Geometry and combinatorics of veering triangulations and Cannon-Thurston maps

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Overview

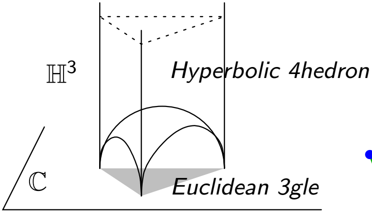
1. Introduction

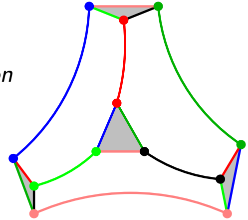
Definition

Ideal tetrahedron: convex hull of 4 non-coplanar points in $\partial_\infty \mathbb{H}^3$.

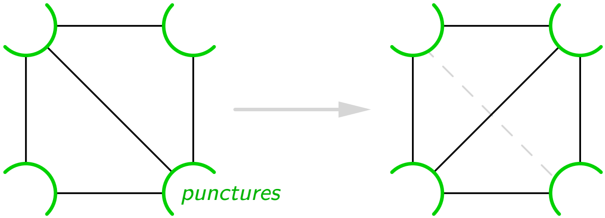
Useful to construct and "manipulate" hyperbolic (cusped) 3-manifolds:

- $\mathbb{S}^3 \setminus$ Fig.-8 knot $\simeq$ gluing of 2 regular ideal 4hedra (Thurston);
- SnapPea: powerful 3-manifold software (Weeks).
- Ideal triangulations can also be studied from a strictly combinatorial viewpoint.





Example of combinatorics: **layered triangulations** of a surface bundle.
Let $\varphi : S \to S$ be a mapping (class) of a punctured surface S.
Given a triangulation τ of S, we can go from τ to $\varphi_*(\tau)$ by a sequence of **diagonal exchanges**.

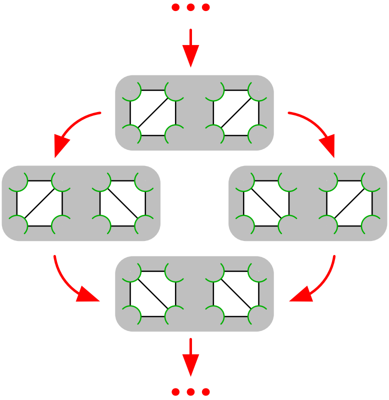


Each such exchange can be seen as a (flattened) ideal tetrahedron.
Yields an **ideal triangulation** of the **mapping torus**

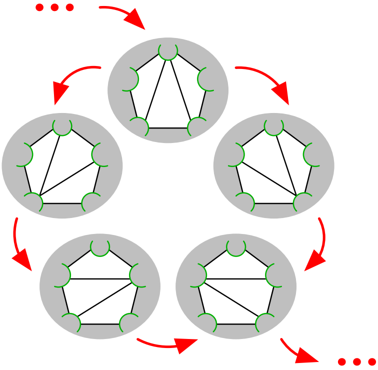
$$M_\varphi := \frac{S \times [0,1]}{\langle (x,1) \sim (\varphi(x),0) \rangle}$$

(provided τ and $\varphi_*^m(\tau)$ share no arc, e.g. φ pseudo-Anosov).

Depending on τ and on the path $\tau \to \cdots \to \varphi_*(\tau)$, there are *many* layered triangulations of a given M_φ.



Square move



Pentagon move

Hyperbolic geometry

Theorem (Thurston)

If $\varphi : S \to S$ is pseudo-Anosov, then M_φ admits a hyperbolic metric g.

Question

Can we find a layered triangulations T of M_φ that is nicer than others?
Can we relate T to the hyperbolic metric g?

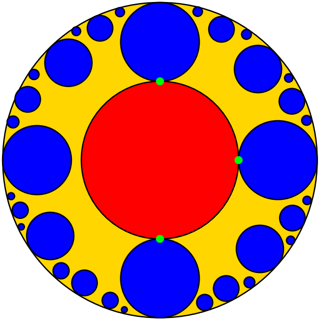
For example:

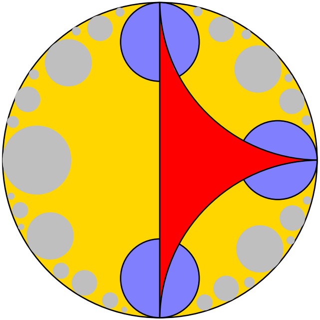
- Can we realize T by geodesic tetrahedra?
- Is the *canonical* Ford-Voronoi-Delaunay triangulation layered?
- Can we find T (layered) that relates to other aspects of g, such as the Cannon-Thurston sphere-filling curve (defined later)?

Recall:

The canonical (Delaunay) cell decomposition of a cusped finite-volume hyperbolic 3-manifold M is given by:

- lifting a small cusp neighborhood to horoballs $(H_i)_{i\in I}$ of $\mathbb{H}^3$;
- finding a maximal ball in $\mathbb{H}^3 \smallsetminus \bigcup_{i\in I} H_i$ (tangent to 4 or more H_i);
- taking the convex hull of the centers of these H_i;
- repeating for all maximal balls, and projecting back to M.

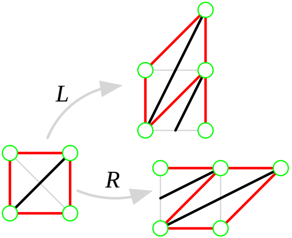




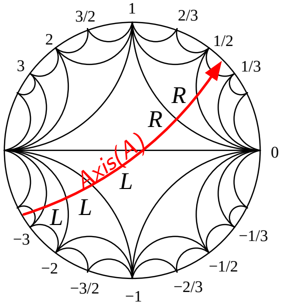
Example

If the fiber S of M_φ is a **once punctured torus**, a layered triangulation T of M_φ arises from $\mathrm{SL}_2(\mathbb{Z})$ / Farey / continued-fraction combinatorics:

Fact: any $A \in \mathrm{SL}_2(\mathbb{Z})$ with $|\mathrm{Tr}(A)| > 2$ is conjugate to $\pm\omega$ for ω a unique cyclic word in $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.



$S \simeq (\mathbb{R}^2 \smallsetminus \mathbb{Z}^2)/\mathbb{Z}^2$



This triangulation $T = T_\varphi$ of an S-bundle M_φ does have all nice features:

Theorem A (Lackenby; G.)

(M_φ, T_φ) is realized geodesically and coincides with the canonical Delaunay triangulation.

... but this fails in various ways for higher-complexity fibers S (Schleimer, Segerman, ...).

Theorem B (Cannon–Dicks; Dicks–Sakuma)

There is a dictionary between the combinatorics of T_φ, and of the Cannon-Thurston sphere-filling curve attached to M_φ.

Today's theme: **Theorem B** survives in higher complexity! (Of course we need to define "T_φ".)

2. The Cannon-Thurston sphere-filling curve

Definition

A **semi-translation structure** on a surface S is an atlas over $\mathbb{R}^2$ with charts valued in $\{\mathrm{Id}, -\mathrm{Id}\} \ltimes \mathbb{R}^2$, the semi-translation group, possibly with:

- cone singularities of angle $m\pi$ where $m \geq 3$, and
- punctures surrounded by an angle $m\pi$ where $m \geq 1$.

Let S be a surface and $M = M_\varphi$ a hyperbolic S-bundle over the circle, of monodromy $\varphi : S \to S$.

Theorem (Thurston)

There exists $\alpha > 1$ and an essentially unique semi-translation structure on S such that φ is given in all charts by $\pm \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix}$.

The axis directions induce two **foliations** λ^+, λ^- (stable/unstable) of S, preserved by φ.

We can lift any injection $S \hookrightarrow M$ (in the correct homotopy class) to universal covers:

$$\begin{array}{ccccccc} \mathbb{H}^2 & \simeq & \Sigma & \overset{\iota}{\hookrightarrow} & \widetilde{M} & \simeq & \mathbb{H}^3 \\ & & \downarrow & & \downarrow & & \\ & & S & \hookrightarrow & M. & & \end{array}$$

Theorem (Cannon-Thurston; Bowditch)

The injection ι extends continuously to a **surjective** map

$$\bar{\iota} \quad : \quad \mathbb{S}^1 \simeq \partial_\infty \Sigma \quad \longrightarrow \quad \partial_\infty \mathbb{H}^3 \simeq \mathbb{S}^2.$$

The map $\bar{\iota}$ collapses the endpoints of any leaf of the stable or unstable foliation of φ, and this generates all identifications occurring under $\bar{\iota}$.

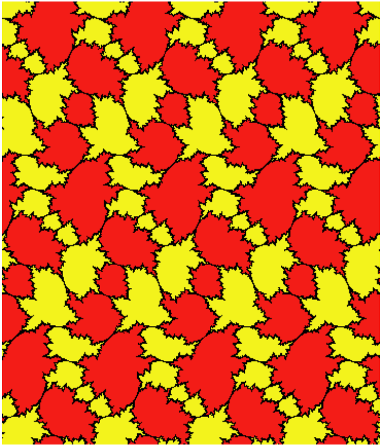
Visualizing the sphere-filling curve $(\bar{\iota}(t))_{t\in\partial_\infty\mathbb{H}^2}$ is a challenge.

Dicks's idea

Assuming S has punctures, select a parabolic fixed point $\Omega \in \partial_\infty\mathbb{H}^3$ and consider $\bar{\iota}$ changes colors each time it goes through Ω.

$S =$ punctured torus

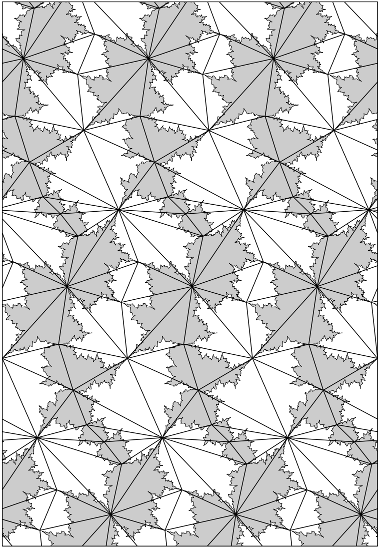
$\varphi = RL^2R^3L^4$



Picture by Warren Dicks

Here is a superposition with the canonical triangulation T_φ, in the same perspective from $\Omega = \infty$.

Vertex sets coincide!



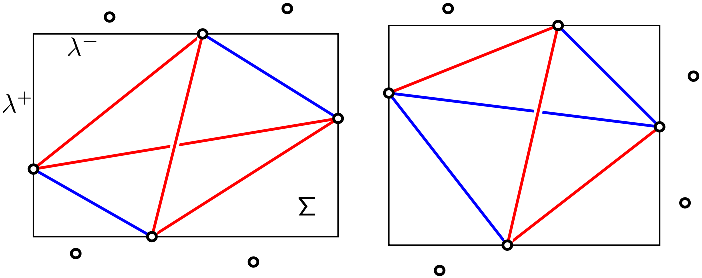
(W. Dicks, M. Sakuma)

3. The Agol triangulation

In 2010, Agol (crediting some ideas to Hamenstädt) constructed a layered ideal triangulation T_φ^A naturally associated to a hyperbolic mapping torus M_φ, **provided all singularities of $(S, \lambda^+, \lambda^-)$ occur at punctures.**

Theorem (G.)

Agol's tetrahedra are obtained by connecting the punctures in the boundary of every **maximal puncture-free rectangles** in the flat foliated surface $(\Sigma, \widetilde{\lambda^+}, \widetilde{\lambda^-})$.

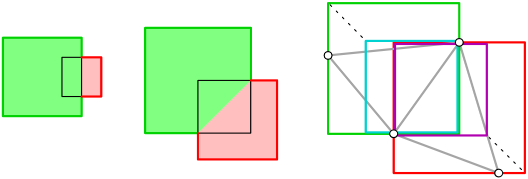


Observation

This triangulation T_{φ}^{A} of M_{φ} has a natural layered structure.

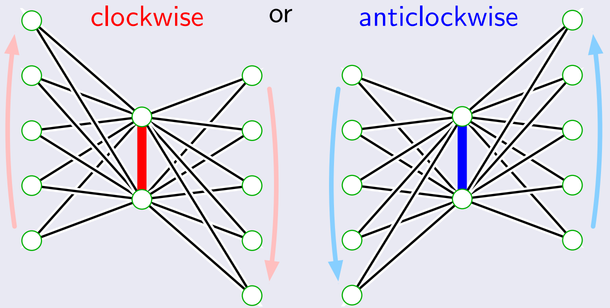
The t-layer ($t \in \mathbb{R}$) is the **square-Delaunay** triangulation of the flat surface $\left(S, \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} g\right)$.

Idea: maximal squares (for generic t) contain 3 punctures in their boundary, spanning a triangle. These triangles do not overlap, and any edge is shared by 2 triangles. Hence they triangulate S. For special t, a diagonal exchange occurs.



T_φ^A enjoys another property: it is **veering**.

A layered triangulation is **veering** (Agol) if any given edge, over its lifespan, sees adjacent triangles shift always towards the same direction:



• Veeringness actually makes sense for any **orientable** triangulation (not necessarily layered) endowed with a $\{0, \pi\}$-valued **angle structure**.

• Here, the color is given by the sign of the **slope** ($+/-$) in the $\frac{1}{2}$-translation surface $(S, \lambda^+, \lambda^-)$.

Properties of veering triangulations:

Theorem (Hodgson, Rubinstein, Segerman, Tillmann)

Any veering triangulation admits a $(0, \pi)$-valued angle structure, i.e. the tetrahedra can be endowed with dihedral angles modelled on those of ideal tetrahedra of $\mathbb{H}^3$, summing to 2π around each edge.

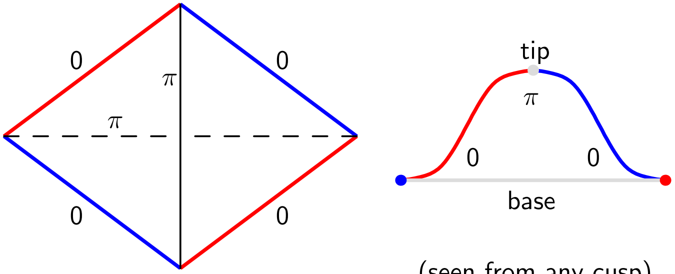
Corollary (Lackenby): any 3-manifold admitting a veering triangulation admits a hyperbolic metric.

Theorem (Futer, G.)

The smallest angle can be made $\geq \dfrac{\pi}{12d^2}$ where d is the largest edge degree. The exponent 2 is optimal (already for punctured torus bundles).

This can be used to give bounds on the number of normal surfaces in terms of their genus.

Combinatorics of veering triangulations (1): one tetrahedron

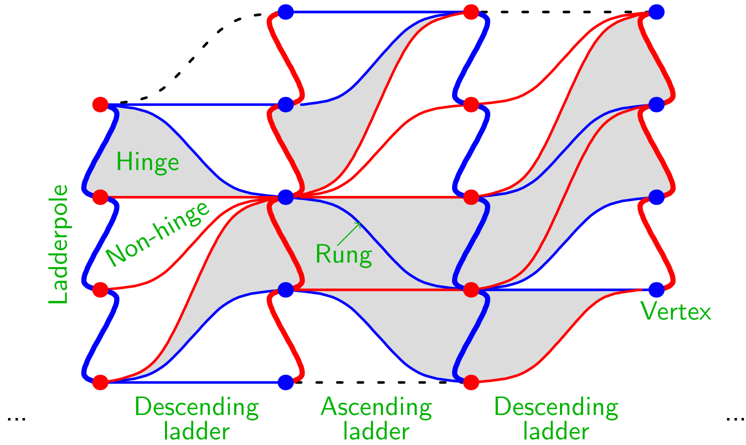


(seen from any cusp)

The tetrahedron is called

- **hinge** if the tip and base ore opposite colors;
- **non-hinge** if the tip and base are the same color.

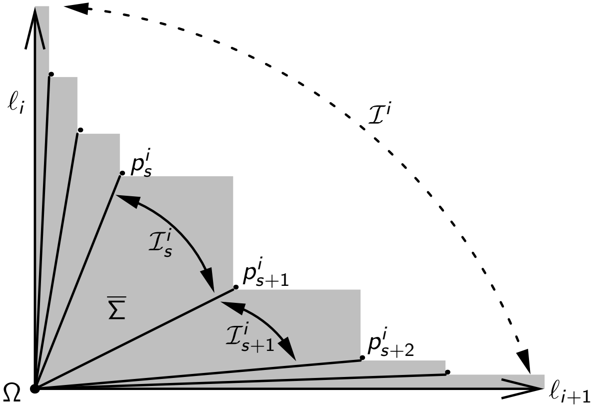
Combinatorics of veering triangulations (2): full cusp link $\mathbb{T}^2$ at Ω.



Each item corresponds to some rectangle(s) in $\overline{\Sigma}$ touching Ω:

e.g. **Triangles** $\leftrightarrow$ Maximal rectangles touching Ω;
Vertices $\leftrightarrow$ Rectangles with 2 singular corners Ω, p.

Agol Vertices $\leftrightarrow$ Rectangles with 2 (opposite) singular corners Ω, p.

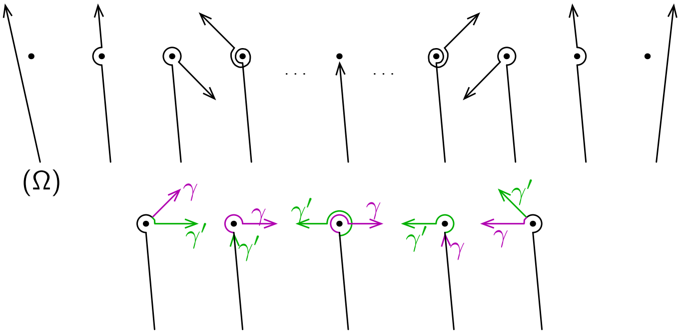


These so-called **ruling rectangles** form several $\mathbb{Z}$-families, one per quadrant $\mathcal{I}$ at Ω.

4. Connecting the combinatorics

- $(S, \varphi, \lambda^{\pm})$ pseudo-Anosov data; S punctured at the singularities.
- Let $\overline{\Sigma}$ be the metric completion of the universal cover of the fiber S, and Ω a singularity of $\overline{\Sigma}$.
- We can identify the domain $\partial_\infty \widetilde{S} \simeq \mathbb{S}^1$ of the Cannon-Thurston map $\bar{\iota}$ with the **space of $\overline{\Sigma}$-geodesic paths issued from Ω** (possibly terminating at a singularity):

Progressing clockwise in the space $\mathbb{S}^1$ of paths



γ' departs from γ to the right (after common prefix)

By the Cannon-Thurston-Bowditch theorem, the paths $\gamma \in \mathbb{S}^1$ mapped to

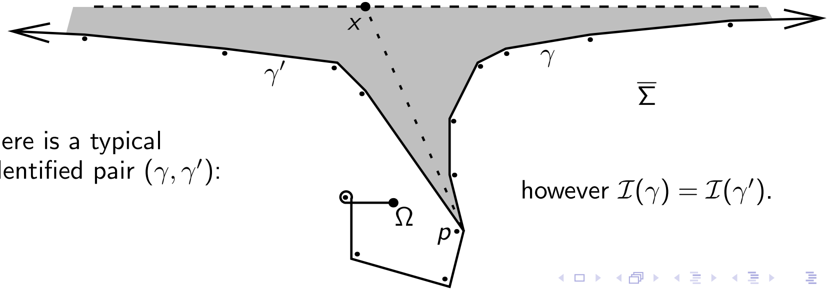
$$\bar{\iota}(\Omega) = \infty \in \mathbb{P}^1\mathbb{C} \simeq \partial_\infty \mathbb{H}^3$$

are exactly the leaves $\{\ell_i\}_{i\in\mathbb{Z}}$ of $\widetilde{\lambda^\pm}$ issued from Ω.
Hence, if $\mathcal{I}(\gamma)$ denotes the quadrant at Ω that $\gamma \in \mathbb{S}^1$ starts off into,

Red/Blue interface in the Dicks coloring of $\mathbb{P}^1\mathbb{C} \simeq \partial_\infty \mathbb{H}^3$

$$\simeq$$

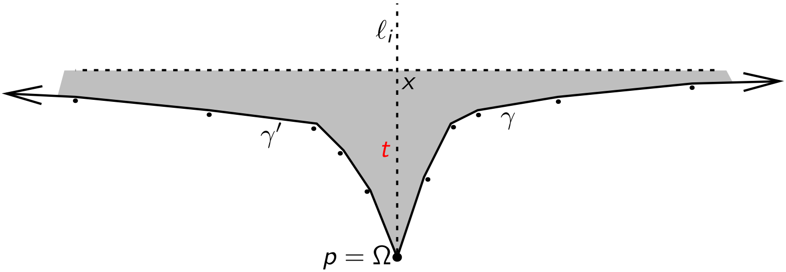
Pairs of paths $\gamma, \gamma' \in \mathbb{S}^1$ identified under $\bar{\iota}$ **but** satisfying $\mathcal{I}(\gamma) \neq \mathcal{I}(\gamma')$.



Here is a typical identified pair (γ, γ'):

however $\mathcal{I}(\gamma) = \mathcal{I}(\gamma')$.

$\mathcal{I}(\gamma) \neq \mathcal{I}(\gamma')$ can in fact only occur if:

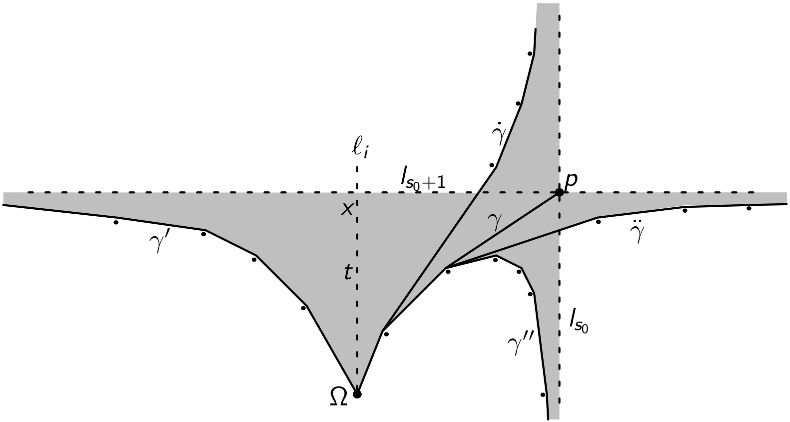


γ, γ' are (the straightenings of) the paths $\Gamma_i^+(t), \Gamma_i^-(t)$ obtained by following some **segment** of length $t \geq 0$ of $\lambda^{\pm}$ (the i-th leaf issued from Ω), then shooting off along **rays** of $\lambda^{\mp}$.

Consequence

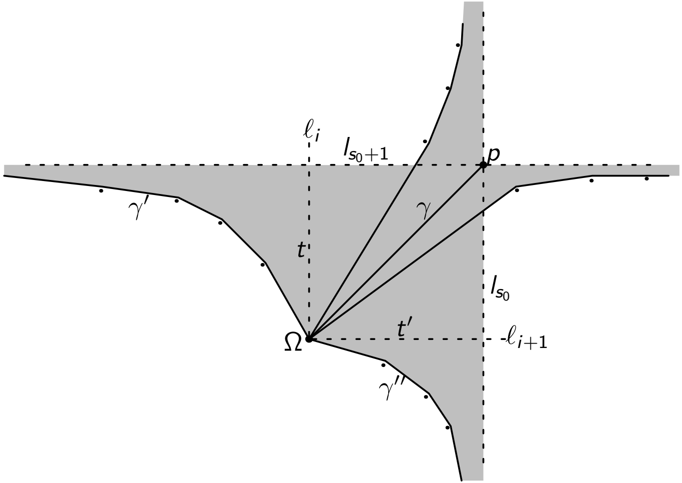
The Red/Blue **interface** in the Dicks coloring of $\mathbb{P}^1\mathbb{C} \simeq \partial_\infty \mathbb{H}^3$ is a $\mathbb{Z}$-collection of **curves** $\quad J_i = \{\bar{\iota}([\Gamma_i^{\pm}(t)])\}_{t \in \overline{\mathbb{R}^+}}$.
Actually **Jordan curves**, by interrogating the Cannon-Thurston-Bowditch criterion for identification under $\bar{\iota}$.

If γ terminates at some singular $p \in \overline{\Sigma}$, then (CTB criterion) **more** than 2 paths are identified under $\bar{\iota}$:



(cf "thorns" in Dicks's exact pictures.)

Exceptionally, one of these extra paths belongs to a **third** quadrant:



This happens exactly for $[\Omega, p]$ a **ruling segment**, and corresponds to intersections $J_i \cap J_{i+1}$ of the Jordan-curve separators $(J_i)_{i\in\mathbb{Z}}$.

Conclusion: the Dicks coloring of $\mathbb{P}^1\mathbb{C} \simeq \partial_\infty \mathbb{H}^3$ has the following aspect.

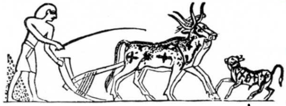
$\delta = \bar{\iota}([\Omega p_s^i, \Omega p_{s+1}^i])$

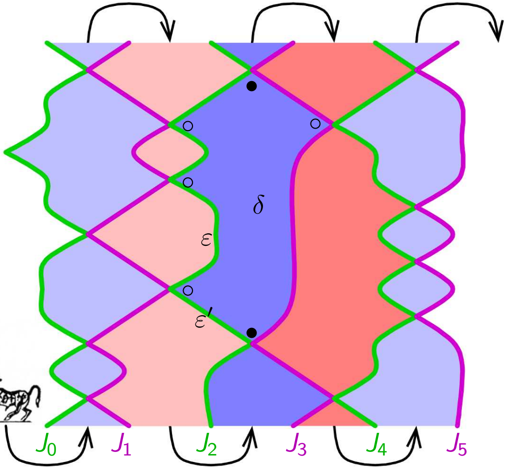
ε in-furrow edge

ε' cross-furrow edge

• gate of δ

∘ spike of δ





The "ox" $\bar{\iota}$ plows the "**furrows**" (solid colors) in sequence.

Vertices, **as in the Agol triangulation**, correspond to ruling segments.

Working out the correpondence more carefully, we get:

Theorem (G.)

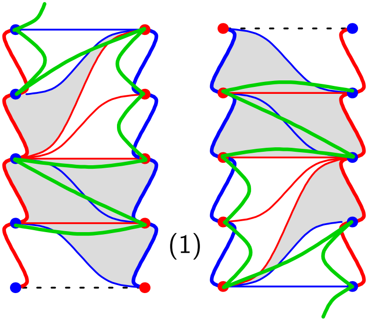
The following are in natural bijection (□ denoting a rectangle):

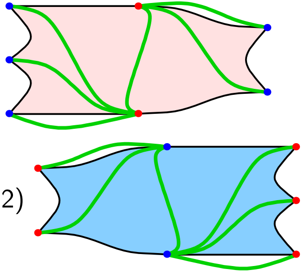
Dicks coloring	Geometry of $(\overline{\Sigma}, \Omega)$	Agol triangulation
Furrows	Quadrants	Ladderpoles
Vertices	□ w. 2 sing. corners Ω, p	Vertices
Disks	□ w. 3 sing., Ω in corner	Ladderpole edges
Spikes	□ w. 3 sing., Ω *not* "	Rungs
In-furrow } edges	□ w. 4 sing., { 4 + 2 edges	Non-hinge } triangles.
Cross-furrow } edges	□ w. 4 sing., { 3 + 3 "	Hinge } triangles.

Any Cannon-Thurston-Dicks edge is isotopic to an edge of the corresponding Agol triangle. In fact:

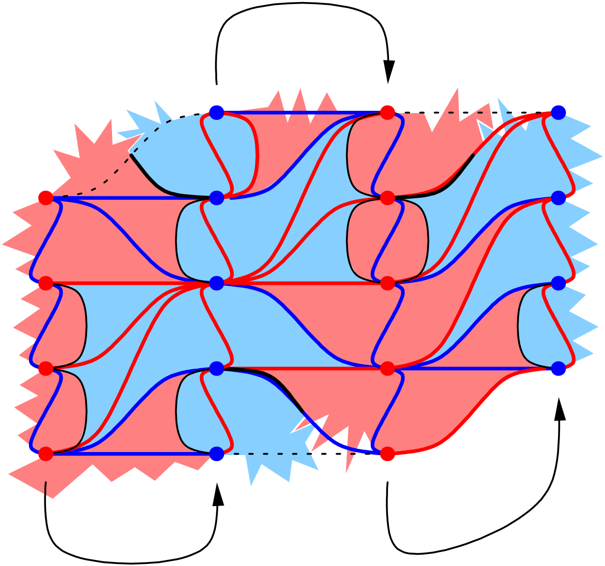
Theorem (G.)

(1) Given the Agol triangulation Δ, the 1-skeleton of the Dicks coloring is obtained by drawing, for each triangle of Δ, an arc from its tip to the tip of the next triangle across the base rung (keep any resulting double edges).
(2) Conversely, given the Dicks coloring, we can obtain the 1-skeleton of Δ by adding edges connecting each gate of a blue (resp. red) cell to all the vertices clockwise (resp. counterclockwise) until the other gate, and deleting redundant edges.

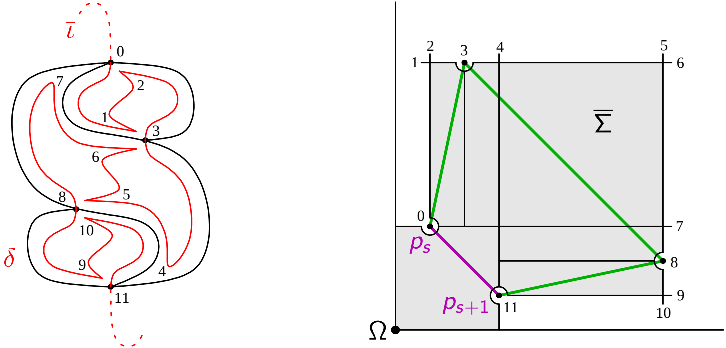

(1)


(2)

Overlay of the two tessellations:

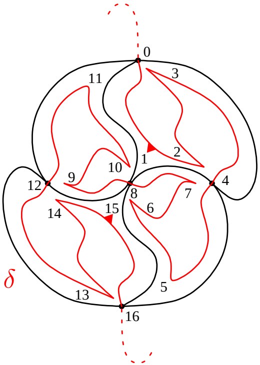


the Agol triangulation, via its tetrahedra *not* incident to Ω, also governs further subdivisions of the image of the Cannon-Thurston map $\bar{\iota}$.
Pick a pair of consecutive ruling singularities p_s, p_{s+1} and call **green** the outer edges of tetrahedra containing $[p_s, p_{s+1}]$.



The Dicks 2-cell $\delta = \{\bar{\iota}(\gamma)\}_{\gamma \in \mathcal{I}_s^i}$ is subdivided according to which green edge the path $\gamma \in \mathbb{S}^1$, issued from $\Omega \in \overline{\Sigma}$, crosses next after $[p_s, p_{s+1}]$.
There could be just **3** green edges...

… or **4**…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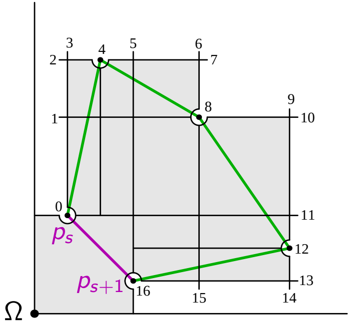
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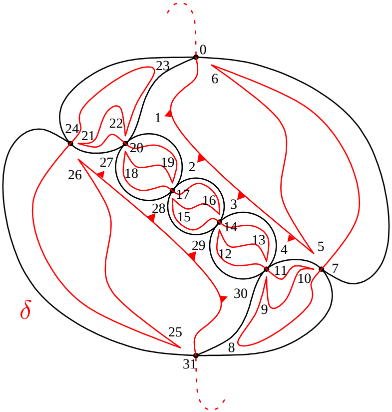
… or **more**.



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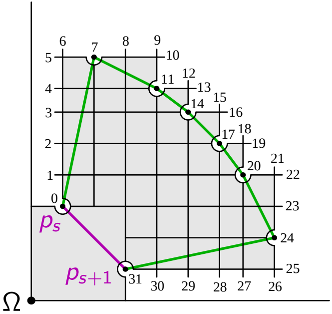

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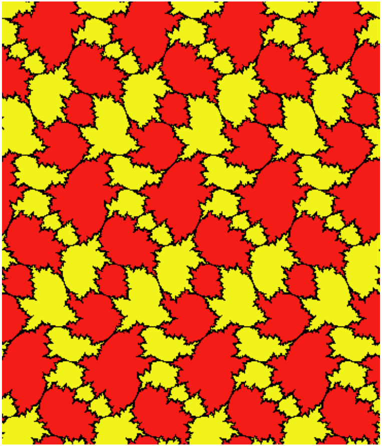
Annex 1: Punctured-torus combinatorics

$\varphi = R\,LL\,RRR\,LLLL$



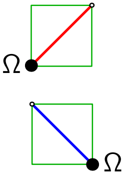
Picture W. Dicks

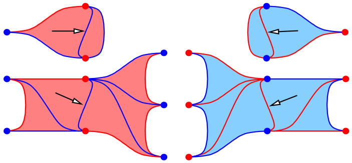
letters R, L $\leftrightarrow$ colored regions

$RL^m\boxed{R}L^nR_{\,(m,n\geq 0)}$ $\leftrightarrow$ red region with $m+n$ spikes.

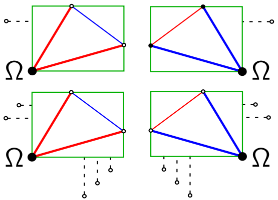
Annex 2: detailed combinatorial correspondence.

Dicks coloring | **Flat surface $\overline{\Sigma}$** | **Agol triangulation**

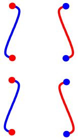




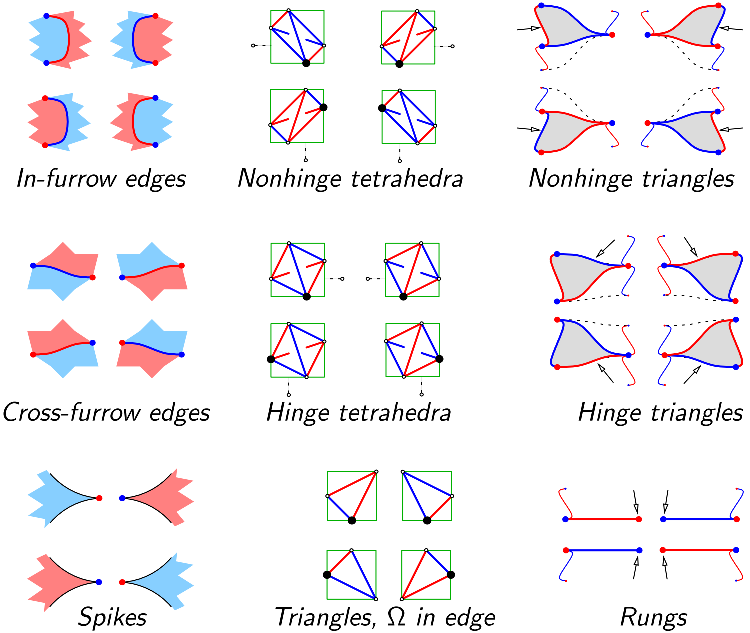
2-cells



Triangles,
Ω in corner



Ladderpole edges


In-furrow edges
Nonhinge tetrahedra
Nonhinge triangles
Cross-furrow edges
Hinge tetrahedra
Hinge triangles
Spikes
Triangles, Ω in edge
Rungs

Future work

Study analogues where the mapping torus M_φ is replaced with any hyperbolic 3-manifold endowed with a **pseudo-Anosov flow**. (Agol)

The End.

Bon anniversaire !