Lorentzian 3-manifolds of constant curvature.

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A Lorentzian 3-manifold is one endowed with a field of symmetric 2-forms of signature $(+,+,-)$.

Examples:

- $\mathbb{R}^{2,1} \simeq (\mathfrak{sl}_2\mathbb{R}, \mathrm{Kill})$ (Minkowski space) : curvature 0.
- $\mathrm{AdS} := (\mathrm{PSL}_2\mathbb{R}, \mathrm{Kill})$ (Anti-de Sitter space) : curvature <0.
- $\mathrm{dS} := \{\text{Planes in } \mathbb{H}^3\}$: constant positive curvature.

The first 2 have interesting quotients, e.g. by free groups.

MAIN QUESTION: Are $\mathbb{R}^{2,1}$-quotients renormalized limits of collapsing AdS quotients? (Yes!)

Let $G := \mathrm{PSL}_2\mathbb{R} = \mathrm{SO}^\circ_{2,1}\mathbb{R} = \mathrm{AdS} = \mathrm{Isom}^\circ(\mathbb{H}^2)$

$\mathfrak{g} := \mathfrak{sl}_2\mathbb{R} = \mathfrak{so}_{2,1}\mathbb{R} = \mathbb{R}^{2+1} = \{\text{Killing fields on } \mathbb{H}^2\}$

$\mathrm{Isom}^\circ(\mathrm{AdS}) \simeq G \times G \quad : \quad (\gamma, \gamma') \cdot g = \gamma' g \gamma^{-1}$

$\mathrm{Isom}^\circ(\mathbb{R}^{2+1}) \simeq G \ltimes \mathfrak{g} \quad : \quad (\gamma, u) \cdot x = \mathrm{Ad}(\gamma)x + u$

(*More later, with pictures!*)

Let Γ be a discrete group.

$\boxed{\text{THM 1}}$ Let $(j,\rho) : \Gamma \longrightarrow G \times G = \mathrm{Isom}^\circ(\mathrm{AdS})$ be a representation with j Fuchsian.

Suppose $\exists\, f : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ such that :

- f is (j,ρ)-equivariant : $f(j(\gamma)\cdot p) = \rho(\gamma)\cdot f(p)$;
- f is C-Lipschitz with $C<1$: $d(f(p), f(q)) \leqslant C\, d(p,q)$.

Then $(j,\rho)(\Gamma)$ acts properly on AdS, with quotient a bundle of geodesic circles over the hyperbolic surface $j(\Gamma)\backslash\mathbb{H}^2$.

[Example : $\rho = \{1\} \rightsquigarrow$ unit tangent bundle !]

MOREOVER, all proper quotients of AdS are virtually of this form.

What happens as $\rho \longrightarrow j$ ... ?

$\boxed{\text{THM 2}}$ Let $(j,u) : \Gamma \longrightarrow G \ltimes \mathfrak{g} = \mathrm{Isom}^\circ(\mathbb{R}^{2+1})$ be a representation with j Fuchsian.

Suppose $\exists$ vector field X on $\mathbb{H}^2$ such that :

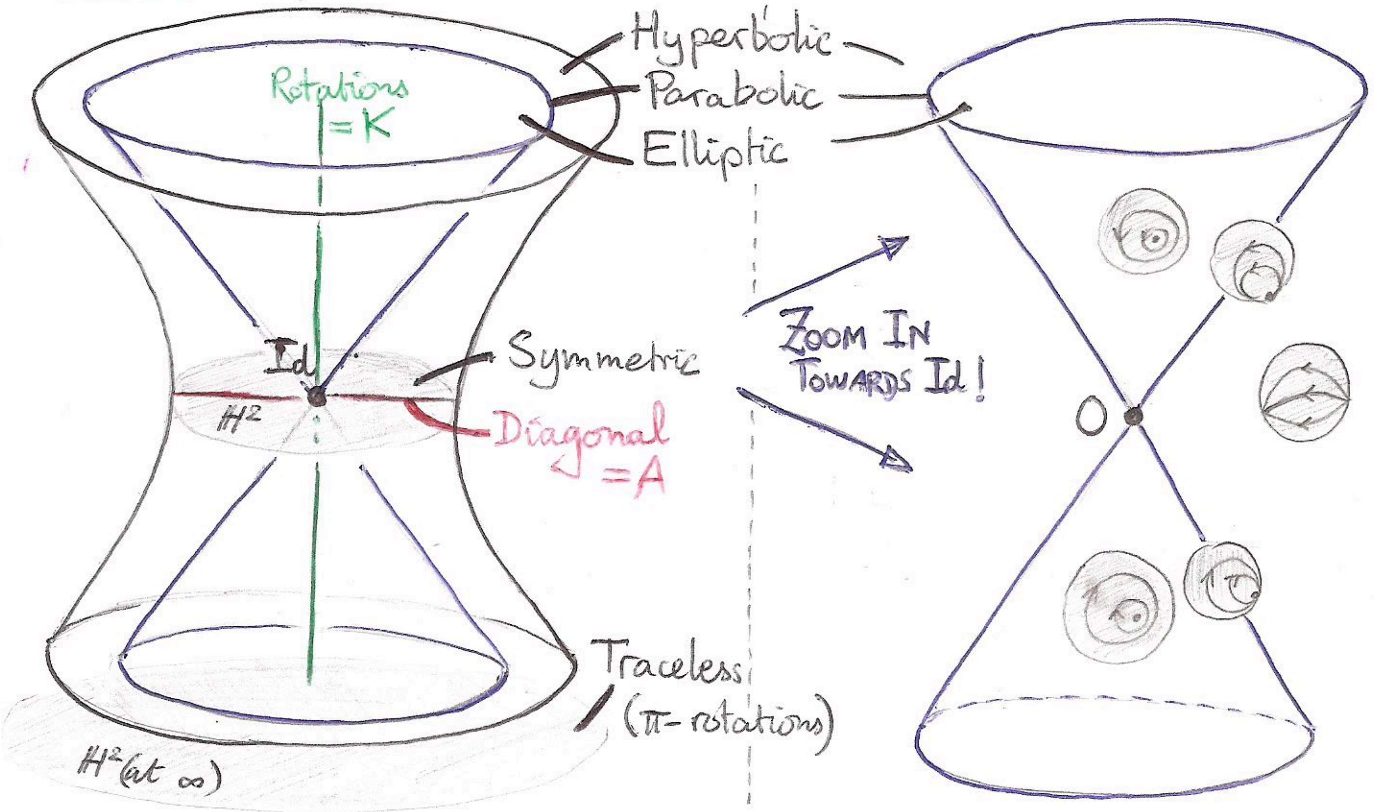
- X is (j,u)-equivariant : $X(j(\gamma)\cdot p) = j(\gamma)_* X(p) + u(\gamma)(j(\gamma)\cdot p)$;
- X is c-contracting with $c<0$: $\underbrace{\frac{d}{dt}\Big|_{t=0} d(\exp_p tX(p), \exp_q tX(q))}_{=:\, d'_X(p,q)} \leqslant c\, d(p,q)$.

Then $(j,u)(\Gamma)$ acts properly on $\mathbb{R}^{2+1}$, with quotient a bundle of geodesic lines over the hyperbolic surface (with boundary) $j(\Gamma)\backslash\mathbb{H}^2$.

MOREOVER, all proper quotients of $\mathbb{R}^{2+1}$ are virtually of this form, provided their linear part j is convex-cocompact.

$\boxed{\text{AdS}} = PSL_2\mathbb{R} \subset \mathbb{P}^3\mathbb{R}$ $\quad = \mathrm{Isom}^\circ \mathbb{H}^2$

$\boxed{\text{Minkowski}}$ space: $\mathbb{R}^{2+1} \simeq \{\text{Killing fields on } \mathbb{H}^2\}$



Conjugation fixes Id. Adjoint action fixes O.

Lines in AdS are of 3 types:

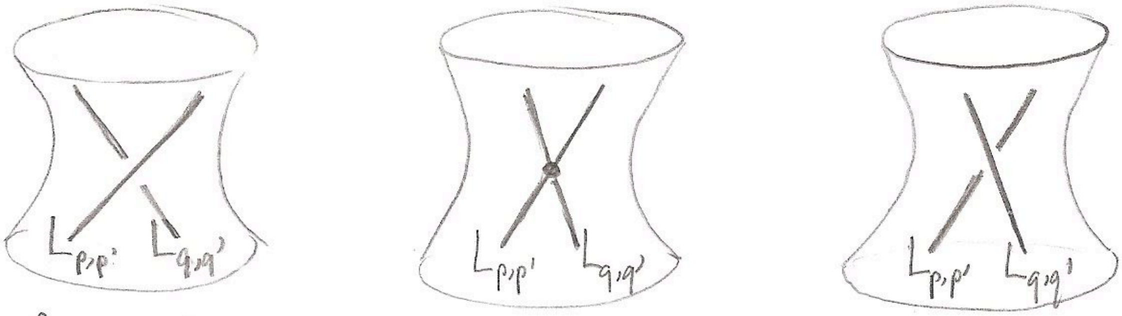
- $L_{p,p'} := \{g \in \mathrm{Isom}\,\mathbb{H}^2 \mid g(p) = p'\}$ for $p, p' \in \mathbb{H}^2$ (timelike: e.g. K)
  $L_{\ell,\ell'} := \{ \text{———} \; g(\ell) = \ell'\}$ — $\ell, \ell' \subset \mathbb{H}^2$ oriented lines (spacelike: e.g. A)
  $L_{\eta,\eta'} := \{ \text{———} \; g(\eta) = \eta'\}$ — $\eta, \eta' \subset \mathbb{H}^2$ horospheres (lightlike).

Similarly in Minkowski space: given $v \in T_p\mathbb{H}^2$,

- $L_{p,v} := \{X \text{ Killing field on } \mathbb{H}^2 \mid X(p) = v\}$ is a timelike line of $\mathbb{R}^{2+1}$
  etc.

Veeringness: configurations of timelike lines.

AdS

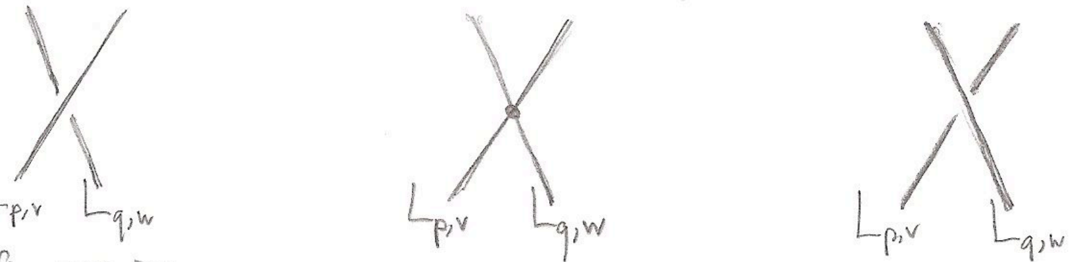


Left-veering: $d(p',q') < d(p,q)$

Intersecting: $d(p',q') = d(p,q)$

Right-veering: $d(p',q') > d(p,q)$

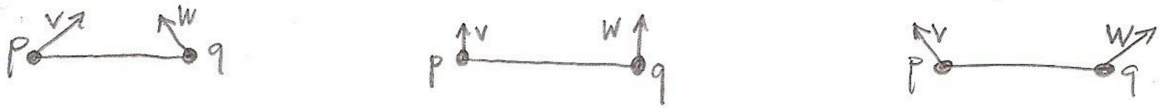
$\mathbb{R}^{2,1}$



Left-veering: $d'_{\{v,w\}}(p,q) < 0$

Intersecting: $d'_{\{v,w\}}(p,q) = 0$

Right-veering: $d'_{\{v,w\}}(p,q) > 0$



Proof of Thm 1 ($\Longrightarrow$)

Suppose $f : \mathbb{H}^2_j \xrightarrow{(C<1)\text{-Lip.}} \mathbb{H}^2_\rho$.

Aim: prove that $(j,\rho)(\Gamma)$ acts properly on $G = PSL_2$.

- The timelike lines

$$\{L_{p,f(p)}\}_{p\in\mathbb{H}^2} = \{g \in AdS \mid g(p) = f(p)\}_{p\in\mathbb{H}^2}$$

are pairwise left-veering (hence disjoint!)

since $d(f(p), f(q)) < d(p,q)$.

- Moreover, the $L_{p,f(p)}$ foliate AdS :

$$g \in L_{p,f(p)} \iff g(p) = f(p) \iff p \in \mathrm{Fix}\,(g^{-1} \circ f)$$

holds for a unique $p \in \mathbb{H}^2$ since $g^{-1} \circ f$ is contracting.

- The map $\mathrm{AdS} \xrightarrow{\pi} \mathbb{H}^2$ collapsing $L_{p,f(p)}$ to p is naturally equivariant for the (j,ρ)- and j- actions. Since $j(\Gamma)$ acts properly on the target $\mathbb{H}^2$,

  $(j,\rho)(\Gamma)$ ———————— source AdS

  and π projects to a fibration into geodesic circles. $\square$

$$\begin{tikzcd} L_{p,f(p)} \arrow[r, hook] & (j,\rho)(\Gamma)\backslash \mathrm{AdS} \arrow[d, "\pi"] \\ & S := j(\Gamma)\backslash \mathbb{H}^2 \end{tikzcd}$$

NB : This circle bundle is oriented (past $\to$ future), hence trivial if $\partial S \neq \emptyset$.
If $\partial S = \emptyset$, its Euler class is that of ρ.

Proof of Thm 2 ($\Rightarrow$)

Identical! Just substitute $L_{p,f(p)} \rightsquigarrow L_{p,X(p)}$
$$= \{ Y \in \mathrm{Kill}\,(\mathbb{H}^2) \mid Y(p) = X(p) \}$$

and " Contracting maps $g^{-1} \circ f$ fix a point "
$\rightsquigarrow$ " Inward-pointing vector fields $X_{\mathrm{Contr.}} - Y_{\mathrm{Kill}}$ have a zero ".

NB: The surface $S = j(\Gamma)\backslash\mathbb{H}^2$ must have boundary [Goldman-Fried], hence the oriented line bundle

$$\begin{array}{ccc} L_{\rho, X(\rho)} & \hookrightarrow & (j,u)(\Gamma)\backslash\mathbb{R}^{2+1} \\ & & \downarrow \\ & & j(\Gamma)\backslash\mathbb{H}^2 = S \end{array}$$

is trivial.

COR: these spacetimes $(j,u)(\Gamma)\backslash\mathbb{R}^{2+1}$ are topologically handlebodies $\simeq S \times \mathbb{R}$ [TAMENESS: of Choi/Goldman]

PROOF OF THM 1 ($\Leftarrow$)

Up to switching $j \leftrightarrow \rho$, we shall assume $\rho(\alpha)$ has smaller hyperbolic translation length than $j(\alpha)$, for at least some $\alpha \in \Gamma$.

To complete the proof of THM 1, we only need to assume the infimal Lipschitz constant for equivariant maps $f : \mathbb{H}^2_j \to \mathbb{H}^2_\rho$ is $C \geqslant 1$, and check that $(j,\rho)(\Gamma)$ acts non-properly on $G = PSL_2\mathbb{R}$.

(i) Let $\mathcal{F} := \left\{ f : \mathbb{H}^2_j \xrightarrow{C\text{-Lip}} \mathbb{H}^2_\rho \right\} \longrightarrow Cl(\mathbb{H}^2)$

$\neq \varnothing$ (Ascoli)

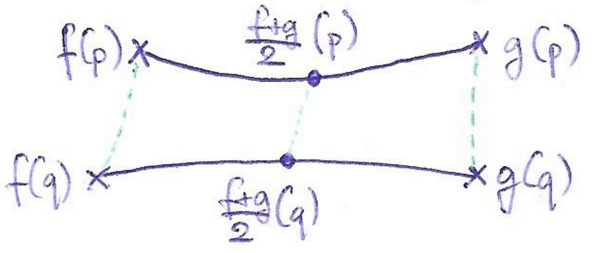
$$f \overset{\Phi}{\longmapsto} \left\{ p \in \mathbb{H}^2 \,\middle|\, \begin{array}{l} \forall \text{ nbhd } U \ni p, \\ \mathrm{Lip}(f|_U) = C \end{array} \right\}$$

We call $\Phi(f)$ the **stretch locus** of f.

(ii) There exists $f_0 \in \mathcal{F}$ with smallest possible stretch locus:

$$\Phi(f_0) = \bigcap_{f \in \mathcal{F}} \Phi(f) =: \Phi(j,\rho)$$

Key idea: $\mathcal{F}$ is convex for geodesic interpolation



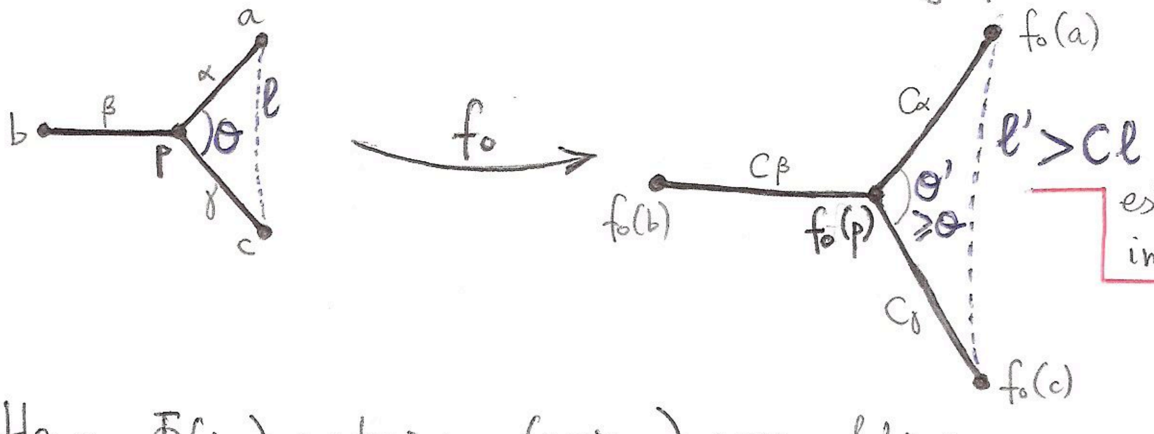
: hence

$$\begin{cases} \mathrm{Lip}\left(\frac{f+g}{2}\right) \leqslant \frac{\mathrm{Lip}(f) + \mathrm{Lip}(g)}{2} \\ \Phi\left(\frac{f+g}{2}\right) \subset \Phi(f) \cap \Phi(g) \end{cases}$$

by convexity of $d : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{R}$.

Thus, $f_0 = \sum_{i \geqslant 1} 2^{-i} f_i$ for $(f_i)_{i \geqslant 1}$ dense in $\mathcal{F}$ will do!

(iii) If $C > 1$, then the optimal stretch locus $\Phi(j,\rho)$ cannot have branching points:



by CAT-estimates: impossible!

Hence, $\Phi(j,\rho)$ contains a (unique) germ of line through any of its points

i.e. $\Phi(j,\rho)$ is a geodesic lamination Λ. [Cf Thurston]

If $C=1$, one similarly finds :

$\Phi(j,\rho)$ is a geodesic lamination, together with some complementary components isometrically preserved by any 1-Lipschitz $f: \mathbb{H}^2_j \to \mathbb{H}^2_\rho$ in $\mathcal{F}$.

(iv) If $C=1$, then $f_0 : \mathbb{H}^2_j \longrightarrow \mathbb{H}^2_\rho$ preserves some line $l \subset \mathbb{H}^2$ isometrically.

Picking orbit points $j(\gamma_n)\cdot p$ near l, we find

$$\boxed{d(p, j(\gamma_n)\cdot p) = d(f(p), \rho(\gamma_n)\cdot f(p)) + O(1)}$$

hence $\rho(\gamma_n)\, K\, j(\gamma_n)^{-1} \cap K \neq \emptyset$

for some $K \subset \mathrm{Isom}(\mathbb{H}^2)$ | compact, independent of n :

i.e. $(j,\rho)(\Gamma)$ acts non-properly on $G = \mathrm{Isom}\,\mathbb{H}^2$.

(v) If $C>1$, by following a (C-stretched) leaf of the lamination Λ, we find

$\beta \in \Gamma$ such that $\rho(\beta)$ has greater translation length than $j(\beta)$.

But $\alpha \in \Gamma$ has the opposite property !

Hence we can find $\boxed{\gamma_n = \beta^{b_n} \alpha^{a_n}}$ such that

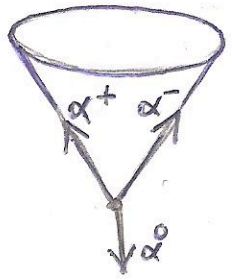
$d(p, j(\gamma_n)\cdot p) = d(f(p), \rho(\gamma_n)\cdot f(p)) + O(1)$ as before :

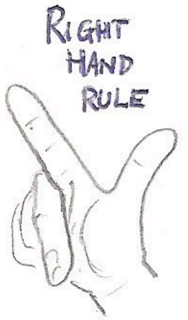
thus $(j,\rho)(\Gamma)$ acts non-properly on G. $\square$

PROOF OF THM 2 ($\Leftarrow$)

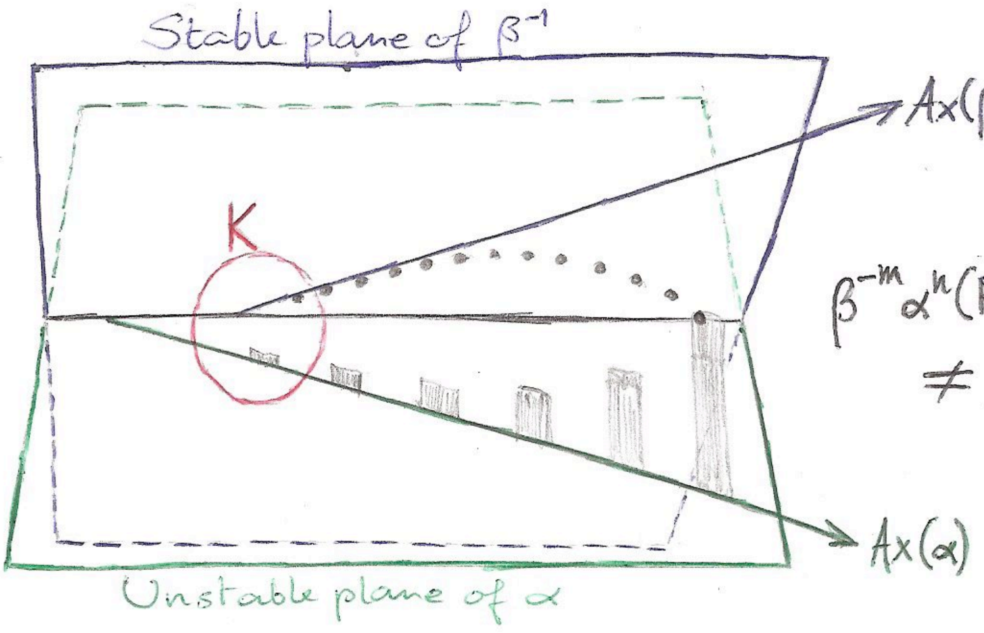
Similar:
- optimal $c \in \mathbb{R}$ and contracting vector field X on $\mathbb{H}^2$ s.t. $\frac{d'_X(p,q)}{d(p,q)} \leqslant c$;
- If $c \geqslant 0$ then $\frac{d'_X(\cdot,\cdot)}{d(\cdot,\cdot)} = c$ along some lamination;
- The last step (v) is then known as "Margulis's opposite sign lemma":

LEM If α, β are affine isometries of $\mathbb{R}^{2+1}$ (eigenvalues $e^a, 1, e^{-a}$ and $e^b, 1, e^{-b}$) whose translations along the neutral axes have opposite sign, then $\langle \alpha, \beta \rangle$ acts non-properly on $\mathbb{R}^{2+1}$.





PROOF:



$\beta^{-m}\alpha^n(K) \cap K \neq \emptyset$ $\square$

GEOMETRIC TRANSITION

Is there more than an analogy between THM 1 & THM 2?

Note: $\mathbb{R}^{2+1}$, AdS $\subset \mathbb{P}^3\mathbb{R}$

THM 3 Let $(j_t)_{t\geq 0}$ be Fuchsian, convex-cocompact representations $\Gamma \longrightarrow G = PSL_2\mathbb{R}$, with $\frac{d}{dt}\Big|_{t=0}(j_t) = u$ and $(j_0, u)(\Gamma)$ acting properly on $\mathbb{R}^{2+1}$.

Then $(j_0, j_t)(\Gamma)$ acts properly on AdS for small enough t.

Moreover, the quotients converge, i.e. $\exists$ developing maps

$$\mathcal{D}_t : \tilde{S} \times \mathbb{S}^1 \longrightarrow \mathbb{P}^3\mathbb{R} \qquad (S := j_0(\Gamma)\backslash \mathbb{H}^2)$$

$(\Gamma, (j_0, j_t))$-equivariant

$t \to 0$ $\Big\downarrow$ converging (after restriction) to

$$\mathcal{D}_0 : \tilde{S} \times (\mathbb{S}^1 \smallsetminus \{*\}) \longrightarrow \mathbb{R}^3 \subset \mathbb{P}^3\mathbb{R}$$

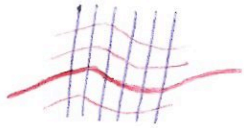
$= \tilde{S} \times \mathbb{R}$ $\qquad$ $(\Gamma, (j, u))$-equivariant

PROOF What is the issue?

⊛ Make sure the AdS fibrations converge to $\mathbb{R}^{2+1}$ fibrations, i.e. find $\left[\begin{array}{l} f_t : \mathbb{H}^2_{j_0} \longrightarrow \mathbb{H}^2_{j_t} \\ \mathrm{Lip}(f_t) < 1 \end{array}\right.$ with $\frac{d}{dt}\Big|_{t=0}(f_t)$ a contracting vector field on $\mathbb{H}^2$.

("Exchange Lip and lim")

⊛⊛ Promote the converging fibrations
to converging product structures (the developing maps!)
i.e. find converging sections



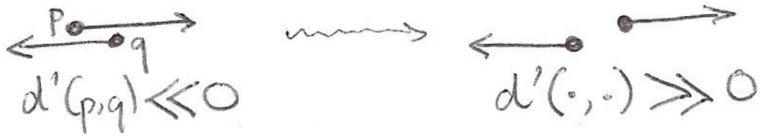
$$\sigma_t : \mathbb{H}^2 \longrightarrow G$$
$$p \longmapsto (\text{Some isometry taking } p \text{ to } f_t(p))$$

For ⊛:

Intuitively, f_t should be some "flow" on $\mathbb{H}^2$ directed by the contracting, (j,u)-equivariant vector field X.

How about $f_t(p) = \exp_p(tX(p))$?

Problem 1: forward flow destroys contractingness!



BUT backward flow is well behaved:

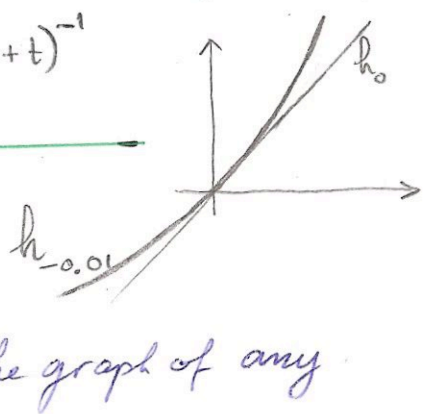
if $\operatorname{lip}(X) := \sup_{p \neq q} d'_X(p,q)/d(p,q)$

and $\Phi_t :=$ geodesic flow on $T\mathbb{H}^2$, then $(t<0)$

CLAIM

$$\forall\, \xi, \eta \in T\mathbb{H}^2, \quad \operatorname{lip}\{\Phi_t(\xi), \Phi_t(\eta)\} \leqslant h_t(\operatorname{lip}\{\xi, \eta\})$$

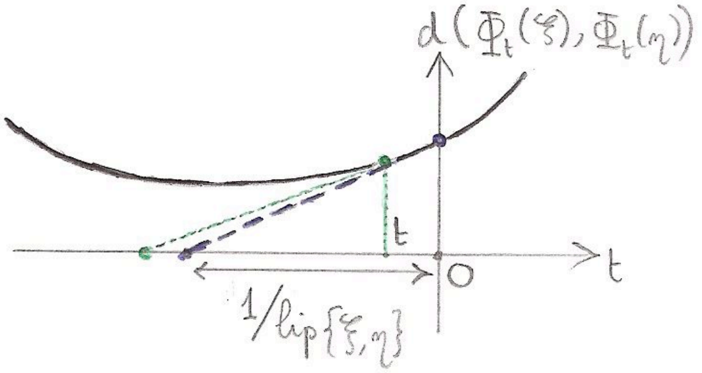
where $h_t(x) = \frac{x}{1+tx} = (x^{-1}+t)^{-1}$



(*In dimension 1, Φ_t applies $\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}$ to the graph of any vector field on $\mathbb{R}$:*

)

Proof of the Claim: Convexity of distance function.



$\square$

Hence $X' := \Phi_{-0.01}(X)$ is almost as contracting,
but much more regular than X.

We may flow forward again:

$$\begin{array}{ll} p \longmapsto \exp_p(tX'(p)) & \text{is } (<1)\text{-Lipschitz} \\ \mathbb{H}^2 \longrightarrow \mathbb{H}^2 & \text{for } t \leqslant 0.01. \end{array}$$

Problem 2:

Flow preserves invariance $X(j(\gamma)\cdot p) = j(\gamma)_* X(p)$,

not equivariance $X(j(\gamma)\cdot p) = j(\gamma)_* X(p) + u(\gamma)(j(\gamma)\cdot p)$.

Solution: decompose

$$X = \frac{d}{dt}\Big|_{t=0} (\tilde{f}_t) + Y$$

contracting (j_0,u)-equivariant poss. non-smooth

smooth (j_0,j_t)-equivariant poss. non-contracting

j_0-invariant correction (poss. non-smooth)

and let only Y flow $\left(\hat{f}_t : \begin{array}{l} \mathbb{H}^2_{j_0} \longrightarrow \mathbb{H}^2_{j_0} \\ p \longmapsto \exp_p tY(p) \end{array} \right)$.

Then pick $\boxed{f_t := \tilde{f}_t \circ \hat{f}_t : \mathbb{H}^2_{j_0} \to \mathbb{H}^2_{j_t}}$ $\square$

$\boxed{\text{For } ⊛⊛ :}$

To promote the fibrations $(L_{p,f_t(p)})_{p\in\mathbb{H}^2}$ of AdS,

given by $f_t = \tilde{f}_t \circ \hat{f}_t : \mathbb{H}^2_{j_0} \longrightarrow \mathbb{H}^2_{j_t}$

$p \longmapsto f_t(p)$,

to sections $p \longmapsto g_t(p) \in L_{p,f_t(p)}$,

(an isometry of $\mathbb{H}^2$ taking p to $f_t(p)$)

any equivariant construction will do.

For example, pick

$$g_t(p) := \underbrace{\left(\begin{array}{c}\text{Osculating isometry of } \tilde{f}_t \\ \text{near } \hat{f}_t(p)\end{array}\right)}_{\text{equivariant motion}} \circ \underbrace{\left(\begin{array}{c}\text{Loxodromy along } \exp_p \mathbb{R}Y(p) \\ \text{of length } \|Y(p)\|\end{array}\right)}_{\text{invariant motion}}$$

□