrow{\sim} & G \times G \\ (g', g) & \longmapsto & (g'g, g) \end{array}$$

is an isomorphism, so (ii) is an infinitesimal version of (i).

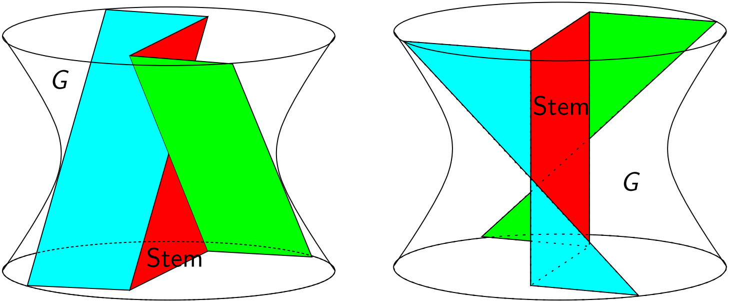
A crooked plane of $\mathbb{R}^{2,1}$ centered at the origin is made of

- a **stem** (red): the rotations centered on ℓ, also equal to $\mathrm{Span}(\ell)$;
- two **wings** (blue, green): lightlike half-planes consisting of loxodromies of $\mathbb{H}^2$ whose attracting fixed point is an endpoint of ℓ;
- two lightlike lines, where the wings are attached to the stem.

$\mathfrak{g}$



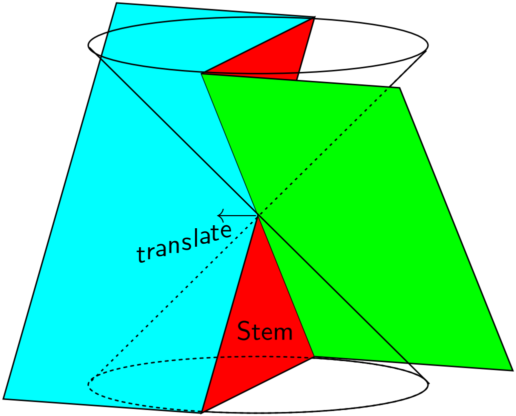
Similarly, an AdS crooked plane also has a stem and two lightlike wings.



But here, each wing terminates on a line of the stem, **and** on a line of $\partial_\infty G$: wings are thus *projective bigons*. In the second view, the center of the stem has been placed at infinity in $\mathbb{P}^3\mathbb{R} \supset G$.

Drumm's observation

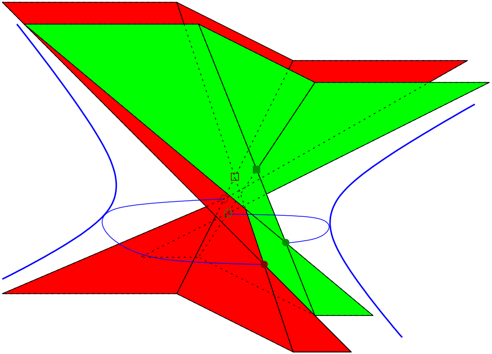
We can pull a crooked $\frac{1}{2}$-space away from its complement, without overlap, by translating into the correct **quadrant of the span of the stem.**



The stem then slides along itself.

Observation (Danciger, G., Kassel)

The same happens with an AdS crooked plane: you can push it off itself by multiplication (on the left) by any element in the correct quadrant of the span of its stem.



This time, the stem moves off itself completely!

- Drumm used his observation to build fundamental domains for properly discontinuous actions of the free group $\mathbb{F}_k$ on $\mathbb{R}^{2,1} = \mathfrak{g}$ bounded by (disjoint) crooked planes.
- We can similarly build fundamental domains for actions on AdS $= G$.
- Do we get all actions?

Properly discontinuous actions are classified by the following two theorems.

Let $G = \mathrm{PSL}_2(\mathbb{R})$ and

$$\lambda : G \to \mathbb{R}_{\geq 0}$$

be the translation length (0 for non-loxodromics), and $\Gamma := \mathbb{F}_k$, $k \geq 2$.

Theorem (G., Kassel)

Given $(j, \rho) : \Gamma \to G \times G$ with j Fuchsian and $\lambda(j(\gamma_0)) > \lambda(\rho(\gamma_0))$ for some γ_0, the following are equivalent:

1. *The Γ-action on G given by $\gamma \cdot g = \rho(\gamma) g j(\gamma)^{-1}$ is prop. disc.;*
2. $\sup_{\gamma \in \Gamma \smallsetminus \{1\}} \frac{\lambda(\rho(\gamma))}{\lambda(j(\gamma))} < 1$;
3. *$\exists$ a (j, ρ)-equivariant, C-Lipschitz map $f : \mathbb{H}^2 \to \mathbb{H}^2$ with $C < 1$:*

$$d(f(p), f(q)) \leq C\, d(p, q) \quad \forall p, q \in \mathbb{H}^2.$$

Moreover, this gives **all** *prop. disc. actions of Γ on G up to swaping $j \leftrightarrow \rho$.*

Theorem (Goldman, Labourie, Margulis; Danciger, G., Kassel)

Given $(j,u) : \Gamma \to G \ltimes \mathfrak{g}$ *(i.e. u is a j-cocycle: $u(\gamma) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} j_t(\gamma) j(\gamma)^{-1}$ for some smooth deformation $(j_t)_{t\geq 0}$ of j),*
if $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \lambda(j_t(\gamma_0)) < 0$ for some γ_0, then the following are equivalent:

1. *The Γ-action on $\mathfrak{g}$ given by $\gamma \cdot X = \mathrm{Ad}(j(\gamma))(X) + u(\gamma)$ is prop. disc.;*
2. $\sup_{\gamma \in \Gamma \smallsetminus \{1\}} \frac{\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \lambda(j_t(\gamma))}{\lambda(j(\gamma))} < 0$;
3. *There exists a vector field Y on $\mathbb{H}^2$ that is (j,u)-equivariant $(Y(j(\gamma)\cdot p) = j(\gamma)_* Y(p) + u(\gamma)(j(\gamma)\cdot p))$, and contracting: $\exists c < 0$,*

$$\frac{\mathrm{d}\left[d(\exp_p(tY(p)), \exp_q(tY(q)))\right]}{\mathrm{d}t|_{t=0}} \leq c\, d(p,q) \;\; \forall p \neq q \in \mathbb{H}^2.$$

Moreover, this gives **all** *prop. disc. actions on $\mathfrak{g}$ up to multiplying u by -1.*

The second condition (involving $\mathrm{d}(\lambda \circ j_t)/\mathrm{d}t$), together with the Strips theorem, implies that (j, u) acts properly discontinuously on $\mathbb{R}^{2,1}$ if and only if $(-u)$ is an infinitesimal strip deformation of j.
In fact, it is straightforward to interpret the relative motions of tiles in a strip deformation as vectors of $\mathbb{R}^{2,1} = \mathrm{Kill}(\mathbb{H}^2)$ that push crooked planes off themselves. Thus

Corollary

All Margulis spacetimes admit fundamental domains bounded by crooked planes. In particular, all Margulis spacetimes are handlebodies ("tameness").

4. A surface group acting properly discontinuously on $\mathbb{R}^6$

Margulis's examples rely on "ping-pong" (free groups).

Question

In dimension $n > 3$, can there be a non-virtually solvable, non-virtually free group Γ acting faithfully, properly discontinuously on $(\mathbb{R}^n, \mathrm{Aff}(\mathbb{R}^n))$?

We may use the third criterion: if $(j, u) : \Gamma \to T(\mathrm{Isom}(\mathbb{H}^k))$ admits a **contracting, equivariant** vector field Y on $\mathbb{H}^k$, then the projection

$$\begin{array}{rccc} \mathfrak{so}_k \hookrightarrow \mathfrak{so}_{k,1} \simeq & \mathrm{Kill}(\mathbb{H}^k) & \ni & X \\ & \downarrow \pi & & \downarrow \\ & \mathbb{H}^k & \ni & \mathrm{Fix}(Y - X) \end{array}$$

is equivariant for the Γ-actions (j, u) and j, hence the action on the source $\mathfrak{so}_{k,1}$ is properly discontinuous because the action on the base $\mathbb{H}^k$ is.

We now exhibit a **Coxeter group** acting on $\mathbb{H}^3$ that admits such a cocycle u and contracting vector field Y. Fix $N \geq 3$. Let

$$\Gamma := \left\langle (\sigma_k)_{k \in \mathbb{Z}/2N\mathbb{Z}} \;\middle|\; \sigma_k^2 = 1 = [\sigma_k, \sigma_{k+1}], \forall k \right\rangle$$

be the group of reflections in the edges of a convex right-angled $2N$-gon (Γ has surface subgroups of finite index, hence is **not** virtually free). For small $t > 0$, taking $\sigma_k \in \Gamma$ to the reflection of $\mathbb{R}^{3,1}$ in the plane

$$S_k(t) := \begin{pmatrix} A(t)\cos\frac{k\pi}{N} \\ A(t)\sin\frac{k\pi}{N} \\ B(t)(-1)^k \\ 1 \end{pmatrix}^{\perp} \quad \text{where} \quad \begin{cases} A(t) &= \cosh(t)/\sqrt{\cos\frac{\pi}{N}} \\ B(t) &= \sinh(t) \end{cases}$$

yields a representation $j_t : \Gamma \to \mathrm{Isom}(\mathbb{H}^3)$ (indeed $S_k \perp S_{k+1}$) acting properly discontinuously on $\mathbb{H}^3$. Here $j_0(\Gamma)$ preserves a copy of $\mathbb{H}^2$.

The polyhedron D_t of $\mathbb{P}^3\mathbb{R}$ bounded by the $S_k(t)$ satisfies for all $0 < t_0 < t$

$$D_t = M_{t,t_0} D_{t_0} \quad \text{where} \quad M_{t,t_0} = \begin{pmatrix} \frac{\cosh t_0}{\cosh t} & & & \\ & \frac{\cosh t_0}{\cosh t} & & \\ & & \frac{\sinh t_0}{\sinh t} & \\ & & & 1 \end{pmatrix}.$$

The projective transformation M_{t,t_0} takes the ellipsoid $\mathbb{H}^3 \subset \mathbb{P}^3\mathbb{R}$ to a **strictly smaller** ellipsoid, hence (by definition of the Hilbert metric) is **contracting** on $\mathbb{H}^3$. The map $M_{t,t_0}|_{D_{t_0} \cap \mathbb{H}^3}$ can be extended (j_{t_0}, j_t)-equivariantly to all of $\mathbb{H}^3$. The resulting maps $\mathcal{M}_{t,t_0}$ have a time-derivative

$$Y = \left. \frac{\mathrm{d}\mathcal{M}_{t,t_0}}{\mathrm{d}t} \right|_{t=t_0}$$

which is a contracting vector field on $\mathbb{H}^3$, and (j_{t_0}, u)-equivariant for $u = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_0} (j_t {j_{t_0}}^{-1})$ as desired. Hence Γ acts properly discontinuously on $\mathfrak{so}_{3,1} \simeq \mathbb{R}^{3,3}$ via (j_{t_0}, u).

Question

Which other subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ have contracting deformations in some $\mathrm{Isom}(\mathbb{H}^m)$, $m \geq n$?

This would yield interesting properly discontinuous affine actions: there are not so many (non-free) examples when one studies Auslander's conjecture!

Thank you!