COMPACTIFICATION OF CERTAIN CLIFFORD–KLEIN FORMS OF REDUCTIVE HOMOGENEOUS SPACES

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Abstract. We describe smooth compactifications of certain families of reductive homogeneous spaces such as group manifolds for classical Lie groups, or pseudo-Riemannian analogues of real hyperbolic spaces and their complex and quaternionic counterparts. We deduce compactifications for Clifford–Klein forms of these homogeneous spaces, namely for quotients by discrete groups Γ acting properly discontinuously, in the case that Γ is word hyperbolic and acts via an Anosov representation. In particular, these Clifford–Klein forms are topologically tame.

1. Introduction

The goal of this note is two-fold. First, we describe compactifications of certain families of reductive homogeneous spaces G/H by embedding G into a larger group G' and realizing G/H as a G-orbit in a flag manifold of G'. These homogeneous spaces include:

- group manifolds for classical Lie groups (Theorems 1.1 and 2.6, see also [He02]),
- certain affine symmetric spaces or reductive homogeneous spaces G/H given in Tables 2 and 3 (Propositions 1.5.(1) and 5.8.(1)),
- pseudo-Riemannian analogues of real hyperbolic spaces and their complex and quaternionic counterparts (see (1.3) in Section 1.4).

Second, we use these compactifications and a construction of domains of discontinuity from [GW12] to compactify Clifford–Klein forms of G/H, i.e. quotient manifolds $\Gamma\backslash G/H$, in the case that Γ is a word hyperbolic group whose action on G/H is given by an Anosov representation $\rho : \Gamma \to G \hookrightarrow G'$. We deduce that these Clifford–Klein forms are topologically tame.

Anosov representations (see Section 3.3) were introduced in [Lab06]. They provide a rich class of quasi-isometric embeddings of word hyperbolic groups into reductive Lie groups with remarkable properties, generalizing convex cocompact representations to higher real rank [Lab06, GW12, KLPb, KLPc, KLPa, KLP16, KL, GGKW16]. Examples include:

(a) The inclusion of convex cocompact subgroups in real semisimple Lie groups of real rank 1 [Lab06, GW12];
(b) Representations of surface groups into split real semisimple Lie groups that belong to the Hitchin component [Lab06, FG06, GW12];

FG and FK were partially supported by the Agence Nationale de la Recherche under the grant DiscGroup (ANR-11-BS01-013) and through the Labex CEMPI (ANR-11-LABX-0007-01). AW was partially supported by the National Science Foundation under agreement DMS-1536017, by the Sloan Foundation, by the Deutsche Forschungsgemeinschaft, by the European Research Council under ERC-Consolidator grant 614733, and by the Klaus-Tschira-Foundation. Part of this work was carried out while OG, FK, and AW were in residence at the MSRI in Berkeley, California, supported by the National Science Foundation under grant 0932078 000. The authors also acknowledge support from U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 "RNMS: GEometric structures And Representation varieties" (the GEAR Network).

(c) Maximal representations of surface groups into semisimple Lie groups of Hermitian type [BILW05, BIW10, GW12];
(d) The inclusion of quasi-Fuchsian subgroups in $\mathrm{SO}(2,d)$ [BM12, Bar15];
(e) Holonomies of compact, strictly convex $\mathbf{R}\mathbb{P}^n$-manifolds [Ben04].

1.1. **Compactifying group manifolds.** Any real reductive Lie group G can be seen as an affine symmetric space $(G\times G)/\mathrm{Diag}(G)$ under the action of $G\times G$ by left and right multiplication. We call G with this structure a *group manifold*. We describe a smooth compactification of the group manifold G when G is a classical group. This compactification is very elementary, in particular when G is the automorphism group of a nondegenerate bilinear form. It shares some common features with the so-called *wonderful compactifications* of algebraic groups over an algebraically closed field constructed by De Concini and Procesi [CP83] or Luna and Vust [LV83], as well as with the compactifications constructed by Neretin [Ner98, Ner03]. After completing this note, we learned that this compactification had first been discovered by He [He02, Th. 0.3 & 0.4]; we still include our original self-contained description for the reader's convenience.

We first consider the case that G is $\mathrm{O}(p,q)$, $\mathrm{O}(m,\mathbf{C})$, $\mathrm{Sp}(2n,\mathbf{R})$, $\mathrm{Sp}(2n,\mathbf{C})$, $\mathrm{U}(p,q)$, $\mathrm{Sp}(p,q)$, or $\mathrm{O}^*(2m)$. In other words, $G=\mathrm{Aut}_{\mathbf{K}}(b)$ is the group of $\mathbf{K}$-linear automorphisms of a nondegenerate $\mathbf{R}$-bilinear form $b: V\otimes_{\mathbf{R}} V\to\mathbf{K}$ on a $\mathbf{K}$-vector space V, for $\mathbf{K}=\mathbf{R}$, $\mathbf{C}$, or the ring $\mathbf{H}$ of quaternions; and we assume that b is $\mathbf{K}$-linear in the second variable, and that b is symmetric or antisymmetric (if $\mathbf{K}=\mathbf{R}$ or $\mathbf{C}$), or Hermitian or anti-Hermitian (if $\mathbf{K}=\mathbf{C}$ or $\mathbf{H}$). We describe a smooth compactification of $G=\mathrm{Aut}_{\mathbf{K}}(b)$ by embedding it into the compact space of maximal $(b\oplus -b)$-isotropic $\mathbf{K}$-subspaces of $(V\oplus V, b\oplus -b)$. Let $n\in\mathbf{N}$ be the real rank of $G=\mathrm{Aut}_{\mathbf{K}}(b)$ and $N=\dim_{\mathbf{K}}(V)\geq 2n$ the real rank of $\mathrm{Aut}_{\mathbf{K}}(b\oplus -b)$. (In other words, n is the dimension over $\mathbf{K}$ of a maximal b-isotropic subspace in V.) For any $0\leq i\leq n$, let $\mathcal{F}_i(b)=\mathcal{F}_i(-b)$ be the space of i-dimensional b-isotropic subspaces of V; it is a smooth manifold with a transitive action of G. We use similar notation for $(V\oplus V, b\oplus -b)$, with $0\leq i\leq N$. For any subspace W of $V\oplus V$, we set

$$\pi(W) := \big(W\cap(V\oplus\{0\}), W\cap(\{0\}\oplus V)\big). \tag{1.1}$$

This defines a map $\pi:\mathcal{F}_N(b\oplus -b)\to\big(\bigcup_{i=0}^n\mathcal{F}_i(b)\big)\times\big(\bigcup_{i=0}^n\mathcal{F}_i(-b)\big)$.

Theorem 1.1. *Let $G=\mathrm{Aut}_{\mathbf{K}}(b)$ be as above. The space $X=\mathcal{F}_N(b\oplus -b)$ of maximal $(b\oplus -b)$-isotropic $\mathbf{K}$-subspaces of $V\oplus V$ is a smooth compactification of the group manifold $(G\times G)/\mathrm{Diag}(G)$ with the following properties:*

(1) X is a real analytic manifold (in fact complex analytic if $\mathbf{K}=\mathbf{C}$ and b is symmetric or antisymmetric). Under the action of a maximal compact subgroup of $\mathrm{Aut}_{\mathbf{K}}(b\oplus -b)$, it identifies with a Riemannian symmetric space of compact type, given explicitly in Table 1.

(2) The $(G\times G)$-orbits in X are the submanifolds $\mathcal{U}_i:=\pi^{-1}(\mathcal{F}_i(b)\times\mathcal{F}_i(-b))$ for $0\leq i\leq n$, of dimension $\dim_{\mathbf{R}}(\mathcal{U}_i)=\dim_{\mathbf{R}}(G)-i^2\dim_{\mathbf{R}}(\mathbf{K})$. The closure of $\mathcal{U}_i$ in X is $\bigcup_{j\geq i}\mathcal{U}_j$.

(3) For $0\leq i\leq n$, the map π defines a fibration of $\mathcal{U}_i$ over $\mathcal{F}_i(b)\times\mathcal{F}_i(-b)$ with fibers isomorphic to $(H_i\times H_i)/\mathrm{Diag}(H_i)$, where $H_i=\mathrm{Aut}_{\mathbf{K}}(b_{V_i})$ is the automorphism group of the bilinear form b_{V_i} induced by b on $V_i^{\perp_b}/V_i$ for some $V_i\in\mathcal{F}_i(b)$.

In particular, $\mathcal{U}_0$ is the unique open $(G\times G)$-orbit and it identifies with $(G\times G)/\mathrm{Diag}(G)$.

G	n	N	X as a Riemannian symmetric space
$\mathrm{O}(p,q)$	$\min(p,q)$	$p+q$	$(\mathrm{O}(p+q)\times\mathrm{O}(p+q))/\mathrm{Diag}(\mathrm{O}(p+q))$
$\mathrm{U}(p,q)$	$\min(p,q)$	$p+q$	$(\mathrm{U}(p+q)\times\mathrm{U}(p+q))/\mathrm{Diag}(\mathrm{U}(p+q))$
$\mathrm{Sp}(p,q)$	$\min(p,q)$	$p+q$	$(\mathrm{Sp}(p+q)\times\mathrm{Sp}(p+q))/\mathrm{Diag}(\mathrm{Sp}(p+q))$
$\mathrm{Sp}(2n,\mathbf{R})$	n	$2n$	$\mathrm{U}(2n)/\mathrm{O}(2n)$
$\mathrm{Sp}(2n,\mathbf{C})$	n	$2n$	$\mathrm{Sp}(2n)/\mathrm{U}(2n)$
$\mathrm{O}(m,\mathbf{C})$	$\lfloor\frac{m}{2}\rfloor$	m	$\mathrm{O}(2m)/\mathrm{U}(m)$
$\mathrm{O}^*(2m)$	$\lfloor\frac{m}{2}\rfloor$	m	$\mathrm{U}(2m)/\mathrm{Sp}(m)$

TABLE 1. The compactification X of Theorem 1.1.

Remark 1.2. For $G=\mathrm{O}(p,q)$, $\mathrm{U}(p,q)$, or $\mathrm{Sp}(p,q)$, the compactification X identifies with the group manifold $(G_c\times G_c)/\mathrm{Diag}(G_c)$ where G_c is the compact real form of a complexification of G. For $G=\mathrm{O}(n,1)$, the embedding of G into $G_c=\mathrm{O}(n+1)$ lifts the embedding of $\mathbb{H}^n_{\mathbf{R}}\sqcup\mathbb{H}^n_{\mathbf{R}}=\mathrm{O}(n,1)/\mathrm{O}(n)$ into $\mathbb{S}^n_{\mathbf{R}}=\mathrm{O}(n+1)/\mathrm{O}(n)$ with image the complement of the equatorial sphere $\mathbb{S}^{n-1}_{\mathbf{R}}$.

Similar compactifications are constructed for general linear groups $\mathrm{GL}_{\mathbf{K}}(V)$ and special linear groups $\mathrm{SL}_{\mathbf{K}}(V)$ in Theorem 2.6 below.

1.2. **Compactifying Clifford–Klein forms of group manifolds.** Let $G=\mathrm{Aut}_{\mathbf{K}}(b)$ be as above. For any discrete group Γ and any representation $\rho:\Gamma\to G$ with discrete image and finite kernel, the action of Γ on G via left multiplication by ρ is properly discontinuous. The quotient $\rho(\Gamma)\backslash G$ is an orbifold, in general non-compact. Suppose that Γ is word hyperbolic and ρ is $P_1(b)$-Anosov (see Section 3 for definitions), where $P_1(b)$ is the stabilizer in G of a b-isotropic line. Considering a suitable subset of the compactification X of G described in Theorem 1.1, we construct a compactification of $\rho(\Gamma)\backslash G$ which is an orbifold, or if Γ is torsion-free, a smooth manifold.

Theorem 1.3. *Let Γ be a word hyperbolic group and $\rho:\Gamma\to G=\mathrm{Aut}_{\mathbf{K}}(b)$ a $P_1(b)$-Anosov representation with boundary map $\xi:\partial_\infty\Gamma\to\mathcal{F}_1(b)$. For any $0\le i\le n$, let $\mathcal{K}^i_\xi$ be the subset of $\mathcal{F}_i(b)$ consisting of subspaces W containing $\xi(\eta)$ for some $\eta\in\partial_\infty\Gamma$, and let $\mathcal{U}^\xi_i$ be the complement in $\mathcal{U}_i$ of $\pi^{-1}(\mathcal{K}^i_\xi\times\mathcal{F}_i(-b))$, where π is the map defined by (1.1). Then $\rho(\Gamma)\times\{e\}\subset\mathrm{Aut}_{\mathbf{K}}(b)\times\mathrm{Aut}_{\mathbf{K}}(b)$ acts properly discontinuously and cocompactly on the open subset*

$$\Omega:=\bigcup_{i=0}^{n}\mathcal{U}^\xi_i$$

of $\mathcal{F}_N(b\oplus -b)$. The quotient orbifold $(\rho(\Gamma)\times\{e\})\backslash\Omega$ is a compactification of

$$\rho(\Gamma)\backslash G\ \simeq\ (\rho(\Gamma)\times\{e\})\backslash(G\times G)/\mathrm{Diag}(G).$$

If Γ is torsion-free, then this compactification is a smooth manifold.

Theorem 1.3 is in fact a special case of Theorem 4.1 below, which gives a procedure to compactify quotients of $G=\mathrm{Aut}_{\mathbf{K}}(b)$ by a word hyperbolic group Γ acting via any $P_1(b\oplus -b)$-Anosov representation

$$\rho:\Gamma\longrightarrow\mathrm{Aut}_{\mathbf{K}}(b)\times\mathrm{Aut}_{\mathbf{K}}(b)\subset\mathrm{Aut}_{\mathbf{K}}(b\oplus -b)\,;$$

the group Γ is thus allowed to act simultaneously by left and right multiplication instead of just left multiplication. We refer to Remark 4.2 in the case that $\mathrm{Aut}_{\mathbf{K}}(b)$ has real rank 1.

Remark 1.4. Let G be an arbitrary real reductive Lie group and P a parabolic subgroup. Composing a P-Anosov representation $\rho : \Gamma \to G$ with an appropriate linear representation $\tau : G \to \mathrm{Aut}_{\mathbf{K}}(b)$ (see Proposition 3.11), we obtain a $P_1(b)$-Anosov representation $\tau \circ \rho : \Gamma \to \mathrm{Aut}_{\mathbf{K}}(b)$. Theorem 1.3 can then be applied to give a compactification of $\rho(\Gamma)\backslash G$: see Corollary 4.5 for a precise statement.

1.3. **Compactifying other families of homogeneous spaces and their Clifford–Klein forms.** The idea of embedding a group G into a larger group G' so that a homogeneous space G/H can be realized explicitly as a G-orbit in an appropriate flag variety G'/P', can be applied in other cases as well. We prove the following.

Proposition 1.5. *Let (G, H, P, G', P') be as in Table 2.*

(1) There exists an open G-orbit $\mathcal{U}$ in G'/P' that is diffeomorphic to G/H; the closure $\overline{\mathcal{U}}$ of $\mathcal{U}$ in G'/P' provides a compactification of G/H.

(2) For any word hyperbolic group Γ and any P-Anosov representation $\rho : \Gamma \to G$, the cocompact domain of discontinuity $\Omega \subset G'/P'$ for $\rho(\Gamma)$ constructed in [GW12] *(see Proposition 3.13) contains $\mathcal{U}$; the quotient $\rho(\Gamma)\backslash(\Omega \cap \overline{\mathcal{U}})$ provides a compactification of $\rho(\Gamma)\backslash G/H$.*

	G	H	P	G'	P'
(i)	$\mathrm{O}(p, q+1)$	$\mathrm{O}(p, q)$	$\mathrm{Stab}_G(W)$	$\mathrm{O}(p+1, q+1)$	$\mathrm{Stab}_{G'}(\ell')$
(ii)	$\mathrm{U}(p, q+1)$	$\mathrm{U}(p, q)$	$\mathrm{Stab}_G(W)$	$\mathrm{U}(p+1, q+1)$	$\mathrm{Stab}_{G'}(\ell')$
(iii)	$\mathrm{Sp}(p, q+1)$	$\mathrm{Sp}(p, q)$	$\mathrm{Stab}_G(W)$	$\mathrm{Sp}(p+1, q+1)$	$\mathrm{Stab}_{G'}(\ell')$
(iv)	$\mathrm{O}(2p, 2q)$	$\mathrm{U}(p, q)$	$\mathrm{Stab}_G(\ell)$	$\mathrm{O}(2p+2q, \mathbf{C})$	$\mathrm{Stab}_{G'}(W')$
(v)	$\mathrm{U}(2p, 2q)$	$\mathrm{Sp}(p, q)$	$\mathrm{Stab}_G(\ell)$	$\mathrm{Sp}(p+q, p+q)$	$\mathrm{Stab}_{G'}(W')$
(vi)	$\mathrm{Sp}(2m, \mathbf{R})$	$\mathrm{U}(p, m-p)$	$\mathrm{Stab}_G(\ell)$	$\mathrm{Sp}(2m, \mathbf{C})$	$\mathrm{Stab}_{G'}(W')$

Table 2. Reductive groups $H \subset G \subset G'$ and parabolic subgroups P of G and P' of G' to which Proposition 1.5 applies. We denote by ℓ or ℓ' an isotropic line and by W or W' a maximal isotropic subspace (over $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$), relative to the form b preserved by G or G'. Here m, p, q are any integers with $m > 0$; we require $p > q + 1$ in case (i) and $p > q$ in cases (ii), (iii), as well as $q > 0$ in cases (iv), (v).

The open G-orbit $\mathcal{U}$ diffeomorphic to G/H is given explicitly in Section 5.

Example (i), example (iv) for $q = 1$, and example (vi) for $p = 0$ were previously described in [GW12, Prop. 13.1, Th. 13.3, and § 12].

In examples (iv), (v), and (vi), the space G/H is an affine symmetric space, which is Riemannian in example (vi) for $p = 0$ or m. In examples (i), (ii), and (iii), the space $G/H = \mathrm{Aut}_{\mathbf{K}}(b^{p,q+1})/\mathrm{Aut}_{\mathbf{K}}(b^{p,q})$ identifies with the quadric

$$\hat{\mathbb{H}}^{p,q}_{\mathbf{K}} = \{x \in \mathbf{K}^{p,q+1} \mid b^{p,q+1}_{\mathbf{K}}(x, x) = -1\}$$

where $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ and $b^{p,q}_{\mathbf{K}}$ is the quadratic form on $\mathbf{K}^{p+q}$ given by

$$b^{p,q}_{\mathbf{K}}(x, x) := \overline{x}_1 x_1 + \cdots + \overline{x}_p x_p - \overline{x}_{p+1} x_{p+1} - \cdots - \overline{x}_{p+q} x_{p+q}. \tag{1.2}$$

Thus G/H fibers over the affine symmetric space

$$\mathbb{H}^{p,q}_{\mathbf{K}} = \mathrm{Aut}_{\mathbf{K}}(b^{p,q+1}_{\mathbf{K}})/\big(\mathrm{Aut}_{\mathbf{K}}(b^{p,q}_{\mathbf{K}}) \times \mathrm{Aut}_{\mathbf{K}}(b^{0,1}_{\mathbf{K}})\big)$$

with compact fibers. This affine symmetric space is Riemannian for $q = 0$.

In Proposition 5.8 below, we treat two other families of reductive homogeneous spaces which are not affine symmetric spaces using the following remark.

Remark 1.6. The cocompact domains of discontinuity Ω of Proposition 1.5.(2) lift to cocompact domains of discontinuity in G'/P'' for any parabolic subgroup P'' of G' contained in P'; in particular, they lift to cocompact domains of discontinuity in $G'/P'_{\min}$ where $P'_{\min}$ is a minimal parabolic subgroup of G'. The compactifications of the quotients $\rho(\Gamma)\backslash\mathcal{U}$ of Proposition 1.5.(2) induce compactifications of the quotients $\rho(\Gamma)\backslash\mathcal{U}'$ for any G-orbit $\mathcal{U}'$ in G'/P'' lifting the G-orbit $\mathcal{U} \subset G'/P'$ diffeomorphic to G/H.

1.4. **Compactifying pseudo-Riemannian analogues of hyperbolic manifolds.** For $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ and $p > q \geq 0$, the space $\mathbb{H}^{p,q}_{\mathbf{K}} = \mathrm{Aut}_{\mathbf{K}}(b^{p,q+1})/(\mathrm{Aut}_{\mathbf{K}}(b^{p,q}) \times \mathrm{Aut}_{\mathbf{K}}(b^{0,1}))$ has a natural realization in projective space as

$$\mathbb{H}^{p,q}_{\mathbf{K}} = \mathbb{P}(\{x \in \mathbf{K}^{p+q+1} \mid b^{p,q+1}_{\mathbf{K}}(x,x) < 0\}) \subset \mathbb{P}(\mathbf{K}^{p+q+1}).$$

The space $\mathbb{H}^{p,q}_{\mathbf{R}}$ is an analogue of the real hyperbolic space $\mathbb{H}^n_{\mathbf{R}}$: it is pseudo-Riemannian of signature (p,q) and has constant negative sectional curvature. Similarly, $\mathbb{H}^{p,q}_{\mathbf{C}}$ and $\mathbb{H}^{p,q}_{\mathbf{H}}$ are analogues of the complex and quaternionic hyperbolic spaces.

The space $\mathbb{H}^{p,q}_{\mathbf{K}}$ has a natural compactification, namely

$$\overline{\mathbb{H}^{p,q}_{\mathbf{K}}} := \mathbb{P}_{\mathbf{K}}(\{x \in \mathbf{K}^{p+q+1} \mid b^{p,q+1}_{\mathbf{K}}(x,x) \leq 0\}). \tag{1.3}$$

This is a manifold with boundary, which is the union of $\mathbb{H}^{p,q}_{\mathbf{K}}$ (open G-orbit) and $\mathcal{F}_1(b^{p,q+1}_{\mathbf{K}})$ (closed G-orbit).

Let $P = P_{q+1}(b^{p,q+1}_{\mathbf{K}})$ be the stabilizer in $\mathrm{Aut}_{\mathbf{K}}(b^{p,q+1}_{\mathbf{K}})$ of a maximal $b^{p,q+1}_{\mathbf{K}}$-isotropic subspace of $\mathbf{K}^{p+q+1}$, so that (G,P) is as in examples (i), (ii), or (iii) of Table 2. Building on Proposition 1.5, we prove the following.

Theorem 1.7. *For $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ and $p > q \geq 0$, let $G = \mathrm{Aut}_{\mathbf{K}}(b^{p,q+1}_{\mathbf{K}})$ and $P = P_{q+1}(b^{p,q+1}_{\mathbf{K}})$. Let Γ be a word hyperbolic group and $\rho : \Gamma \to G$ a P-Anosov representation.*

1. *The action of Γ on $\mathbb{H}^{p,q}_{\mathbf{K}}$ via ρ is properly discontinuous, except possibly if $\mathbf{K} = \mathbf{R}$ and $p = q+1$.*
2. *Assume that the action is properly discontinuous. Let $\xi : \partial_\infty\Gamma \to \mathcal{F}_{q+1}(b^{p,q+1}_{\mathbf{K}})$ be the boundary map of ρ and $\mathcal{K}_\xi$ the subset of $\partial\mathbb{H}^{p,q}_{\mathbf{K}} = \mathcal{F}_1(b^{p,q+1}_{\mathbf{K}})$ consisting of lines ℓ contained in $\xi(\eta)$ for some $\eta \in \partial_\infty\Gamma$. Then Γ acts properly discontinuously and cocompactly, via ρ, on $\overline{\mathbb{H}^{p,q}_{\mathbf{K}}} \smallsetminus \mathcal{K}_\xi$. In particular, if Γ is torsion-free, then $\rho(\Gamma)\backslash(\overline{\mathbb{H}^{p,q}_{\mathbf{K}}} \smallsetminus \mathcal{K}_\xi)$ is a smooth manifold with boundary compactifying $\rho(\Gamma)\backslash\mathbb{H}^{p,q}_{\mathbf{K}}$.*

Remark 1.8. For $\mathbf{K} = \mathbf{R}$ and $p = q+1$, the fact that ρ is $P_{q+1}(b^{p,q+1}_{\mathbf{K}})$-Anosov does not imply that the action of Γ on $\mathbb{H}^{p,q}_{\mathbf{K}}$ is properly discontinuous: see Example 5.4. In the case that $\mathbf{K} = \mathbf{R}$ and $p = q+1$ is odd, the action of Γ on $\mathbb{H}^{p,q}_{\mathbf{K}}$ can actually never be properly discontinuous unless Γ is virtually cyclic, by [Kas08].

1.5. **Tameness.** We establish the topological tameness of the Clifford–Klein forms $\rho(\Gamma)\backslash G/H$ of Sections 1.2, 1.3, and 1.4. Recall that a manifold is said to be *topologically tame* if it is homeomorphic to the interior of a compact manifold with boundary. Here is an immediate consequence of Theorem 1.7.

Corollary 1.9. *For $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ and $p > q \geq 0$, let $G = \mathrm{Aut}_{\mathbf{K}}(b^{p,q+1}_{\mathbf{K}})$ and $P = P_{q+1}(b^{p,q+1}_{\mathbf{K}})$. For any torsion-free word hyperbolic group Γ and any P-Anosov* *representation $\rho : \Gamma \to G$, if the quotient $\rho(\Gamma)\backslash\mathbb{H}^{p,q}_{\mathbf{K}}$ is a manifold (which is always the case except possibly if $\mathbf{K} = \mathbf{R}$ and $p = q + 1$, see Theorem 1.7), then this manifold is topologically tame.*

In order to prove topological tameness in more general cases, we establish the following useful fact.

Lemma 1.10. *Let $G \subset G'$ be two real reductive algebraic groups and Γ a torsion-free discrete subgroup of G. Let X be a G'-homogeneous space and Ω an open subset of X on which Γ acts properly discontinuously and cocompactly. For any G-orbit $\mathcal{U} \subset \Omega$, the quotient $\Gamma\backslash\mathcal{U}$ is a topologically tame manifold.*

Proposition 1.5 and Lemma 1.10 immediately imply the following, by taking $\mathcal{U}$ to be a G-orbit in G'/P' that identifies with G/H.

Corollary 1.11. *Let Γ be a torsion-free word hyperbolic group and let $H \subset G \supset P$ be as in Table 2. For any P-Anosov representation $\rho : \Gamma \to G$, the quotient $\rho(\Gamma)\backslash G/H$ is a topologically tame manifold.*

Using Theorem 1.3 and Lemma 1.10, we also prove the following.

Theorem 1.12. *Let Γ be a torsion-free word hyperbolic group, G a real reductive algebraic group, and P a proper parabolic subgroup of G. For any P-Anosov representation $\rho : \Gamma \to G$, the quotient $\rho(\Gamma)\backslash G$ is a topologically tame manifold.*

Remark 1.13. Let K be a maximal compact subgroup of G. Compactifications of the Riemannian locally symmetric spaces $\rho(\Gamma)\backslash G/K$ for P-Anosov representations $\rho : \Gamma \to G$ have recently been constructed in [KL] and [GKW]. They also induce compactifications of $\rho(\Gamma)\backslash G$.

1.6. **Organization of the paper.** In Section 2 we establish Theorem 1.1 and its analogue for $\mathrm{GL}_{\mathbf{K}}(V)$ (Theorem 2.6). In Section 3 we recall the notion of Anosov representation, the construction of domains of discontinuity from [GW12], and a few facts from [GGKW16] on Anosov representations into $\mathrm{Aut}_{\mathbf{K}}(b) \times \mathrm{Aut}_{\mathbf{K}}(b)$. This allows us, in Section 4, to establish Theorem 1.3 and some generalization (Theorem 4.1). In Section 5 we prove Proposition 1.5 and Theorem 1.7. Finally, Section 6 is devoted to topological tameness, with a proof of Lemma 1.10 and Theorem 1.12.

Acknowledgements. We are grateful to Jeff Danciger, Adolfo Guillot, and Pablo Solis for useful comments and discussions. We thank the referees for pertinent remarks which helped improve the paper.

2. Compactification of group manifolds

In this section we provide a short proof of Theorem 1.1 and of its analogue for general linear groups $\mathrm{GL}_{\mathbf{K}}(V)$ with $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ (Theorem 2.6).

2.1. **The case $G = \mathrm{Aut}_{\mathbf{K}}(b)$.** Let us prove Theorem 1.1. We use the notation of Section 1.1. In particular,

$$\pi : \mathcal{F}_N(b \oplus -b) \longrightarrow \left(\bigcup_{i=0}^{n} \mathcal{F}_i(b)\right) \times \left(\bigcup_{i=0}^{n} \mathcal{F}_i(-b)\right)$$

is the map defined by (1.1). The group

$$\mathrm{Aut}_{\mathbf{K}}(b) \times \mathrm{Aut}_{\mathbf{K}}(b) = \mathrm{Aut}_{\mathbf{K}}(b) \times \mathrm{Aut}_{\mathbf{K}}(-b)$$

naturally embeds into $\mathrm{Aut}_{\mathbf{K}}(b\oplus -b)$. For $0\leq i\leq n$, the set

$$\mathcal{U}_i := \pi^{-1}\big(\mathcal{F}_i(b)\times\mathcal{F}_i(-b)\big)$$

is clearly invariant under $\mathrm{Aut}_{\mathbf{K}}(b)\times\mathrm{Aut}_{\mathbf{K}}(-b)$.

Lemma 2.1. *The space $X=\mathcal{F}_N(b\oplus -b)$ of maximal $(b\oplus -b)$-isotropic $\mathbf{K}$-subspaces of $V\oplus V$ is the union of the sets $\mathcal{U}_i$ for $0\leq i\leq n$.*

Proof. It is enough to prove that for any $W\in\mathcal{F}_N(b\oplus -b)$,

$$\dim_{\mathbf{K}}(W\cap(\{0\}\oplus V)) = \dim_{\mathbf{K}}(W\cap(V\oplus\{0\})). \tag{2.1}$$

We have

$$\begin{aligned}\dim_{\mathbf{K}}(W\cap(\{0\}\oplus V)) &= \dim_{\mathbf{K}}(W)+\dim_{\mathbf{K}}(\{0\}\oplus V)-\dim_{\mathbf{K}}(W+(\{0\}\oplus V))\\ &= \dim_{\mathbf{K}}(V\oplus V)-\dim_{\mathbf{K}}(W+(\{0\}\oplus V))\\ &= \dim_{\mathbf{K}}(W+(\{0\}\oplus V))^{\perp},\end{aligned}$$

where $(W+(\{0\}\oplus V))^{\perp}$ denotes the orthogonal complement of $W+(\{0\}\oplus V)$ in $V\oplus V$ with respect to $b\oplus -b$. But

$$(W+(\{0\}\oplus V))^{\perp} = W^{\perp}\cap(\{0\}\oplus V)^{\perp} = W^{\perp}\cap(V\oplus\{0\}),$$

hence $\dim_{\mathbf{K}}(W\cap(\{0\}\oplus V)) = \dim_{\mathbf{K}}(W^{\perp}\cap(V\oplus\{0\}))$. Since W is maximal isotropic for $b\oplus -b$, we have $W=W^{\perp}$, and so (2.1) holds. □

For any $0\leq i\leq n$, let

$$\pi_i : \mathcal{U}_i \longrightarrow \mathcal{F}_i(b)\times\mathcal{F}_i(-b)$$

be the map induced by π. By construction, π_i is $(\mathrm{Aut}_{\mathbf{K}}(b)\times\mathrm{Aut}_{\mathbf{K}}(b))$-equivariant.

Let us describe the fiber of π_i above (V_i,V_i) for some given $V_i\in\mathcal{F}_i(b)$. We denote by b_{V_i} the $\mathbf{R}$-bilinear form induced by b on $V_i^{\perp_b}/V_i\simeq\mathbf{K}^{\dim_{\mathbf{K}}(V)-2i}$. If b is symmetric, antisymmetric, Hermitian, or anti-Hermitian, then so is b_{V_i}. For instance, if b is symmetric over $\mathbf{R}$ with signature (p,q), then b_{V_i} has signature $(p-i,q-i)$.

Lemma 2.2. *For any $V_i\in\mathcal{F}_i(b)$, the fiber $\pi_i^{-1}(V_i,V_i)\subset\mathcal{F}_N(b\oplus -b)$ is the set of maximal $(b\oplus -b)$-isotropic $\mathbf{K}$-subspaces of $V_i^{\perp_b}\oplus V_i^{\perp_b}$ that contain $V_i\oplus V_i$ and project to maximal isotropic subspaces of $(V_i^{\perp_b}/V_i)\oplus(V_i^{\perp_b}/V_i)$ transverse to both factors $(V_i^{\perp_b}/V_i)\oplus\{0\}$ and $\{0\}\oplus(V_i^{\perp_b}/V_i)$. As an $(\mathrm{Aut}_{\mathbf{K}}(b_{V_i})\times\mathrm{Aut}_{\mathbf{K}}(b_{V_i}))$-space, $\pi_i^{-1}(V_i,V_i)$ is isomorphic to*

$$(\mathrm{Aut}_{\mathbf{K}}(b_{V_i})\times\mathrm{Aut}_{\mathbf{K}}(b_{V_i}))/\mathrm{Diag}(\mathrm{Aut}_{\mathbf{K}}(b_{V_i})).$$

In particular, $\mathcal{U}_i$ is nonempty. Taking $i=0$, we obtain that $\mathcal{U}_0$ is an $(\mathrm{Aut}_{\mathbf{K}}(b)\times\mathrm{Aut}_{\mathbf{K}}(b))$-space isomorphic to

$$(\mathrm{Aut}_{\mathbf{K}}(b)\times\mathrm{Aut}_{\mathbf{K}}(b))/\mathrm{Diag}(\mathrm{Aut}_{\mathbf{K}}(b)).$$

Proof of Lemma 2.2. By definition, any $W\in\pi_i^{-1}(V_i,V_i)$ satisfies $W\cap(V\oplus\{0\}) = V_i\oplus\{0\}$ and $W\cap(\{0\}\oplus V)=\{0\}\oplus V_i$, hence W contains $V_i\oplus V_i$ and $W\subset V_i^{\perp_b}\oplus V_i^{\perp_b}$ since W is $(b\oplus -b)$-isotropic. Thus $\pi_i^{-1}(V_i,V_i)$ is the set of maximal $(b\oplus -b)$-isotropic subspaces of $V_i^{\perp_b}\oplus V_i^{\perp_b}$ that contain $V_i\oplus V_i$ and correspond to maximal isotropic subspaces of $(V_i^{\perp_b}/V_i)\oplus(V_i^{\perp_b}/V_i)$ transverse to both factors. In particular, $\pi_i^{-1}(V_i,V_i)$ identifies with its image in $\mathcal{F}_{N-2i}(b_{V_i}\oplus -b_{V_i})$ and is endowed with an action of $\mathrm{Aut}_{\mathbf{K}}(b_{V_i})\times\mathrm{Aut}_{\mathbf{K}}(b_{V_i})$.

We first check that this action of $\mathrm{Aut}_{\mathbf{K}}(b_{V_i}) \times \mathrm{Aut}_{\mathbf{K}}(b_{V_i})$ is transitive. Let W'_0 be the image in $(V_i^{\perp_b}/V_i) \oplus (V_i^{\perp_b}/V_i)$ of

$$\{(v,v) \mid v \in V_i^{\perp_b}\} \subset V_i^{\perp_b} \oplus V_i^{\perp_b}.$$

The image W' in $(V_i^{\perp_b}/V_i) \oplus (V_i^{\perp_b}/V_i)$ of any element of $\pi_i^{-1}(V_i, V_i)$ meets the second factor $V_i^{\perp_b}/V_i$ trivially, hence is the graph of some linear endomorphism h of $V_i^{\perp_b}/V_i$. This h belongs to $\mathrm{Aut}_{\mathbf{K}}(b_{V_i})$ since W' is $(b_{V_i} \oplus -b_{V_i})$-isotropic. Thus $W' = (e,h) \cdot W'_0$ lies in the $(\mathrm{Aut}_{\mathbf{K}}(b_{V_i}) \times \mathrm{Aut}_{\mathbf{K}}(b_{V_i}))$-orbit of W'_0, proving transitivity.

Let us check that the stabilizer of W'_0 in $\mathrm{Aut}_{\mathbf{K}}(b_{V_i}) \times \mathrm{Aut}_{\mathbf{K}}(b_{V_i})$ is the diagonal $\mathrm{Diag}(\mathrm{Aut}_{\mathbf{K}}(b_{V_i}))$. For any $(g_1, g_2) \in \mathrm{Aut}_{\mathbf{K}}(b_{V_i}) \times \mathrm{Aut}_{\mathbf{K}}(b_{V_i})$,

$$(g_1, g_2) \cdot W'_0 = \{(g_1(v), g_2(v)) \mid v \in V_i^{\perp_b}/V_i\} = \{(v, g_2 g_1^{-1}(v)) \mid v \in V_i^{\perp_b}/V_i\},$$

and so $(g_1, g_2) \cdot W'_0 = W'_0$ if and only if $g_1 = g_2$. ☐

Lemma 2.3. *For any $0 \leq i \leq n$, the map π_i is surjective and the action of $\mathrm{Aut}_{\mathbf{K}}(b) \times \mathrm{Aut}_{\mathbf{K}}(b)$ on $\mathcal{U}_i$ is transitive.*

Proof. The map π_i is $(\mathrm{Aut}_{\mathbf{K}}(b) \times \mathrm{Aut}_{\mathbf{K}}(b))$-equivariant and the action of $\mathrm{Aut}_{\mathbf{K}}(b) \times \mathrm{Aut}_{\mathbf{K}}(b)$ on $\mathcal{F}_i(b) \times \mathcal{F}_i(-b)$ is transitive, hence π_i is surjective. To see that the action of $\mathrm{Aut}_{\mathbf{K}}(b) \times \mathrm{Aut}_{\mathbf{K}}(b)$ on $\mathcal{U}_i$ is transitive, it is enough to check that for any $V_i \in \mathcal{F}_i(b)$ the action of the stabilizer of (V_i, V_i) in $\mathrm{Aut}_{\mathbf{K}}(b) \times \mathrm{Aut}_{\mathbf{K}}(b)$ is transitive on the fiber $\pi_i^{-1}(V_i, V_i)$. This follows from Lemma 2.2. ☐

In particular, the fiber of π_i above any point of $\mathcal{F}_i(b) \times \mathcal{F}_i(b)$ is the image, by some element of $\mathrm{Aut}_{\mathbf{K}}(b) \times \mathrm{Aut}_{\mathbf{K}}(b)$, of the fiber $\pi_i^{-1}(V_i, V_i)$ described in Lemma 2.2.

Lemma 2.4. *For any $0 \leq i \leq n$, the dimension of the manifold $\mathcal{U}_i$ is*

$$\dim_{\mathbf{R}}(\mathcal{U}_i) = \dim_{\mathbf{R}}(\mathrm{Aut}_{\mathbf{K}}(b)) - i^2 \dim_{\mathbf{R}}(\mathbf{K}).$$

Proof. Consider two elements $V_i, V'_i \in \mathcal{F}_i(b)$ such that $V_i^{\perp_b} \cap V'_i = \{0\}$. Let

$$T = V_i^{\perp_b} \cap V_i'^{\perp_b} \simeq V_i^{\perp_b}/V_i.$$

The parabolic subgroups $P_i = \mathrm{Stab}_{\mathrm{Aut}_{\mathbf{K}}(b)}(V_i)$ and $P'_i = \mathrm{Stab}_{\mathrm{Aut}_{\mathbf{K}}(b)}(V'_i)$ are conjugate in $\mathrm{Aut}_{\mathbf{K}}(b)$. Let $(e_1, \dots, e_N)$ be a basis adapted to b and to the decomposition of V as a direct sum of V_i, T, and V'_i, i.e.

$$\begin{cases} V_i &= \mathrm{span}_{\mathbf{K}}(e_1, \dots, e_i), \\ T &= \mathrm{span}_{\mathbf{K}}(e_{i+1}, \dots, e_{N-i}), \\ V'_i &= \mathrm{span}_{\mathbf{K}}(e_{N-i+1}, \dots, e_N), \end{cases}$$

and $b(e_k, e_{N-i+\ell}) = \delta_{k,\ell}$ for all $k, \ell \in \{1, \dots, i\}$ (where $\delta_{\cdot,\cdot}$ is the Kronecker symbol). In this basis, the Lie algebra of $\mathrm{Aut}_{\mathbf{K}}(b)$ is given by block matrices as

$$\mathfrak{Aut}_{\mathbf{K}}(b) = \left\{ \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix} \in \mathfrak{gl}_N(\mathbf{K}) \;\middle|\; \begin{array}{l} BQ = {}^tF^\sigma,\ D = Q^tH^\sigma, \\ C = -\varepsilon^tC^\sigma,\ G = -\varepsilon^tG^\sigma, \\ I = -{}^tA^\sigma,\ EQ = -Q^tE^\sigma \end{array} \right\},$$

where $\varepsilon = 1$ if b is symmetric or Hermitian, $\varepsilon = -1$ if b is antisymmetric or anti-Hermitian, σ is the identity if b is symmetric or antisymmetric, σ is the conjugation (in $\mathbf{C}$ or $\mathbf{H}$) if b is Hermitian or anti-Hermitian, and Q is the matrix of the bilinear form $b|_T$ (so that ${}^tQ^\sigma = \varepsilon Q$). The Lie algebras $\mathfrak{p}_i$ of P_i and $\mathfrak{p}'_i$ of P'_i are given by

$$\mathfrak{p}_i = \left\{ \begin{pmatrix} A & B & C \\ 0 & E & F \\ 0 & 0 & I \end{pmatrix} \in \mathfrak{Aut}_{\mathbf{K}}(b) \right\} \quad \text{and} \quad \mathfrak{p}'_i = \left\{ \begin{pmatrix} A & 0 & 0 \\ D & E & 0 \\ G & H & I \end{pmatrix} \in \mathfrak{Aut}_{\mathbf{K}}(b) \right\}.$$

Their sum is thus equal to $\mathfrak{Aut}_{\mathbf{K}}(b)$ and $\mathfrak{p}_i \cap \mathfrak{p}_i' \simeq \mathfrak{gl}_{\mathbf{K}}(V_i) \times \mathfrak{Aut}_{\mathbf{K}}(b|_T)$. This implies

$$\begin{aligned} 2\dim_{\mathbf{R}}(\mathfrak{p}_i) &= \dim_{\mathbf{R}}(\mathfrak{p}_i) + \dim_{\mathbf{R}}(\mathfrak{p}_i') \\ &= \dim_{\mathbf{R}}(\mathfrak{p}_i + \mathfrak{p}_i') + \dim_{\mathbf{R}}(\mathfrak{p}_i \cap \mathfrak{p}_i') \\ &= \dim_{\mathbf{R}}(\mathrm{Aut}_{\mathbf{K}}(b)) + \dim_{\mathbf{R}}(\mathrm{GL}_{\mathbf{K}}(V_i)) + \dim_{\mathbf{R}}(\mathrm{Aut}_{\mathbf{K}}(b|_T)) \\ &= \dim_{\mathbf{R}}(\mathrm{Aut}_{\mathbf{K}}(b)) + i^2 \dim_{\mathbf{R}}(\mathbf{K}) + \dim_{\mathbf{R}}(\mathrm{Aut}_{\mathbf{K}}(b_{V_i})). \end{aligned}$$

Using Lemma 2.2, we obtain

$$\begin{aligned} \dim_{\mathbf{R}}(\mathcal{U}_i) =& 2\dim_{\mathbf{R}}(\mathcal{F}_i(b)) + \dim_{\mathbf{R}}(\mathrm{Aut}_{\mathbf{K}}(b_{V_i})) \\ =& 2\dim_{\mathbf{R}}(\mathrm{Aut}_{\mathbf{K}}(b)) - 2\dim_{\mathbf{R}}(P_i) + \dim_{\mathbf{R}}(\mathrm{Aut}_{\mathbf{K}}(b_{V_i})) \\ =& \dim_{\mathbf{R}}(\mathrm{Aut}_{\mathbf{K}}(b)) - i^2 \dim_{\mathbf{R}}(\mathbf{K}). \end{aligned}$$ ☐

By Lemma 2.4, we have $\dim_{\mathbf{R}}(\mathcal{U}_i) > \dim_{\mathbf{R}}(\mathcal{U}_j)$ for all $0 \le i < j \le n$.

Lemma 2.5. *For any $0 \le i \le n$, the closure S_i of $\mathcal{U}_i$ in $\mathcal{F}_N(b \oplus -b)$ is the union of the submanifolds $\mathcal{U}_j$ for $i \le j \le n$.*

Proof. The inclusion $S_i \subset \bigcup_{j \ge i} \mathcal{U}_j$ follows from the upper semicontinuity of the function $W \mapsto \dim_{\mathbf{R}}(W \cap (V \oplus \{0\}))$ on $\mathcal{F}_N(b \oplus -b)$. In order to prove the reverse inclusion, it is sufficient to show that $\mathcal{U}_{i+1} \subset S_i$: we can then conclude by descending induction on i. Let us establish this last inclusion.

Let V_i, V_i' and T be as in the proof of Lemma 2.4 and let $e, f \in T$ satisfy $b(e,e) = b(f,f) = 0$ and $b(e,f) = 1$. Let $S = T \cap \{e,f\}^{\perp_b}$, let b_S be the restriction of b to S, and let $R \in \mathcal{F}_{N-2i-2}(b_S \oplus -b_S)$ be transverse to the factors $S \oplus \{0\}$ and $\{0\} \oplus S$. We denote elements of $V \oplus V$ as pairs (v, v') with $v, v' \in V$.

The vectors (e,e) and (f,f) span a $(b \oplus -b)$-isotropic plane P. The direct sum of $V_i \oplus V_i'$, of R, and of P is a subspace $W \in \mathcal{F}_N(b \oplus -b)$ which belongs to $\mathcal{U}_i$ since its intersection with $V \oplus \{0\}$ is equal to $V_i \oplus \{0\}$.

For any $\lambda \in \mathbf{R}^*$, the linear map $g_\lambda : V \to V$ defined by

$$\begin{cases} g_\lambda(e) &= \lambda e, \\ g_\lambda(f) &= \lambda^{-1} f, \\ g_\lambda(v) &= v \qquad \text{for } v \in \{e,f\}^{\perp_b} \end{cases}$$

belongs to $\mathrm{Aut}_{\mathbf{K}}(b)$; furthermore, the element (g_λ, id) fixes pointwise $V_i \oplus V_i'$ and R, and sends (e,e) to $(\lambda e, e)$ and (f,f) to $(\lambda^{-1} f, f)$. The limit W' of $(g_\lambda, \mathrm{id}) \cdot W$ as $\lambda \to +\infty$ is thus spanned by $V_i \oplus V_i'$, by R, by $(e,0)$, and by $(0,f)$, and it belongs to S_i. The intersection $W' \cap (V \oplus \{0\})$ is spanned by $V_i \oplus \{0\}$ and $(e,0)$, hence W' belongs to $\mathcal{U}_{i+1}$. The $(\mathrm{Aut}_{\mathbf{K}}(b) \times \mathrm{Aut}_{\mathbf{K}}(b))$-orbit of W' is therefore equal to $\mathcal{U}_{i+1}$ and is contained in S_i. This completes the proof. ☐

By the Iwasawa decomposition, any maximal compact subgroup of $\mathrm{Aut}_{\mathbf{K}}(b \oplus -b)$ acts transitively on the flag variety $\mathcal{F}_N(b \oplus -b)$. By computing the stabilizer of a point in each case, we see that $\mathcal{F}_N(b \oplus -b)$ identifies with a Riemannian symmetric space of the compact type as in Table 1. This completes the proof of Theorem 1.1.

2.2. **The case $G = \mathrm{GL}_{\mathbf{K}}(V)$.** We now establish an analogue of Theorem 1.1 when $G = \mathrm{GL}_{\mathbf{K}}(V)$ is the full group of invertible $\mathbf{K}$-linear transformations of V. Here we use the notation $\mathcal{F}_i(V)$ to denote the Grassmannian of i-dimensional $\mathbf{K}$-subspaces of V, and N to denote $\dim_{\mathbf{K}}(V)$. Then (1.1) defines a map

$$\pi : \mathcal{F}_N(V \oplus V) \longrightarrow \Big(\bigcup_{i=0}^{N} \mathcal{F}_i(V)\Big) \times \Big(\bigcup_{i=0}^{N} \mathcal{F}_i(V)\Big).$$

Theorem 2.6. *Let V be an N-dimensional vector space over $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$, and $G = \mathrm{GL}_{\mathbf{K}}(V)$. The Grassmannian $X = \mathcal{F}_N(V \oplus V)$ of N-dimensional $\mathbf{K}$-subspaces of $V \oplus V$ is a smooth compactification of the group manifold $(G \times G)/\mathrm{Diag}(G)$ with the following properties:*

1. *X is a real analytic manifold (in fact complex analytic if $\mathbf{K} = \mathbf{C}$). Under the action of a maximal compact subgroup of $\mathrm{GL}_{\mathbf{K}}(V \oplus V)$, it identifies with a Riemannian symmetric space of the compact type, namely*
   - $\mathrm{O}(2N)/(\mathrm{O}(N) \times \mathrm{O}(N))$ *if* $\mathbf{K} = \mathbf{R}$,
   - $\mathrm{U}(2N)/(\mathrm{U}(N) \times \mathrm{U}(N))$ *if* $\mathbf{K} = \mathbf{C}$,
   - $\mathrm{Sp}(2N)/(\mathrm{Sp}(N) \times \mathrm{Sp}(N))$ *if* $\mathbf{K} = \mathbf{H}$.
2. *The $(G \times G)$-orbits in X are the submanifolds $\mathcal{U}_{i,j} := \pi^{-1}(\mathcal{F}_i(V) \times \mathcal{F}_j(V))$ for $i, j \geq 0$ with $i + j \leq N$; there are $(N + 1)(N + 2)/2$ of them. They have dimension $\dim_{\mathbf{K}}(\mathcal{U}_{i,j}) = \dim_{\mathbf{K}}(G) - i^2 - j^2 = N^2 - i^2 - j^2$. The closure of $\mathcal{U}_{i,j}$ in X is $\bigcup_{k \geq i,\, \ell \geq j} \mathcal{U}_{k,\ell}$.*
3. *For $0 \leq i+j \leq N$, the map π defines a fibration $\pi_{i,j}$ of $\mathcal{U}_{i,j}$ over $\mathcal{F}_i(V) \times \mathcal{F}_j(V)$ with fibers given by Lemma 2.8 below.*

In particular, $\mathcal{U}_{0,0}$ is the unique open $(G \times G)$-orbit in X and it identifies with $(G \times G)/\mathrm{Diag}(G)$.

Any $(\mathrm{SL}_{\mathbf{K}}(V) \times \mathrm{SL}_{\mathbf{K}}(V))$-orbit $\mathcal{O} \subset \mathcal{U}_{0,0}$ identifies with

$$(\mathrm{SL}_{\mathbf{K}}(V) \times \mathrm{SL}_{\mathbf{K}}(V))/\mathrm{Diag}(\mathrm{SL}_{\mathbf{K}}(V));$$

the closure of $\mathcal{O}$ in X is the union of $\mathcal{O}$ and of the $\mathcal{U}_{i,j}$ for $i, j \geq 1$ with $i + j \leq N$.

Remark 2.7. For $\mathbf{K} = \mathbf{C}$ and $N = 2$, the $(\mathrm{SL}_2(\mathbf{C}) \times \mathrm{SL}_2(\mathbf{C}))$-equivariant compactification of $\mathrm{SL}_2(\mathbf{C})$ given by Theorem 2.6 was previously described by Guillot [Gui07], who showed that this is the only $(\mathrm{SL}_2(\mathbf{C}) \times \mathrm{SL}_2(\mathbf{C}))$-equivariant compactification of $\mathrm{SL}_2(\mathbf{C})$ as a complex manifold. It identifies with the compactification of $\mathrm{Sp}(2, \mathbf{C})$ from Theorem 1.1.

The proof of Theorem 2.6 is similar to Theorem 1.1: the group $\mathrm{GL}_{\mathbf{K}}(V) \times \mathrm{GL}_{\mathbf{K}}(V)$ naturally embeds into $\mathrm{GL}_{\mathbf{K}}(V \oplus V)$. For $i, j \geq 0$ with $i + j \leq N$, the set

$$\mathcal{U}_{i,j} := \pi^{-1}\big(\mathcal{F}_i(V) \times \mathcal{F}_j(V)\big) \subset \mathcal{F}_N(V \oplus V)$$

is invariant under $\mathrm{GL}_{\mathbf{K}}(V) \times \mathrm{GL}_{\mathbf{K}}(V)$, and $X = \mathcal{F}_N(V \oplus V)$ is the union of these sets $\mathcal{U}_{i,j}$. Here it is clear that $\mathcal{U}_{i,j}$ is nonempty for all i, j. Let

$$\pi_{i,j} : \mathcal{U}_{i,j} \longrightarrow \mathcal{F}_i(V) \times \mathcal{F}_j(V)$$

be the map induced by π. By construction, $\pi_{i,j}$ is $(\mathrm{GL}_{\mathbf{K}}(V) \times \mathrm{GL}_{\mathbf{K}}(V))$-equivariant, hence surjective (because the action of $\mathrm{GL}_{\mathbf{K}}(V)$ on $\mathcal{F}_i(V)$ and $\mathcal{F}_j(V)$ is transitive). As above, it is enough to determine the fiber of $\pi_{i,j}$ above one particular point of $\mathcal{F}_i(V) \times \mathcal{F}_j(V)$. Let $(e_1, \dots, e_N)$ be a basis of V. We set

$$(2.2)\qquad \begin{cases} V_i & := & \mathrm{span}_{\mathbf{K}}(e_1, \dots, e_i), \\ V'_i & := & \mathrm{span}_{\mathbf{K}}(e_{i+1}, \dots, e_N), \\ V_j & := & \mathrm{span}_{\mathbf{K}}(e_{N-j+1}, \dots, e_N), \\ V'_j & := & \mathrm{span}_{\mathbf{K}}(e_1, \dots, e_{N-j}), \\ V'_{i,j} & := & V'_i \cap V'_j = \mathrm{span}_{\mathbf{K}}(e_{i+1}, \dots, e_{N-j}), \end{cases}$$

so that V is the direct sum of V_i and V'_i, and also of V'_j and V_j. By assumption, $i + j \leq N$, hence $V_i \cap V_j = \{0\}$. The quotient V/V_i identifies with V'_i, which is the direct sum of $V'_{i,j}$ and V_j. Similarly, the quotient V/V_j identifies with V'_j, which is the direct sum of V_i and $V'_{i,j}$. We see (V_i, V_j) as an element of $\mathcal{F}_i(V) \times \mathcal{F}_j(V)$.

Lemma 2.8. *The fiber $\pi_{i,j}^{-1}(V_i,V_j)\subset\mathcal{F}_N(V\oplus V)$ is the set of N-dimensional $\mathbf{K}$-subspaces of $V\oplus V$ that contain $V_i\oplus V_j$ and project to $(N-i-j)$-dimensional $\mathbf{K}$-subspaces of $(V/V_i)\oplus(V/V_j)$ transverse to both factors $(V/V_i)\oplus\{0\}$ and $\{0\}\oplus(V/V_j)$. As a $(\mathrm{GL}_{\mathbf{K}}(V/V_i)\times\mathrm{GL}_{\mathbf{K}}(V/V_j))$-space, $\pi_i^{-1}(V_i,V_j)$ is isomorphic to the quotient of*

$$\mathrm{GL}_{\mathbf{K}}(V/V_i)\times\mathrm{GL}_{\mathbf{K}}(V/V_j)\simeq\mathrm{GL}_{\mathbf{K}}(V_i')\times\mathrm{GL}_{\mathbf{K}}(V_j')$$

by the subgroup consisting of the pairs of block matrices

$$\left\{\left(\begin{pmatrix}A&B\\0&C\end{pmatrix},\begin{pmatrix}D&0\\E&A\end{pmatrix}\right)\;\middle|\;\begin{matrix}A\in\mathrm{GL}_{\mathbf{K}}(V_{i,j}'),\,C\in\mathrm{GL}_{\mathbf{K}}(V_j),\,D\in\mathrm{GL}_{\mathbf{K}}(V_i),\\ B\in\mathrm{Hom}_{\mathbf{K}}(V_j,V_{i,j}'),\,E\in\mathrm{Hom}_{\mathbf{K}}(V_i,V_{i,j}')\end{matrix}\right\}. \tag{2.3}$$

Proof. The first statement is clear. For the second statement, one easily checks that $\pi_{i,j}^{-1}(V_i,V_j)$ is the $(\mathrm{GL}_{\mathbf{K}}(V_i')\times\mathrm{GL}_{\mathbf{K}}(V_j'))$-orbit of

$$W_0:=(\{0\}\oplus V_j)+(V_i\oplus\{0\})+\big\{(v,v)\mid v\in V_{i,j}'\big\}$$

and that the stabilizer of W_0 in $\mathrm{GL}_{\mathbf{K}}(V_i')\times\mathrm{GL}_{\mathbf{K}}(V_j')$ is (2.3). □

In particular, $\mathcal{U}_{0,0}$ is a $(\mathrm{GL}_{\mathbf{K}}(V)\times\mathrm{GL}_{\mathbf{K}}(V))$-space isomorphic to

$$(\mathrm{GL}_{\mathbf{K}}(V)\times\mathrm{GL}_{\mathbf{K}}(V))/\mathrm{Diag}(\mathrm{GL}_{\mathbf{K}}(V)).$$

Similarly to Lemma 2.3, for any $i,j\geq 0$ with $i+j\leq N$, the action of $\mathrm{GL}_{\mathbf{K}}(V)\times\mathrm{GL}_{\mathbf{K}}(V)$ on $\mathcal{U}_{i,j}$ is transitive. Note that $\dim_{\mathbf{K}}(\mathcal{F}_i(V))=i(N-i)$. From Lemma 2.8 we compute $\dim_{\mathbf{K}}(\pi_{i,j}^{-1}(V_i,V_j))=N^2-(i+j)N$, and so

$$\begin{aligned}\dim_{\mathbf{K}}(\mathcal{U}_{i,j})&=\dim_{\mathbf{K}}(\mathcal{F}_i(V))+\dim_{\mathbf{K}}(\mathcal{F}_j(V))+\dim_{\mathbf{K}}(\pi_{i,j}^{-1}(V_i,V_j))\\&=N^2-i^2-j^2.\end{aligned}$$

In particular, $\dim_{\mathbf{K}}(\mathcal{U}_{i,j})>\dim_{\mathbf{K}}(\mathcal{U}_{k,\ell})$ for all $(i,j)\neq(k,\ell)$ with $i\leq k$ and $j\leq\ell$. By upper semicontinuity of the functions $W\mapsto\dim_{\mathbf{K}}(W\cap(V\oplus\{0\}))$ and $W\mapsto\dim_{\mathbf{K}}(W\cap(\{0\}\oplus V))$, the closure $S_{i,j}$ of $\mathcal{U}_{i,j}$ in $\mathcal{F}_N(V\oplus V)$ is the union of the submanifolds $\mathcal{U}_{k,\ell}$ for $k\geq i$ and $\ell\geq j$.

By the Iwasawa decomposition, any maximal compact subgroup of $\mathrm{GL}_{\mathbf{K}}(V\oplus V)$ acts transitively on the flag variety $\mathcal{F}_N(V\oplus V)$. By computing the stabilizer of a point, we see that $\mathcal{F}_N(V\oplus V)$ identifies with a Riemannian symmetric space of the compact type as in Theorem 2.6.(1).

We now determine the closure in X of the $(\mathrm{SL}_{\mathbf{K}}(V)\times\mathrm{SL}_{\mathbf{K}}(V))$-orbit $\mathcal{O}$ of

$$W_0:=\{(v,v)\mid v\in V\}\in\mathcal{U}_{0,0}.$$

For this we use a Cartan decomposition $\mathrm{SL}_{\mathbf{K}}(V)=K(\exp\overline{\mathfrak{a}}^+)K$ where K is a maximal compact subgroup of G and, in some basis $(e_1,\dots,e_N)$ of V, the set $\exp\overline{\mathfrak{a}}^+$ consists of the diagonal $(N\times N)$-matrices of determinant 1 whose entries are positive and in nonincreasing order, see Example 3.1 below. Consider a sequence $(g_m,g_m')\in(\mathrm{SL}_{\mathbf{K}}(V)\times\mathrm{SL}_{\mathbf{K}}(V))^{\mathbf{N}}$. For any $m\in\mathbf{N}$, we may write $g_m'g_m^{-1}=k_ma_mk_m'$ where $k_m,k_m'\in K$ and $a_m=\mathrm{diag}(\lambda_{1,m},\dots,\lambda_{N,m})\in\exp\overline{\mathfrak{a}}^+$; then

$$(g_m,g_m')\cdot W_0=\big\{\big(k'^{-1}_m\cdot v,k_ma_m\cdot v\big)\mid v\in V\big\}=\big\{\big(k'^{-1}_m a_m^{-1}\cdot v,k_m\cdot v\big)\mid v\in V\big\}.$$

Up to passing to a subsequence, by compactness of K, we may assume that the sequences $(k_m)_{m\in\mathbf{N}}$ and $(k_m')_{m\in\mathbf{N}}$ converge to some $k,k'\in K$, respectively. If $(a_m)_{m\in\mathbf{N}}$ is bounded, then all accumulation points of $((g_m,g_m')\cdot W_0)_{m\in\mathbf{N}}$ belong to $\mathcal{O}$. Otherwise, up to passing again to a subsequence, we may assume that for any $1 \leq \ell \leq N$ we have $\lambda_{\ell,m} \to \lambda_\ell$ where, for some $i, j \geq 1$,

$$\begin{cases} \lambda_\ell = +\infty & \text{for } 1 \leq \ell \leq i, \\ \lambda_\ell \in (0, +\infty) & \text{for } i < \ell \leq N - j, \\ \lambda_\ell = 0 & \text{for } N - j < \ell. \end{cases}$$

As in (2.2), let

$$\begin{cases} V_i & := & \mathrm{span}_{\mathbf{K}}(e_1, \dots, e_i), \\ V'_{i,j} & := & \mathrm{span}_{\mathbf{K}}(e_{i+1}, \dots, e_{N-j}), \\ V_j & := & \mathrm{span}_{\mathbf{K}}(e_{N-j+1}, \dots, e_N), \end{cases}$$

and let a be the endomorphism of $V'_{i,j}$ given by the matrix $\mathrm{diag}(\lambda_{i+1}, \dots, \lambda_{N-j})$ in the basis $(e_{i+1}, \dots, e_{N-j})$. Then $(g_m, g'_m) \cdot W_0$ tends to

$$(\{0\} \oplus k \cdot V_i) + \{(k'^{-1} \cdot v, ka \cdot v) \mid v \in V'_{i,j}\} + \big(k'^{-1} \cdot V_j \oplus \{0\}\big) \in \mathcal{U}_{i,j}.$$

For $i, j \geq 1$ the action of $\mathrm{SL}_{\mathbf{K}}(V) \times \mathrm{SL}_{\mathbf{K}}(V)$ on $\mathcal{U}_{i,j}$ is transitive, and so the closure of $\mathcal{O}$ in X is the union of $\mathcal{O}$ and of the $\mathcal{U}_{i,j}$ for $i, j \geq 1$.

This completes the proof of Theorem 2.6.

3. Reminders on Anosov representations and their domains of discontinuity

In this section we recall the definition of an Anosov representation into a reductive Lie group, see [Lab06, GW12, GGKW16], and the construction of domains of discontinuity given in [GW12]. We first introduce some notation.

3.1. **Notation.** Let G be a real reductive Lie group with Lie algebra $\mathfrak{g}$. We assume G to be noncompact, equal to a finite union of connected components (for the real topology) of $\mathbf{G}(\mathbf{R})$ for some algebraic group $\mathbf{G}$. Recall that $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) + \mathfrak{g}_s$, where $\mathfrak{z}(\mathfrak{g})$ is the Lie algebra of the center of G and $\mathfrak{g}_s$ the Lie algebra of the derived subgroup of G, which is semisimple. Let K be a maximal compact subgroup of G, with Lie algebra $\mathfrak{k}$. Let $\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{z}(\mathfrak{g})) + (\mathfrak{a} \cap \mathfrak{g}_s)$ be a maximal abelian subspace of the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ for the Killing form; we shall call $\mathfrak{a}$ a *Cartan subspace* of $\mathfrak{g}$. The *real rank* of G is by definition the dimension of $\mathfrak{a}$. Let Σ be the set of restricted roots of $\mathfrak{a}$ in $\mathfrak{g}$, i.e. the set of nonzero linear forms $\alpha \in \mathfrak{a}^*$ for which

$$\mathfrak{g}_\alpha := \{z \in \mathfrak{g} \mid \mathrm{ad}(a)(z) = \langle \alpha, a \rangle\, z \quad \forall a \in \mathfrak{a}\}$$

is nonzero. Choose a system of *simple roots* $\Delta \subset \Sigma$, i.e. any element of Σ is expressed uniquely as a linear combination of elements of Δ with coefficients all of the same sign. Let

$$\overline{\mathfrak{a}}^+ := \{Y \in \mathfrak{a} \mid \langle \alpha, Y \rangle \geq 0 \quad \forall \alpha \in \Delta\}$$

be the closed positive Weyl chamber of $\mathfrak{a}$ associated with Δ. The *Weyl group* of $\mathfrak{a}$ in $\mathfrak{g}$ is the group $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$, where $N_K(\mathfrak{a})$ (resp. $Z_K(\mathfrak{a})$) is the normalizer (resp. centralizer) of $\mathfrak{a}$ in K. There is a unique element $w_0 \in W$ such that $w_0 \cdot \overline{\mathfrak{a}}^+ = -\overline{\mathfrak{a}}^+$; the involution of $\mathfrak{a}$ defined by $Y \mapsto -w_0 \cdot Y$ is called the *opposition involution*. The corresponding dual linear map preserves Δ; we shall denote it by

$$\begin{aligned} \mathfrak{a}^* &\longrightarrow \mathfrak{a}^* \\ \alpha &\longmapsto \alpha^\star = -\alpha \circ w_0. \end{aligned} \tag{3.1}$$

Recall that the *Cartan decomposition* $G = K(\exp \overline{\mathfrak{a}}^+)K$ holds: any $g \in G$ may be written $g = k(\exp \mu(g))k'$ for some $k, k' \in K$ and a unique $\mu(g) \in \overline{\mathfrak{a}}^+$ (see [Hel01, Ch. IX, Th. 1.1]). This defines a map

$$\mu : G \longrightarrow \overline{\mathfrak{a}}^+ \tag{3.2}$$

called the *Cartan projection*, inducing a homeomorphism $K\backslash G/K \simeq \overline{\mathfrak{a}}^+$. We refer to [GGKW16, § 2.3] for more details.

Example 3.1. For $\mathbf{K} = \mathbf{R}$ (resp. $\mathbf{C}$, resp. $\mathbf{H}$), the real Lie group $G = \mathrm{SL}_d(\mathbf{K})$ admits the Cartan decomposition $G = K(\exp \overline{\mathfrak{a}}^+)K$ where $K = \mathrm{O}(d)$ (resp. $\mathrm{U}(d)$, resp. $\mathrm{Sp}(d)$), and $\mathfrak{a} \subset \mathfrak{gl}_d(\mathbf{K})$ is the set of traceless real diagonal matrices of size $d \times d$. For $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, the diagonal entries of $\mu(g)$ are the logarithms of the singular values of $g \in G$ (i.e. of the square roots of the eigenvalues of ${}^t\bar{g}g$, where $\bar{g}$ is the complex conjugate of g), in nonincreasing order.

Let $\Sigma^+ \subset \Sigma$ be the set of positive roots with respect to Δ, i.e. restricted roots that are nonnegative linear combinations of elements of Δ. For any nonempty subset θ of Δ, we denote by P_θ the normalizer in G of the Lie algebra $\mathfrak{u}_\theta = \bigoplus_{\alpha \in \Sigma^+ \smallsetminus \mathrm{span}(\Delta \smallsetminus \theta)} \mathfrak{g}_\alpha$. Explicitly,

$$\mathrm{Lie}(P_\theta) = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Sigma^+ \cap \mathrm{span}(\Delta \smallsetminus \theta)} \mathfrak{g}_{-\alpha}.$$

In particular, $P_\emptyset = G$ and P_Δ is a minimal parabolic subgroup of G.[1] Any parabolic subgroup of G is conjugate to P_θ for some $\theta \subset \Delta$.

3.2. **Proper actions and sharp actions.** Fix a W-invariant Euclidean norm $\|\cdot\|$ on $\mathfrak{a}$. For any point $x \in \mathfrak{a}$ and any subset $S \subset \mathfrak{a}$, we denote by

$$\mathrm{dist}_{\mathfrak{a}}(x, S) = \inf_{s \in S} \|x - s\|$$

the corresponding distance from x to S. The following properness criterion of Benoist and Kobayashi shows that the Cartan projection μ of (3.2) can be used to understand properly discontinuous actions on homogeneous spaces of G.

Fact 3.2 ([Ben96, Kob96])**.** *Let Γ be a discrete subgroup of G and H a closed subgroup of G. The action of Γ on G/H is properly discontinuous if and only if*

$$\lim_{\gamma \to \infty} \mathrm{dist}_{\mathfrak{a}}(\mu(\gamma), \mu(H)) = +\infty.$$

This condition means that $\lim_{n \to +\infty} \mathrm{dist}_{\mathfrak{a}}(\mu(\gamma_n), \mu(H)) = +\infty$ for any sequence $(\gamma_n)_{n \in \mathbf{N}}$ of pairwise distinct elements of Γ.

A quantitative way of understanding proper actions is given by the notion of *sharpness*, which was introduced by Kassel and Kobayashi [KK16].

Definition 3.3. Let $\Gamma < G$ be a discrete subgroup and let $H < G$ be a closed subgroup. The action of Γ on G/H is *sharp* it there exist $c, C > 0$ such that for any $\gamma \in \Gamma$,

$$\mathrm{dist}_{\mathfrak{a}}(\mu(\gamma), \mu(H)) \geq c\,\|\mu(\gamma)\| - C.$$

Besides its geometric content, this notion is also relevant to the spectral theory of the Laplacian on pseudo-Riemannian locally symmetric spaces, see [KK16].

The following estimates are useful when manipulating the Cartan projection (see e.g. [Kas08, Lem. 2.3]): for any $g_1, g_2, g_3, g \in G$,

$$\|\mu(g_1g_2g_3) - \mu(g_2)\| \leq \|\mu(g_1)\| + \|\mu(g_3)\| \quad \text{and} \quad \|\mu(g^{-1})\| = \|\mu(g)\|. \tag{3.3}$$

[1] This is the same convention as in [GGKW16]. In [GW12] however, θ and $\Delta \smallsetminus \theta$ are switched.

3.3. **Anosov representations.** The following definition of Anosov representations is not the original one from [Lab06, GW12], but an equivalent one taken from [GGKW16, Th. 1.3] (see also [KLPc]).

Definition 3.4. Let Γ be a word hyperbolic group, with boundary at infinity $\partial_\infty\Gamma$. Let $\theta \subset \Delta$ be a nonempty subset of the simple restricted roots. A representation $\rho : \Gamma \to G$ is *P_θ-Anosov* if there exists a pair of continuous ρ-equivariant boundary maps

$$\xi^+ : \partial_\infty\Gamma \to G/P_\theta \quad \text{and} \quad \xi^- : \partial_\infty\Gamma \to G/P_{\theta^\star}$$

that are dynamics-preserving for ρ and transverse, and if for any $\alpha \in \theta$,

$$\lim_{\gamma\to\infty} \langle \alpha, \mu(\rho(\gamma))\rangle = +\infty. \tag{3.4}$$

By *dynamics-preserving* we mean that for any $\gamma \in \Gamma$ of infinite order with attracting fixed point $\eta_\gamma^+ \in \partial_\infty\Gamma$, the point $\xi^+(\eta_\gamma^+)$ (resp. $\xi^-(\eta_\gamma^+)$ is an attracting fixed point for the action of $\rho(\gamma)$ on G/P_θ (resp. $G/P_{\theta^\star}$). By *transverse* we mean that pairs of distinct points in $\partial_\infty\Gamma$ are sent to transverse pairs in $G/P_\theta \times G/P_{\theta^\star}$, i.e. to pairs belonging to the unique open G-orbit in $G/P_\theta \times G/P_{\theta^\star}$ (for the diagonal action of G). Condition (3.4) means that $\lim_{n\to+\infty}\langle \alpha, \mu(\rho(\gamma_n))\rangle$ for any sequence $(\gamma_n)_{n\in\mathbf{N}}$ of pairwise distinct elements of Γ.

The maps ξ^+, ξ^- are unique, entirely determined by ρ.

Remark 3.5. We will often use the definition when $\theta = \theta^\star$, in which case $G/P_\theta = G/P_{\theta^\star}$ and $\xi^+ = \xi^-$ by the aforementioned uniqueness. This common map $\xi^+ = \xi^-$ will be then denoted by ξ and called the equivariant boundary map associated with ρ.

By [Lab06, GW12], any P_θ-Anosov representation is a quasi-isometric embedding; in particular, it has discrete image and finite kernel. The set of P_θ-Anosov representations is open in $\mathrm{Hom}(\Gamma, G)$. Any P_θ-Anosov representation is $P_{\theta'}$-Anosov for any $\theta' \subset \theta$ [GW12, Lem. 3.18]; thus the strongest form of Anosov is with respect to the minimal proper parabolic subgroup P_Δ.

We shall use the following fact from [GGKW16, Th. 1.3 & Cor. 1.9], which also follows from [KLPc].

Lemma 3.6. *If $\rho : \Gamma \to G$ is P_θ-Anosov, then the following strengthening of* (3.4) *is satisfied: there exist $c, C > 0$ such that for any $\alpha \in \theta$ and any $\gamma \in \Gamma$,*

$$\mathrm{dist}_{\mathfrak{a}}(\mu(\gamma), \mathrm{Ker}(\alpha)) \geq c\,\|\mu(\rho(\gamma))\| - C.$$

In particular, Γ acts sharply, via ρ, on G/H for any closed subgroup H of G with $\mu(H) \subset \bigcup_{\alpha\in\theta} \mathrm{Ker}(\alpha)$.

3.4. **Uniform domination.** Let $\lambda : G \to \overline{\mathfrak{a}}^+$ be the *Lyapunov projection* of G, i.e. the projection induced by the Jordan decomposition: any $g \in G$ can be written uniquely as the commuting product $g = g_h g_e g_u$ of a hyperbolic, an elliptic, and a unipotent element (see e.g. [Ebe96, Th. 2.19.24]), and $\exp(\lambda(g))$ is the unique element of $\exp(\overline{\mathfrak{a}}^+)$ in the conjugacy class of g_h. For any $g \in G$,

$$\lambda(g) = \lim_{n\to+\infty} \frac{1}{n}\mu(g^n). \tag{3.5}$$

For any simple restricted root $\alpha \in \Delta$, let $\omega_\alpha \in \mathfrak{a}^*$ be the fundamental weight associated with α: by definition, for any $\beta \in \Delta$,

$$2\frac{(\omega_\alpha, \beta)}{(\alpha, \alpha)} = \delta_{\alpha,\beta},$$

where $(\cdot,\cdot)$ is a W-invariant inner product on $\mathfrak{a}^*$ and $\delta_{\cdot,\cdot}$ is the Kronecker symbol. We shall use the following terminology from [GGKW16].

Definition 3.7. Let $\alpha \in \Delta$. A representation $\rho_L : \Gamma \to G$ *uniformly ω_α-dominates* a representation $\rho_R : \Gamma \to G$ if there exists $c < 1$ such that for any $\gamma \in \Gamma$,

$$\langle \omega_\alpha, \lambda(\rho_R(\gamma))\rangle \leq c\,\langle \omega_\alpha, \lambda(\rho_L(\gamma))\rangle.$$

Remark 3.8. Uniform ω_α-domination implies uniform $\omega_{\alpha^\star}$-domination. Indeed, for any $g \in G$ we have $\langle \omega_{\alpha^\star}, \lambda(g)\rangle = \langle \omega_\alpha^\star, \lambda(g)\rangle = \langle \omega_\alpha, \lambda(g^{-1})\rangle$.

3.5. **Anosov representations into $\mathrm{Aut}_{\mathbf{K}}(b)$ and $\mathrm{Aut}_{\mathbf{K}}(b\oplus -b)$.** Let $G = \mathrm{Aut}_{\mathbf{K}}(b)$ where b is a nondegenerate $\mathbf{R}$-bilinear form on a $\mathbf{K}$-vector space V as in Section 1.1.

In all cases except when $\mathbf{K} = \mathbf{R}$ and b is a symmetric bilinear form of signature (n,n), the restricted root system is of type B_n, C_n, or BC_n. (See [Hel01, Ch. X, Th. 3.28] for definitions of the types.) We can choose the system of simple restricted roots $\Delta = \{\alpha_i(b) \mid 1 \leq i \leq n\}$ so that for any $1 \leq i \leq n$ the parabolic subgroup $P_i(b) := P_{\{\alpha_i(b)\}}$ is the stabilizer of an i-dimensional b-isotropic $\mathbf{K}$-subspace of V. The space $\mathcal{F}_i(b)$ of i-dimensional b-isotropic $\mathbf{K}$-subspaces of V then identifies with $G/P_i(b)$. We have $\alpha_i(b) = \alpha_i(b)^\star$ for all $1 \leq i \leq n$.

In the case that $\mathbf{K} = \mathbf{R}$ and b is a symmetric bilinear form of signature (n,n), the restricted root system is of type D_n. We can still choose the system of simple restricted roots $\Delta = \{\alpha_i(b) \mid 1 \leq i \leq n\}$ so that for any $1 \leq i \leq n-2$ the parabolic subgroup $P_i(b) := P_{\{\alpha_i(b)\}}$ is the stabilizer of an i-dimensional b-isotropic subspace of V. We have $\alpha_i(b) = \alpha_i(b)^\star$ for all $1 \leq i \leq n-2$. When n is even, $\alpha_{n-1}(b) = \alpha_{n-1}(b)^\star$ and $\alpha_n(b) = \alpha_n(b)^\star$ whereas when n is odd, $\alpha_{n-1}(b) = \alpha_n(b)^\star$. The parabolic subgroups $P_{n-1}(b) := P_{\{\alpha_{n-1}(b)\}}$ and $P_n(b) := P_{\{\alpha_n(b)\}}$ are both stabilizers of n-dimensional b-isotropic subspaces of V, and they are conjugate by some element $g \in \mathrm{Aut}_{\mathbf{K}}(b) \smallsetminus \mathrm{Aut}_{\mathbf{K}}(b)_0$. The stabilizer of an $(n-1)$-dimensional b-isotropic subspace is conjugate to $P_{n-1}(b) \cap P_n(b) = P_{\{\alpha_{n-1}(b),\alpha_n(b)\}}$.

We shall use the following result.

Lemma 3.9 ([GGKW16, Th. 7.3])**.** *For $\rho_L, \rho_R \in \mathrm{Hom}(\Gamma, \mathrm{Aut}_{\mathbf{K}}(b))$, the representation $\rho_L \oplus \rho_R : \Gamma \to \mathrm{Aut}_{\mathbf{K}}(b) \times \mathrm{Aut}_{\mathbf{K}}(-b) \hookrightarrow \mathrm{Aut}_{\mathbf{K}}(b \oplus -b)$ is $P_1(b\oplus -b)$-Anosov if and only if one of the two representations ρ_L or ρ_R is $P_1(b)$-Anosov and uniformly $\omega_{\alpha_1(b)}$-dominates the other.*

Since the boundary map of an Anosov representation is dynamics-preserving, Lemma 3.9 immediately implies the following.

Corollary 3.10. *If $\rho_L \oplus \rho_R : \Gamma \to \mathrm{Aut}_{\mathbf{K}}(b) \times \mathrm{Aut}_{\mathbf{K}}(-b) \hookrightarrow \mathrm{Aut}_{\mathbf{K}}(b \oplus -b)$ is $P_1(b \oplus -b)$-Anosov, then its boundary map*

$$\xi : \partial_\infty\Gamma \longrightarrow \mathcal{F}_1(b \oplus -b)$$

is, up to switching ρ_L and ρ_R, the composition of the boundary map $\xi_L : \partial_\infty\Gamma \to \mathcal{F}_1(b)$ of ρ_L with the natural embedding $\mathcal{F}_1(b) \hookrightarrow \mathcal{F}_1(b \oplus -b)$.

We will always be able to reduce to $P_1(b)$-Anosov representations into $\mathrm{Aut}_{\mathbf{K}}(b)$ using the following result.

Proposition 3.11 ([GGKW16, Prop. 3.5 & 7.8, Fact 2.34, and § 7.3])**.** *Let $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$, or the ring $\mathbf{H}$ of quaternions. For any real reductive Lie group G and any nonempty subset $\theta \subset \Delta$ of the simple restricted roots, there exist a nondegenerate $\mathbf{R}$-bilinear form b on a $\mathbf{K}$-vector space V and an irreducible linear representation $\tau : G \to \mathrm{Aut}_{\mathbf{K}}(b)$ with the following properties:*

(1) an arbitrary representation $\rho : \Gamma \to G$ is P_θ-Anosov if and only if the composition $\tau \circ \rho : \Gamma \to \mathrm{Aut}_{\mathbf{K}}(b)$ is $P_1(b)$-Anosov;
(2) if a representation $\rho_L : \Gamma \to G$ uniformly ω_α-dominates another representation $\rho_R : \Gamma \to G$ for all $\alpha \in \theta$, then $\tau \circ \rho_L : \Gamma \to \mathrm{Aut}_{\mathbf{K}}(b)$ uniformly $\omega_{\alpha_1(b)}$-dominates $\tau \circ \rho_R : \Gamma \to \mathrm{Aut}_{\mathbf{K}}(b)$.

The existence of such b and τ satisfying (1) was first proved in [GW12, § 4] for $\mathbf{K} = \mathbf{R}$. In fact, the irreducible representations τ satisfying (1) and (2) are exactly those for which the highest restricted weight χ of τ satisfies

$$\{\alpha \in \Delta \mid (\alpha, \chi) > 0\} = \theta \cup \theta^\star$$

and for which the weight space corresponding to χ is a line; there are infinitely many such τ.

Example 3.12. For $G = \mathrm{GL}_d(\mathbf{R})$ and $\theta = \{\varepsilon_1 - \varepsilon_2\}$, we can take τ to be the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}_{\mathbf{R}}(\mathfrak{g})$ and b to be the Killing form of $\mathfrak{g}$.

3.6. **Domains of discontinuity.** We shall use the following result.

Proposition 3.13 ([GW12, Th. 8.6])**.** *Let Γ be a word hyperbolic group.*

(1) For any $P_1(b)$-Anosov representation $\rho : \Gamma \to \mathrm{Aut}_{\mathbf{K}}(b)$ with boundary map $\xi : \partial_\infty\Gamma \to \mathcal{F}_1(b)$, the group Γ acts properly discontinuously and cocompactly, via ρ, on the complement Ω in $\mathcal{F}_n(b)$ of

$$\mathcal{K}_\xi := \bigcup_{\eta \in \partial_\infty\Gamma} \{W \in \mathcal{F}_n(b) \mid \xi(\eta) \subset W\} \subset \mathcal{F}_n(b).$$

(2) Suppose we are not *in the case that $\mathbf{K} = \mathbf{R}$ and b is a symmetric bilinear form of signature (n, n). For any $P_n(b)$-Anosov representation $\rho : \Gamma \to \mathrm{Aut}_{\mathbf{K}}(b)$ with boundary map $\xi : \partial_\infty\Gamma \to \mathcal{F}_n(b)$, the group Γ acts properly discontinuously and cocompactly, via ρ, on the complement Ω in $\mathcal{F}_1(b)$ of*

$$\mathcal{K}_\xi := \bigcup_{\eta \in \partial_\infty\Gamma} \{\ell \in \mathcal{F}_1(b) \mid \ell \subset \xi(\eta)\} \subset \mathcal{F}_1(b).$$

Contrary to what is stated in [GW12, Th. 8.6], the case of $\mathrm{O}(n, n)$ (i.e. of a restricted root system of type D_n) has to be excluded in point (2) of the proposition.

4. Properly discontinuous actions on group manifolds

Let $G = \mathrm{Aut}_{\mathbf{K}}(b)$ where b is a nondegenerate $\mathbf{R}$-bilinear form on a $\mathbf{K}$-vector space V as in Section 1.1. By Theorem 1.1, the $(G \times G)$-orbits in the space $\mathcal{F}_N(b \oplus -b)$ of maximal $(b \oplus -b)$-isotropic $\mathbf{K}$-subspaces of V are the $\mathcal{U}_i := \pi^{-1}(\mathcal{F}_i(b) \times \mathcal{F}_i(-b))$, for $0 \leq i \leq n$, where

$$\pi : \mathcal{F}_N(b \oplus -b) \longrightarrow \left(\bigcup_{i=0}^{n} \mathcal{F}_i(b)\right) \times \left(\bigcup_{i=0}^{n} \mathcal{F}_i(-b)\right)$$

is the map defined by (1.1). The following generalization of Theorem 1.3 is an immediate consequence of Theorem 1.1, Corollary 3.10, and Proposition 3.13.(1).

Theorem 4.1. *Let Γ be a torsion-free word hyperbolic group and $\rho_L, \rho_R : \Gamma \to G = \mathrm{Aut}_{\mathbf{K}}(b)$ two representations. Suppose that ρ_L is $P_1(b)$-Anosov and uniformly $\omega_{\alpha_1(b)}$-dominates ρ_R (Definition 3.7). Then Γ acts properly discontinuously, via $\rho_L \oplus \rho_R$, on $(G \times G)/\mathrm{Diag}(G)$.*

Let $\xi_L : \partial_\infty\Gamma \to \mathcal{F}_1(b)$ be the boundary map of ρ_L. For any $0 \leq i \leq n$, let $\mathcal{K}^i_{\xi_L}$ be the subset of $\mathcal{F}_i(b)$ consisting of subspaces W containing $\xi_L(\eta)$ for some $\eta \in \partial_\infty\Gamma$, and let $\mathcal{U}_i^{\xi_L}$ be the complement in $\mathcal{U}_i$ of $\pi^{-1}(\mathcal{K}^i_{\xi_L} \times \mathcal{F}_i(-b))$. Then Γ acts properly discontinuously and cocompactly, via $\rho_L \oplus \rho_R$, on the open subset

$$\Omega := \bigcup_{i=0}^{n} \mathcal{U}_i^{\xi_L}$$

of $\mathcal{F}_N(b \oplus -b)$, and the quotient orbifold $(\rho_L \oplus \rho_R)(\Gamma)\backslash\Omega$ is a compactification of

$$(\rho_L \oplus \rho_R)(\Gamma)\backslash(G \times G)/\mathrm{Diag}(G).$$

If Γ is torsion-free, then this compactification is a smooth manifold.

Recall from Lemma 3.9 that the condition that one of the representations ρ_L or ρ_R be $P_1(b)$-Anosov and uniformly $\omega_{\alpha_1(b)}$-dominate the other is equivalent to the condition that

$$\rho := \rho_L \oplus \rho_R : \Gamma \longrightarrow G \times G = \mathrm{Aut}_{\mathbf{K}}(b) \times \mathrm{Aut}_{\mathbf{K}}(-b) \longhookrightarrow \mathrm{Aut}_{\mathbf{K}}(b \oplus -b)$$

be $P_1(b \oplus -b)$-Anosov [GGKW16, Th. 7.3].

Proof of Theorem 4.1. By Corollary 3.10, the boundary map $\xi : \partial_\infty\Gamma \to \mathcal{F}_1(b \oplus -b)$ of $\rho = \rho_L \oplus \rho_R$ is the composition of ξ_L with the natural embedding $\mathcal{F}_1(b) \hookrightarrow \mathcal{F}_1(b \oplus -b)$. By Proposition 3.13.(1), the group Γ acts properly discontinuously and cocompactly, via ρ, on the open set Ω. Note that Ω contains $\mathcal{U}_0$, hence the action of Γ on $\mathcal{U}_0$ via ρ is properly discontinuous. By Theorem 1.1, the set $\mathcal{U}_0$ is an open and dense $(G \times G)$-orbit in $\mathcal{F}_N(b \oplus -b)$, isomorphic to $(G \times G)/\mathrm{Diag}(G)$. Therefore, Γ acts properly discontinuously via ρ on $(G \times G)/\mathrm{Diag}(G)$ and $\rho(\Gamma)\backslash\mathcal{U}_0 \simeq \rho(\Gamma)\backslash(G \times G)/\mathrm{Diag}(G)$ is open and dense in the compact orbifold $\rho(\Gamma)\backslash\Omega$. This orbifold is a manifold if Γ is torsion-free. ☐

Remark 4.2. In the case that $\mathrm{Aut}_{\mathbf{K}}(b)$ has real rank 1, *all* properly discontinuous actions via a quasi-isometric embedding $\rho_L \oplus \rho_R : \Gamma \to \mathrm{Aut}_{\mathbf{K}}(b) \times \mathrm{Aut}_{\mathbf{K}}(b)$ fall into the setting of Theorem 4.1, by [GGKW16, Th. 7.3]. For $\mathrm{Aut}_{\mathbf{K}}(b) = \mathrm{O}(2,1)$ we obtain compactifications of *anti-de Sitter* 3-manifolds, and for $\mathrm{Aut}_{\mathbf{K}}(b) = \mathrm{O}(3,1)$ compactifications of *holomorphic Riemannian* complex 3-manifolds of constant nonzero curvature. We refer to [Gol85, Ghy95, Kob98, Sal00, Kas, GK16, GKW15, DT15, Tho16, DGK16] for examples of such pairs (ρ_L, ρ_R).

Remark 4.3. Suppose $G = \mathrm{Aut}_{\mathbf{K}}(b) = \mathrm{Sp}(2, \mathbf{C}) \simeq \mathrm{SL}_2(\mathbf{C})$. For $\rho_R : \Gamma \to G$ constant, the compactification of Theorem 4.1 is naturally endowed with a holomorphic action of G; by [Gui07], all other holomorphic equivariant compactifications are bimeromorphically equivalent to this one. For $\rho_R : \Gamma \to G \simeq \mathrm{SL}_2(\mathbf{C})$ not necessarily constant but close enough to the constant representation, a compactification similar to Theorem 4.1 has recently been worked out by Mayra Méndez in her ongoing PhD thesis, building on [Gui07].

Corollary 4.4 ([GGKW16, Th. 7.3, (1)⇒(6)])**.** *Let Γ be a word hyperbolic group, G an arbitrary real reductive Lie group, and $\rho_L, \rho_R : \Gamma \to G$ two representations. Let $\alpha \in \Delta$ be a simple restricted root of G. If ρ_L is $P_{\{\alpha\}}$-Anosov and uniformly ω_α-dominates ρ_R, then the action of Γ on $(G \times G)/\mathrm{Diag}(G)$ via $(\rho_L, \rho_R) : \Gamma \to G \times G$ is properly discontinuous.*

Recall that any P_θ-Anosov representation is $P_{\{\alpha\}}$-Anosov for all $\alpha \in \theta$ (see Section 3.3).

Proof. By Proposition 3.11, there exist a nondegenerate bilinear form b on a real vector space V and a linear representation $\tau : G \to \mathrm{Aut}_{\mathbf{R}}(b)$ such that $\tau \circ \rho_L : \Gamma \to \mathrm{Aut}_{\mathbf{R}}(b)$ is $P_1(b)$-Anosov and uniformly $\omega_{\alpha_1(b)}$-dominates $\tau \circ \rho_R$. By Theorem 4.1, the action of Γ on

$$(\mathrm{Aut}_{\mathbf{R}}(b) \times \mathrm{Aut}_{\mathbf{R}}(b))/\mathrm{Diag}(\mathrm{Aut}_{\mathbf{R}}(b))$$

via $\tau \circ \rho_L \oplus \tau \circ \rho_R$ is properly discontinuous. Since $(\tau(G) \times \tau(G))/\mathrm{Diag}(\tau(G))$ embeds into $(\mathrm{Aut}_{\mathbf{R}}(b) \times \mathrm{Aut}_{\mathbf{R}}(b))/\mathrm{Diag}(\mathrm{Aut}_{\mathbf{R}}(b))$ as the $(\tau(G) \times \tau(G))$-orbit of (e, e), the action of Γ on $(\tau(G) \times \tau(G))/\mathrm{Diag}(\tau(G))$ via $\tau \circ \rho_L \oplus \tau \circ \rho_R$ is also properly discontinuous. Thus the action of Γ on $(G \times G)/\mathrm{Diag}(G)$ via (ρ_L, ρ_R) is properly discontinuous. $\square$

As above, the condition that one of the representations $\tau \circ \rho_L$ or $\tau \circ \rho_R$ be $P_1(b)$-Anosov and uniformly $\omega_{\alpha_1(b)}$-dominate the other is equivalent to the condition that

$$\tau \circ \rho_L \oplus \tau \circ \rho_R : \Gamma \longrightarrow \mathrm{Aut}_{\mathbf{K}}(b) \times \mathrm{Aut}_{\mathbf{K}}(-b) \hookrightarrow \mathrm{Aut}_{\mathbf{K}}(b \oplus -b)$$

be $P_1(b \oplus -b)$-Anosov.

Corollary 4.5. *Let Γ be a word hyperbolic group, G an arbitrary real reductive Lie group, and $\rho_L, \rho_R : \Gamma \to G$ two representations of Γ. Let b be a nondegenerate $\mathbf{R}$-bilinear form on a $\mathbf{K}$-vector space V as above, for $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$, and let $\tau : G \to \mathrm{Aut}_{\mathbf{K}}(b)$ be a linear representation of G such that $\tau \circ \rho_L : \Gamma \to \mathrm{Aut}_{\mathbf{K}}(b)$ is $P_1(b)$-Anosov and uniformly $\omega_{\alpha_1(b)}$-dominates $\tau \circ \rho_R$ (see Proposition 3.11). Let Ω be the cocompact domain of discontinuity of $(\tau \circ \rho_L \oplus \tau \circ \rho_R)(\Gamma)$ in $\mathcal{F}_N(b \oplus -b)$ provided by Proposition 3.13.(1). A compactification of*

$$(\tau \circ \rho_L \oplus \tau \circ \rho_R)(\Gamma)\backslash(\tau(G) \times \tau(G))/\mathrm{Diag}(\tau(G))$$

is given by its closure in $(\tau \circ \rho_L \oplus \tau \circ \rho_R)(\Gamma)\backslash\Omega$. If $\tau : G \to \mathrm{Aut}_{\mathbf{K}}(b)$ has compact kernel, this provides a compactification of $(\rho_L, \rho_R)(\Gamma)\backslash(G \times G)/\mathrm{Diag}(G)$.

In the special case where $\rho_R : \Gamma \to \{e\} \subset G$ is the trivial representation, the action of Γ on $(G \times G)/\mathrm{Diag}(G)$ via $\rho_L \oplus \rho_R$ is the action of Γ on G via left multiplication by ρ_L and Corollary 4.5 yields, when τ has compact kernel, a compactification of $\rho_L(\Gamma)\backslash G \simeq (\rho_L(\Gamma) \times \{e\})\backslash(G \times G)/\mathrm{Diag}(G)$.

We refer to Theorem 6.5 for the tameness of $(\rho_L, \rho_R)(\Gamma)\backslash(G \times G)/\mathrm{Diag}(G)$ for general ρ_L, ρ_R.

5. Properly discontinuous actions on other homogeneous spaces

This section is devoted to the proof of Proposition 1.5 and Theorem 1.7.

5.1. **Notation.** For $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ and $p, q \in \mathbf{N}$, we denote by $\mathbf{K}^{p,q}$ the vector space $\mathbf{K}^{p+q}$ endowed with the $\mathbf{R}$-bilinear form $b_{\mathbf{K}}^{p,q}$ of (1.2), so that $\mathrm{Aut}_{\mathbf{K}}(b_{\mathbf{K}}^{p,q}) = \mathrm{O}(p, q)$, $\mathrm{U}(p, q)$, or $\mathrm{Sp}(p, q)$. We use the notation $P_i(b_{\mathbf{K}}^{p,q})$ of Section 3.5 for parabolic subgroups. For $m \in \mathbf{N}$ and $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, we denote by

$$\omega_{\mathbf{K}}^{2m} : (x, y) \longmapsto x_1 y_{m+1} - x_{m+1} y_1 + \cdots + x_m y_{2m} - x_{2m} y_m$$

the standard symplectic form on $\mathbf{K}^{2m}$, so that $\mathrm{Aut}_{\mathbf{K}}(\omega_{\mathbf{K}}^{2m}) = \mathrm{Sp}(2m, \mathbf{K})$ for $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ and $\mathrm{Aut}_{\mathbf{R}}(\omega_{\mathbf{R}}^{2m}) \subset \mathrm{Aut}_{\mathbf{C}}(\omega_{\mathbf{C}}^{2m})$.

Recall that a Hermitian form h on a $\mathbf{C}$-vector space V is completely determined by its real part b: for any $v, v' \in V$,

$$h(v, v') = b(v, v') - \sqrt{-1}\, b(v, \sqrt{-1} v').$$

If the signature of h is (p,q), then the signature of b is $(2p,2q)$. Similarly, an $\mathbf{H}$-Hermitian form $h_{\mathbf{H}}$ on a right $\mathbf{H}$-vector space V is completely determined by its complex part h: for any $v,v'\in V$,

$$h_{\mathbf{H}}(v,v') = h(v,v') - h\big(v,v'j\big)\,j.$$

Thus an $\mathbf{H}$-Hermitian form is completely determined by its real part. If the signature of $h_{\mathbf{H}}$ is (p,q), then the signature of h is $(2p,2q)$, and the signature of the real part b of h and $h_{\mathbf{H}}$ is $(4p,4q)$.

5.2. **Compactifying pseudo-Riemannian analogues of (locally) hyperbolic spaces.** We first prove Proposition 1.5 in cases (i), (ii), and (iii) of Table 2. Let $\mathbf{K}=\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ and $p,q\in\mathbf{N}$. As in Sections 1.3 and 1.4, the quadric

$$\hat{\mathbb{H}}^{p,q}_{\mathbf{K}} = \{x\in\mathbf{K}^{p,q+1} \mid b^{p,q+1}_{\mathbf{K}}(x,x)=-1\}$$

identifies with the homogeneous space G/H where $G=\mathrm{Aut}_{\mathbf{K}}(b^{p,q+1}_{\mathbf{K}})$ and $H=\mathrm{Aut}_{\mathbf{K}}(b^{p,q}_{\mathbf{K}})$, the embedding $H\hookrightarrow G$ being given by the splitting $\mathbf{K}^{p,q+1}=\mathbf{K}^{p,q}\oplus\mathbf{K}^{0,1}$. Let Z be the center of G, i.e. the set of multiples of the identity $\lambda\,\mathrm{id}$ for $\lambda\in\mathbf{K}$ satisfying $\bar\lambda\lambda=1$, so that $\mathrm{Aut}_{\mathbf{K}}(b^{p,q}_{\mathbf{K}})\times\mathrm{Aut}_{\mathbf{K}}(b^{0,1}_{\mathbf{K}})=H\times Z$. The quadric $\hat{\mathbb{H}}^{p,q}_{\mathbf{K}}$ fibers, with compact fiber, over the affine symmetric space $\mathbb{H}^{p,q}_{\mathbf{K}}=G/(H\times Z)$, which can be realized as

$$\mathbb{H}^{p,q}_{\mathbf{K}} = \mathbb{P}(\{x\in\mathbf{K}^{p,q+1} \mid b^{p,q+1}_{\mathbf{K}}(x,x)<0\}) \subset \mathbb{P}(\mathbf{K}^{p,q+1}).$$

The splitting $\mathbf{K}^{p+1,q+1}=\mathbf{K}^{1,0}\oplus\mathbf{K}^{p,q+1}$ induces an embedding $\iota : G\hookrightarrow G'=\mathrm{Aut}_{\mathbf{K}}(b^{p+1,q+1}_{\mathbf{K}})$ and a projection $\mathrm{pr}:\mathbf{K}^{p+1,q+1}\to\mathbf{K}^{p,q+1}$. Proposition 1.5.(1) in cases (i), (ii), and (iii) of Table 2 is contained in the following elementary remarks.

Lemma 5.1. *For $\mathbf{K}=\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ and $p>q\geq 0$, let $G=\mathrm{Aut}_{\mathbf{K}}(b^{p,q+1}_{\mathbf{K}})$ and $H=\mathrm{Aut}_{\mathbf{K}}(b^{p,q}_{\mathbf{K}})$.*

1. *The space $\mathcal{F}_1(b^{p+1,q+1}_{\mathbf{K}})$ is a smooth compactification of $\hat{\mathbb{H}}^{p,q}_{\mathbf{K}}=G/H$. It is the union of two G-orbits: an open one $\mathcal{U}$ isomorphic to $\hat{\mathbb{H}}^{p,q}_{\mathbf{K}}$ and a closed one, namely $\mathcal{F}_1(b^{p,q+1}_{\mathbf{K}})$.*
2. *The space $\overline{\mathbb{H}^{p,q}_{\mathbf{K}}} := \mathbb{P}_{\mathbf{K}}(\{x\in\mathbf{K}^{p+q+1} \mid b^{p,q+1}_{\mathbf{K}}(x,x)\leq 0\})$ is a compactification of $\mathbb{H}^{p,q}_{\mathbf{K}}=G/(H\times Z)$ as a manifold with boundary. It is the union of two G-orbits: an open one, namely $\mathbb{H}^{p,q}_{\mathbf{K}}$, and a closed one, namely $\mathcal{F}_1(b^{p,q+1}_{\mathbf{K}})$.*
3. *The map*

$$\begin{array}{ccc} \mathcal{F}_1\big(b^{p+1,q+1}_{\mathbf{K}}\big) & \longrightarrow & \mathbb{P}_{\mathbf{K}}(\mathbf{K}^{p+q+1}) \\ \ell & \longmapsto & \mathrm{pr}(\ell) \end{array}$$

is well defined, proper, and G-equivariant. Its image is $\overline{\mathbb{H}^{p,q}_{\mathbf{K}}}$, and its fibers are exactly the Z-orbits in $\mathcal{F}_1(b^{p+1,q+1}_{\mathbf{K}})$. In restriction to $\hat{\mathbb{H}}^{p,q}_{\mathbf{K}}$ it is the natural projection $\hat{\mathbb{H}}^{p,q}_{\mathbf{K}}\to\mathbb{H}^{p,q}_{\mathbf{K}}$, and in restriction to $\mathcal{F}_1(b^{p,q+1}_{\mathbf{K}})$ it is the identity.

Proof. The group $G=\mathrm{Aut}_{\mathbf{K}}(b^{p,q+1}_{\mathbf{K}})$ acts transitively on the closed submanifold $\mathcal{F}_1(b^{p,q+1}_{\mathbf{K}})$ of the smooth compact manifold $\mathcal{F}_1(b^{p+1,q+1}_{\mathbf{K}})$, which has positive codimension. The complement $\mathcal{U}=\mathcal{F}_1(b^{p+1,q+1}_{\mathbf{K}})\smallsetminus\mathcal{F}_1(b^{p,q+1}_{\mathbf{K}})$ is open and dense in $\mathcal{F}_1(b^{p+1,q+1}_{\mathbf{K}})$, and identifies with $\hat{\mathbb{H}}^{p,q}_{\mathbf{K}}$ since G acts transitively on $\mathcal{U}$ and the stabilizer in G of $[1:0:\ldots:0:1]\in\mathcal{U}\subset\mathbb{P}(\mathbf{K}^{p+1,q+1})$ is $H=\mathrm{Aut}_{\mathbf{K}}(b^{p,q}_{\mathbf{K}})$. Thus $\mathcal{F}_1(b^{p+1,q+1}_{\mathbf{K}})$ is a smooth compactification of $\hat{\mathbb{H}}^{p,q}_{\mathbf{K}}$. This proves (1). Point (2) easily follows from the definition. In (3), the map is well defined since the restriction of the form $b^{p+1,q+1}_{\mathbf{K}}$ to the kernel of pr is positive definite. The other claims in (3) are checked by a direct calculation. □

Proposition 1.5.(2) in cases (i), (ii), and (iii) of Table 2 is contained in the following result, which will be proved in Section 5.3.

Theorem 5.2. *For* $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$, *or* $\mathbf{H}$ *and* $p > q \geq 0$, *let* $G = \mathrm{Aut}_{\mathbf{K}}(b^{p,q+1}_{\mathbf{K}})$ *and* $G' = \mathrm{Aut}_{\mathbf{K}}(b^{p+1,q+1}_{\mathbf{K}})$. *Let* Γ *be a word hyperbolic group and* $\rho : \Gamma \to G$ *a* $P_{q+1}(b^{p,q+1}_{\mathbf{K}})$*-Anosov representation with boundary map* $\xi : \partial_\infty\Gamma \to \mathcal{F}_{q+1}(b^{p,q+1}_{\mathbf{K}})$. *Let*

$$\mathcal{K}_\xi := \bigcup_{\eta\in\partial_\infty\Gamma} \left\{\ell \in \mathcal{F}_1\big(b^{p,q+1}_{\mathbf{K}}\big) \mid \ell \subset \xi(\eta)\right\}$$

and let $i_1 : \mathcal{F}_1(b^{p,q+1}_{\mathbf{K}}) \hookrightarrow \mathcal{F}_1(b^{p+1,q+1}_{\mathbf{K}})$ *and* $i_{q+1} : \mathcal{F}_{q+1}(b^{p,q+1}_{\mathbf{K}}) \hookrightarrow \mathcal{F}_{q+1}(b^{p+1,q+1}_{\mathbf{K}})$ *be the natural inclusions.*

(1) *The composition* $\iota \circ \rho : \Gamma \to G'$ *is* $P_{q+1}(b^{p+1,q+1}_{\mathbf{K}})$*-Anosov with boundary map* $\xi' = i_{q+1} \circ \xi$, *except possibly if* $\mathbf{K} = \mathbf{R}$ *and* $p = q + 1$.

(2) *If* $\mathbf{K} = \mathbf{R}$ *and* $p = q + 1$, *then the composition* $\iota \circ \rho$ *is* $P_{q+1}(b^{p+1,q+1}_{\mathbf{K}})$*-Anosov if and only if the action of* Γ *via* ρ *on* $\mathbb{H}^{p,q}_{\mathbf{K}} = \mathbb{H}^{p,p-1}_{\mathbf{R}}$ *is properly discontinuous; in this case the boundary map of* $\iota \circ \rho$ *is* $\xi' = i_{q+1} \circ \xi$.

(3) *Assume that* $\iota \circ \rho : \Gamma \to G'$ *is* $P_{q+1}(b^{p+1,q+1}_{\mathbf{K}})$*-Anosov. Then the cocompact domain of discontinuity* Ω *of Proposition 3.13.(2) for* $\iota \circ \rho$ *is* $\mathcal{F}_1(b^{p+1,q+1}_{\mathbf{K}}) \smallsetminus i_1(\mathcal{K}_\xi)$, *which contains the dense* G*-orbit* $\mathcal{U}$ *of* $\mathcal{F}_1(b^{p+1,q+1}_{\mathbf{K}})$ *isomorphic to* $\hat{\mathbb{H}}^{p,q}_{\mathbf{K}}$. *In particular, the action of* Γ *on* $\hat{\mathbb{H}}^{p,q}_{\mathbf{K}}$ *via* $\iota \circ \rho$ *is properly discontinuous and, if* Γ *is torsion-free, then* $\rho(\Gamma)\backslash\Omega$ *is a smooth compactification of* $\rho(\Gamma)\backslash\hat{\mathbb{H}}^{p,q}_{\mathbf{K}}$. *Let* $C_\xi = \mathcal{F}_1(b^{p,q+1}_{\mathbf{K}}) \smallsetminus \mathcal{K}_\xi$. *Then the action of* Γ *via* ρ *on* $\mathbb{H}^{p,q}_{\mathbf{K}} \cup C_\xi \subset \overline{\mathbb{H}^{p,q}_{\mathbf{K}}}$ *is properly discontinuous and cocompact. The action of* Γ *via* ρ *on* $\mathbb{H}^{p,q}_{\mathbf{K}}$ *is in fact sharp (Definition 3.3).*

Suppose $\mathbf{K} = \mathbf{R}$ and $q = 0$. Then Lemma 5.1.(1) describes the usual compactification of the disjoint union of two copies of the real hyperbolic space $\mathbb{H}^p_{\mathbf{R}}$, obtained by embedding them as two open hemispheres into the visual boundary $\partial\mathbb{H}^{p+1}_{\mathbf{R}} = \mathcal{F}_1(b^{p+1,1}_{\mathbf{R}}) \simeq \mathbb{S}^p_{\mathbf{R}}$ of $\mathbb{H}^{p+1}_{\mathbf{R}}$. A representation $\rho : \Gamma \to \mathrm{O}(p, 1)$ is $P_1(b^{p,1}_{\mathbf{R}})$-Anosov if and only if it is convex cocompact, in which case Theorem 5.2.(3) states that Γ acts properly discontinuously, via ρ, on the complement in $\partial\mathbb{H}^{p+1}_{\mathbf{R}}$ of the limit set $\mathcal{K}_\xi$ of ρ in $\partial\mathbb{H}^p_{\mathbf{R}}$. When $\rho(\Gamma) \subset \mathrm{SO}(p, 1)$, Theorem 5.2.(3) describes the compactification of two copies of the convex cocompact hyperbolic manifold $\rho(\Gamma)\backslash\mathbb{H}^p_{\mathbf{R}}$ obtained by gluing them along their common boundary.

For $\mathbf{K} = \mathbf{R}$ and $q = 1$ (Lorentzian case), Theorem 5.2.(1) describes the usual compactification of the double cover of the anti-de Sitter space AdS^{p+1}, obtained by embedding it into the Einstein universe Ein^{p+1}.

In general, the compactification $\mathcal{F}_1(b^{p+1,q+1}_{\mathbf{K}})$ of $\hat{\mathbb{H}}^{p,q}_{\mathbf{K}}$ of Lemma 5.1.(1) is homeomorphic to

$$(\mathbb{S}^p_{\mathbf{K}} \times \mathbb{S}^q_{\mathbf{K}})/\{z \in \mathbf{K} \mid \overline{z}z = 1\}.$$

Remark 5.3. Identifying $\mathbf{R}^{2n+2}$ with $\mathbf{C}^{n+1}$ gives a $\mathrm{U}(n, 1)$-equivariant identification of $\hat{\mathbb{H}}^{2n,2}_{\mathbf{R}}$ with $\hat{\mathbb{H}}^{n,1}_{\mathbf{C}}$. Examples of $P_2(b^{2n,2}_{\mathbf{R}})$-Anosov representations $\rho : \Gamma \to \mathrm{O}(2n, 2)$ include the composition of any convex cocompact representation $\rho_1 : \Gamma \to \mathrm{U}(n, 1)$ with the natural inclusion of $\mathrm{U}(n, 1)$ into $\mathrm{O}(2n, 2)$; the manifold $\rho(\Gamma)\backslash\hat{\mathbb{H}}^{2n,2}_{\mathbf{R}}$ then identifies with $\rho_1(\Gamma)\backslash\hat{\mathbb{H}}^{n,1}_{\mathbf{C}}$, and the compactifications of these two manifolds given by

Theorem 5.2.(3) coincide. The same holds if we replace

$$\begin{array}{rl} & (\hat{\mathbb{H}}^{2n,2}_{\mathbf{R}}, \hat{\mathbb{H}}^{n,1}_{\mathbf{C}}, \mathrm{O}(2n,2), \mathrm{U}(n,1), P_2(b^{2n,2}_{\mathbf{R}})) \\ \text{with} & (\hat{\mathbb{H}}^{4n,4}_{\mathbf{R}}, \hat{\mathbb{H}}^{n,1}_{\mathbf{H}}, \mathrm{O}(4n,4), \mathrm{Sp}(n,1), P_4(b^{4n,4}_{\mathbf{R}})) \\ \text{or with} & (\hat{\mathbb{H}}^{2n,2}_{\mathbf{C}}, \hat{\mathbb{H}}^{n,1}_{\mathbf{H}}, \mathrm{U}(2n,2), \mathrm{Sp}(n,1), P_2(b^{2n,2}_{\mathbf{C}})). \end{array}$$

The following examples show that for $\mathbf{K} = \mathbf{R}$ and $p = q + 1$, the fact that $\rho : \Gamma \to G = \mathrm{Aut}_{\mathbf{K}}(b^{p,q+1}_{\mathbf{K}})$ is $P_{q+1}(b^{p,q+1}_{\mathbf{K}})$-Anosov does not imply that the action of Γ on $\hat{\mathbb{H}}^{p,q}_{\mathbf{K}}$ via ρ is properly discontinuous.

Example 5.4. Let $\mathbf{K} = \mathbf{R}$ and $p = q + 1 = 2$. Then the identity component G_0 of $G = \mathrm{O}(2,2)$ identifies with $\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R})$ and $\hat{\mathbb{H}}^{2,1}_{\mathbf{R}}$ is a covering of order two of $(\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R}))/\mathrm{Diag}(\mathrm{PSL}_2(\mathbf{R}))$. A representation $\rho : \Gamma \to G_0$ is $P_2(b^{2,2}_{\mathbf{R}})$-Anosov if and only if the projection of ρ to the first (or second, depending on the numbering of the simple roots) $\mathrm{PSL}_2(\mathbf{R})$ factor is convex cocompact. However, the action of Γ via ρ is properly discontinuous on $\hat{\mathbb{H}}^{2,1}_{\mathbf{R}}$ if and only if the projection of ρ to one $\mathrm{PSL}_2(\mathbf{R})$ factor is convex cocompact *and* uniformly dominates the other, by [GGKW16, Th. 7.3] (see Remark 4.2).

Example 5.5. Let $\mathbf{K} = \mathbf{R}$ and $p = q + 1 \geq 2$. Any Hitchin representation $\rho : \Gamma \to \mathrm{O}(p,p)$ of a closed surface group Γ or any Schottky representation $\rho : \Gamma \to \mathrm{O}(p,p)$ representation of a nonabelian free group is $P_p(b^{p,p}_{\mathbf{R}})$-Anosov. However, for odd p the action of such groups Γ on $\hat{\mathbb{H}}^{p,p-1}_{\mathbf{R}}$ via ρ is never properly discontinuous [Kas08].

5.3. **Proof of Theorem 5.2.** We first prove (1). Consider a Cartan subspace $\mathfrak{a}'$ for $G' = \mathrm{Aut}_{\mathbf{K}}(b^{p+1,q+1}_{\mathbf{K}})$ that contains a Cartan subspace $\mathfrak{a}$ for $G = \mathrm{Aut}_{\mathbf{K}}(b^{p,q+1}_{\mathbf{K}})$. If $\mathbf{K} = \mathbf{R}$ and $p > q + 1$, then G and G' both have restricted root systems of type B_{q+1}, hence the restriction of $\alpha_{q+1}(b^{p+1,q+1}_{\mathbf{K}})$ to $\mathfrak{a}$ is $\alpha_{q+1}(b^{p,q+1}_{\mathbf{K}})$. (Recall that $\alpha_{q+1}(b)$ is the simple restricted root such that $P_{\{\alpha_{q+1}(b)\}}$ is the stabilizer of a $(q+1)$-dimensional isotropic space, see Section 3.9.) If $\mathbf{K} = \mathbf{C}$ or $\mathbf{H}$ and if $p > q+1$, then G and G' both have restricted root systems of type $(BC)_{q+1}$, hence the restriction of $\alpha_{q+1}(b^{p+1,q+1}_{\mathbf{K}})$ to $\mathfrak{a}$ is $\alpha_{q+1}(b^{p,q+1}_{\mathbf{K}})$. If $\mathbf{K} = \mathbf{C}$ or $\mathbf{H}$ and if $p = q + 1$, then G has a restricted root system of type C_{q+1} and G' of type $(BC)_{q+1}$, hence the restriction of $\alpha_{q+1}(b^{p+1,q+1}_{\mathbf{K}})$ to $\mathfrak{a}$ is $\frac{1}{2}\alpha_{q+1}(b^{p,q+1}_{\mathbf{K}})$. In all three cases, it follows from Definition 3.4 that if $\rho : \Gamma \to G = \mathrm{Aut}_{\mathbf{K}}(b^{p,q+1}_{\mathbf{K}})$ is a $P_{q+1}(b^{p,q+1}_{\mathbf{K}})$-Anosov representation with boundary map $\xi : \partial_\infty\Gamma \to \mathcal{F}_{q+1}(b^{p,q+1}_{\mathbf{K}})$, then the composed representation $\iota \circ \rho$ is $P_{q+1}(b^{p+1,q+1}_{\mathbf{K}})$-Anosov with boundary map $\xi' = i_{q+1} \circ \xi$. This proves (1).

We now assume $\mathbf{K} = \mathbf{R}$ and $p = q + 1$, and prove (2). The group G has a restricted root system of type D_p and G' of type B_p, hence the restriction of $\alpha_p(b^{p+1,p}_{\mathbf{R}})$ to $\mathfrak{a}$ is $\frac{1}{2}(\alpha_p(b^{p,p}_{\mathbf{R}}) - \alpha_{p-1}(b^{p,p}_{\mathbf{R}}))$. The boundary map $\xi : \partial_\infty\Gamma \to \mathcal{F}_p(b^{p,p}_{\mathbf{R}})$ of the $P_p(b^{p,p}_{\mathbf{R}})$-Anosov representation $\rho : \Gamma \to \mathrm{Aut}_{\mathbf{R}}(b^{p+1,p}_{\mathbf{R}})$ induces, by composition with $i : \mathcal{F}_p(b^{p,p}_{\mathbf{R}}) \hookrightarrow \mathcal{F}_p(b^{p+1,p}_{\mathbf{R}})$, a continuous, $(\iota \circ \rho)$-equivariant, transverse boundary map $\xi' : \partial_\infty\Gamma \to \mathcal{F}_p(b^{p+1,p}_{\mathbf{R}})$. Note that $\mathbb{H}^{p,p-1}_{\mathbf{R}} = G/(H \times Z)$ where $H \times Z = \mathrm{Aut}_{\mathbf{R}}(b^{p,p-1}_{\mathbf{R}}) \times \mathrm{Aut}_{\mathbf{R}}(b^{0,1}_{\mathbf{R}})$ satisfies

$$\mu(H \times Z) = \overline{\mathfrak{a}}^+ \cap \mathrm{Ker}(\alpha_p(b^{p+1,p}_{\mathbf{R}})).$$

If $\iota \circ \rho : \Gamma \to G'$ is $P_p(b^{p+1,p}_{\mathbf{R}})$-Anosov, then the action of Γ on $\mathbb{H}^{p,p-1}_{\mathbf{R}}$ is sharp by Lemma 3.6; in particular, it is properly discontinuous. Conversely, suppose that the action of Γ on $\mathbb{H}^{p,p-1}_{\mathbf{R}}$ is properly discontinuous. The properness criterion of Benoist and Kobayashi (Fact 3.2) implies

$$\left|\langle \alpha_p(b_{\mathbf{R}}^{p+1,p}), \mu(\rho(\gamma))\rangle\right| \underset{\gamma\to\infty}{\longrightarrow} +\infty.$$

Using Lemma 5.6 below, we deduce that for any $\gamma \in \Gamma$ of infinite order,

$$\langle \alpha_p(b_{\mathbf{R}}^{p+1,p}), \lambda(\iota\circ\rho(\gamma))\rangle > 0,$$

and so $\iota\circ\rho(\gamma)$ has a unique attracting fixed point in $\mathcal{F}_p(b_{\mathbf{R}}^{p+1,p})$, see [GGKW16, Prop. 3.3.(c)]. Since $\rho(\gamma) \in \mathrm{Aut}_{\mathbf{R}}(b_{\mathbf{R}}^{p,p})$, this attracting fixed point lies in $\mathcal{F}_p(b_{\mathbf{R}}^{p,p})$ and is thus the image by ξ of the attracting fixed point of γ in $\partial_\infty\Gamma$. We conclude that ξ' is dynamics-preserving. Therefore, the composed representation $\iota\circ\rho$ is $P_p(b_{\mathbf{R}}^{p+1,p})$-Anosov with boundary map $\xi' = i_{q+1}\circ\xi$. This concludes the proof of (2).

We finally prove (3). Suppose that $\iota\circ\rho : \Gamma \to G'$ is $P_{q+1}(b_{\mathbf{K}}^{p+1,q+1})$-Anosov. By (1) and (2), the boundary map $\xi' : \partial_\infty\Gamma \to \mathcal{F}_p(b_{\mathbf{R}}^{p+1,q+1})$ of $\iota\circ\rho$ is the composition of the boundary map $\xi : \partial_\infty\Gamma \to \mathcal{F}_{q+1}(b_{\mathbf{R}}^{p,q+1})$ of ρ with the natural inclusion $i_{q+1} : \mathcal{F}_{q+1}(b_{\mathbf{K}}^{p,q+1}) \hookrightarrow \mathcal{F}_{q+1}(b_{\mathbf{K}}^{p+1,q+1})$. By Proposition 3.13.(2), the group Γ acts properly discontinuously and cocompactly, via $\iota\circ\rho$, on $\Omega = \mathcal{F}_1(b_{\mathbf{K}}^{p+1,q+1}) \smallsetminus \mathcal{K}_{\xi'}$, where

$$\mathcal{K}_{\xi'} = \bigcup_{\eta\in\partial_\infty\Gamma} \left\{\ell \in \mathcal{F}_1\big(b_{\mathbf{K}}^{p+1,q+1}\big) \mid \ell \subset i_{q+1}(\xi(\eta))\right\} = i_1(\mathcal{K}_\xi).$$

This set Ω contains the dense G-orbit $\mathcal{U}$ of $\mathcal{F}_1(b_{\mathbf{K}}^{p+1,q+1})$ isomorphic to $\hat{\mathbb{H}}_{\mathbf{K}}^{p,q}$ described in Lemma 5.1.(1). Since the surjective map $\mathcal{F}_1(b_{\mathbf{K}}^{p+1,q+1}) \to \overline{\mathbb{H}_{\mathbf{K}}^{p,q}}$ of Lemma 5.1.(3) is proper and G-equivariant, the group Γ acts properly discontinuously and cocompactly, via ρ, on the image of Ω in $\overline{\mathbb{H}_{\mathbf{K}}^{p,q}}$, which is $\mathbb{H}_{\mathbf{K}}^{p,q} \cup \mathcal{C}_\xi$. Recall that $\mathbb{H}_{\mathbf{K}}^{p,q} = G/(H\times Z)$. We have $\mu(H\times Z) \subset \mathrm{Ker}(\alpha_{q+1}(b_{\mathbf{K}}^{p,q+1}))$, and so Lemma 3.6 shows that the properly discontinuous action of Γ on $\mathbb{H}_{\mathbf{K}}^{p,q}$ is in fact sharp. This completes the proof of Theorem 5.2.

Lemma 5.6. *Let $g \in \mathrm{Aut}_{\mathbf{R}}(b_{\mathbf{R}}^{p,p})$ satisfy*

$$\langle \alpha_{p-1}(b_{\mathbf{R}}^{p,p}), \lambda(g)\rangle = \langle \alpha_p(b_{\mathbf{R}}^{p,p}), \lambda(g)\rangle > 0. \tag{5.1}$$

Then the sequence $(\langle \alpha_p(b_{\mathbf{R}}^{p,p}) - \alpha_{p-1}(b_{\mathbf{R}}^{p,p}), \mu(g^n)\rangle)_{n\in\mathbf{N}}$ is bounded.

Proof. To make computations easier, we replace $b_{\mathbf{R}}^{p,p}$ with the equivalent symmetric bilinear form b given, for all $x, y \in \mathbf{R}^{2p}$, by

$$b(x,y) = \sum_{i=1}^{p} x_i\, y_{p+i} + x_{p+i}\, y_i.$$

With this bilinear form, the Lie algebra of $\mathrm{O}(p,p)$ is

$$\mathfrak{o}(p,p) = \left\{ \begin{pmatrix} B & C \\ D & -{}^tB \end{pmatrix} \;\middle|\; B, C, D \in M_p(\mathbf{R}),\ C + {}^tC = D + {}^tD = 0 \right\}.$$

A Cartan subspace of $\mathfrak{o}(p,p)$ is

$$\mathfrak{a} = \{\mathrm{diag}(\lambda_1,\dots,\lambda_p,-\lambda_1,\dots,-\lambda_p) \mid \lambda_1,\dots,\lambda_p \in \mathbf{R}\}.$$

The corresponding set of roots is $\Sigma = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le p\}$, where $\varepsilon_i \in \mathfrak{a}^*$ is given by $\varepsilon_i(\mathrm{diag}(\lambda_1,\dots,-\lambda_p)) = \lambda_i$. A system of simple roots is given by $\Delta = \{\alpha_1(b),\dots,\alpha_p(b)\}$, where $\alpha_i(b) = \varepsilon_i - \varepsilon_{i+1}$ for $1 \le i \le p-1$ and $\alpha_p(b) = \varepsilon_{p-1} + \varepsilon_p$. The corresponding set of positive roots is $\Sigma^+ = \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le p\}$. Using the notation of Section 3.1, we take $\mu : \mathrm{O}(p,p) \to \overline{\mathfrak{a}}^+$ to be the Cartan decomposition associated with the Cartan decomposition $\mathrm{O}(p,p) = K(\exp\overline{\mathfrak{a}}^+)K$ where $K = \mathrm{O}(p)\times\mathrm{O}(p)$.

Let $g = g_e g_u g_h$ be the Jordan decomposition of g. Using the Jacobson–Morozov theorem [Hel01, Th. 7.4] and (3.3), we may assume $g_e = 1$ and $g_h \in \overline{\mathfrak{a}}^+$ and $g_u \in \exp\mathfrak{u}_\Delta$. Assuming this, let us check that $\langle \alpha_p(b_{\mathbf{R}}^{p,p}) - \alpha_{p-1}(b_{\mathbf{R}}^{p,p}), \mu(g^n)\rangle = 0$ for all $n \in \mathbf{N}$.

Let $x := \log(g_h) = \lambda(g) \in \overline{\mathfrak{a}}^+$ and $y := \log(g_u) \in \mathfrak{u}_\Delta$. The assumption (5.1) on g implies $\langle \varepsilon_p, x\rangle = 0$ and $\langle \varepsilon_1, x\rangle \geq \cdots \geq \langle \varepsilon_{p-1}, x\rangle > 0$. In particular, $\langle \varepsilon_i + \varepsilon_j, x\rangle > 0$ and $\langle \varepsilon_i - \varepsilon_p, x\rangle > 0$ for all $1 \leq i < j \leq p$. Since g_h and g_u commute, so do x and y, hence

$$y \in \bigoplus_{1\leq i<j\leq p-1} \mathfrak{u}_{\varepsilon_i - \varepsilon_j}.$$

We deduce that $g = g_h g_u$ belongs to the connected subgroup of $\mathrm{O}(p,p)$ whose Lie algebra is

$$\left\{ \begin{pmatrix} B & 0 \\ 0 & -{}^tB \end{pmatrix} \;\middle|\; B \in M_{p-1}(\mathbf{R}) \subset M_p(\mathbf{R}) \right\},$$

where $M_{p-1}(\mathbf{R})$ is embedded in the upper left corner of $M_p(\mathbf{R})$. This subgroup is isomorphic to $\mathrm{GL}_{p-1}^+(\mathbf{R})$ and admits a Cartan decomposition

$$\mathrm{GL}_{p-1}^+(\mathbf{R}) = \big(K \cap \mathrm{GL}_{p-1}^+(\mathbf{R})\big)\big(\exp\overline{\mathfrak{a}}^+ \cap \mathrm{GL}_{p-1}^+(\mathbf{R})\big)\big(K \cap \mathrm{GL}_{p-1}(\mathbf{R})^+\big)$$

compatible with that of $\mathrm{O}(p,p)$, from which we see that $\langle \varepsilon_p, \mu(\mathrm{GL}_{p-1}^+(\mathbf{R}))\rangle = \{0\}$. In particular, $\langle \alpha_p(b_{\mathbf{R}}^{p,p}) - \alpha_{p-1}(b_{\mathbf{R}}^{p,p}), \mu(g^n)\rangle = \langle 2\varepsilon_p, \mu(g^n)\rangle = 0$ for all $n \in \mathbf{N}$. ☐

5.4. **Proof of Proposition 1.5.** Cases (i), (ii), and (iii) of Table 2 are covered by Lemma 5.1 and Theorem 5.2. We now treat the remaining cases. Let $(\mathbf{K}, \mathbf{L}, N, b_{\mathbf{K}})$ be:

- in case (iv), $\mathbf{K} = \mathbf{R}$, $\mathbf{L} = \mathbf{C}$, $N = 2p + 2q$, and $b_{\mathbf{K}} = b_{\mathbf{R}}^{2p,2q}$ on $\mathbf{K}^N$;
- in case (v), $\mathbf{K} = \mathbf{C}$, $\mathbf{L} = \mathbf{H}$, $N = 2p + 2q$, and $b_{\mathbf{K}} = b_{\mathbf{C}}^{2p,2q}$ on $\mathbf{K}^N$;
- in case (vi), $\mathbf{K} = \mathbf{R}$, $\mathbf{L} = \mathbf{C}$, $N = 2m$, and $b_{\mathbf{K}} = \omega_{\mathbf{R}}^{2m}$ on $\mathbf{K}^N$.

In all three cases, the group G of Table 2 is $\mathrm{Aut}_{\mathbf{K}}(b_{\mathbf{K}})$. Consider $j \in \mathbf{L} \smallsetminus \mathbf{K}$ such that $j^2 = -1$ and let $\sigma : \mathbf{K} \to \mathbf{K}$ be the conjugation by j, namely $z^\sigma = -jzj$ for all $z \in \mathbf{K}$. (In cases (iv) and (vi) we have $\sigma = \mathrm{id}_{\mathbf{R}}$ and in case (v) we have $z^\sigma = \bar{z}$.) Let $b_{\mathbf{L}}$ be the bilinear form on $\mathbf{L}^N = \mathbf{K}^N + \mathbf{K}^N j$ given by

$$b_{\mathbf{L}}(v + v'j, w + w'j) = b_{\mathbf{K}}(v, w) - b_{\mathbf{K}}(v', w')^\sigma + (b_{\mathbf{K}}(v', w)^\sigma + b_{\mathbf{K}}(v, w'))j.$$

The group G' of Table 2 is $\mathrm{Aut}_{\mathbf{L}}(b_{\mathbf{L}})$ and the natural injection $\mathrm{Aut}_{\mathbf{K}}(b_{\mathbf{K}}) \hookrightarrow \mathrm{Aut}_{\mathbf{L}}(b_{\mathbf{L}})$ defines the injection $\iota : G \hookrightarrow G'$.

As in Section 3.9, we denote by $P_1(b_{\mathbf{K}})$ the stabilizer of an isotropic line in $(\mathbf{K}^N, b_{\mathbf{K}})$ and by $\mathcal{F}_1(b_{\mathbf{K}}) = G/P_1(b_{\mathbf{K}})$ the set of isotropic lines. We use similar notation $P_1(b_{\mathbf{L}})$ and $\mathcal{F}_1(b_{\mathbf{L}})$ for G'. There is a natural ι-equivariant embedding $i : \mathcal{F}_1(b_{\mathbf{K}}) \hookrightarrow \mathcal{F}_1(b_{\mathbf{L}})$. Let Γ be a word hyperbolic group and $\rho : \Gamma \to G$ a $P_1(b_{\mathbf{K}})$-Anosov representation with boundary map $\xi : \partial_\infty\Gamma \to \mathcal{F}_1(b_{\mathbf{K}})$. It easily follows from Definition 3.4 (see also [GGKW16, Prop. 3.5]) that the composition $\iota \circ \rho : \Gamma \to G'$ is $P_1(b_{\mathbf{L}})$-Anosov with boundary map $\xi' = i \circ \xi : \partial_\infty\Gamma \to \mathcal{F}_1(b_{\mathbf{L}})$. For any $\eta \in \partial_\infty\Gamma$, the $\mathbf{L}$-line $\xi'(\eta)$ intersects $\mathbf{K}^N \subset \mathbf{L}^N$ nontrivially (the intersection is $\xi(\eta)$). Therefore, the cocompact domain of discontinuity Ω of Proposition 3.13.(1) contains

$$\mathcal{V} = \big\{ W \in \mathcal{F}_N(b_{\mathbf{L}}) \mid W \cap \mathbf{K}^N = \{0\} \big\},$$

which is a G-invariant, open, and dense subset of $\mathcal{F}_N(b_{\mathbf{L}})$. In particular, the action of Γ on $\mathcal{V}$ via $\iota \circ \rho$ is properly discontinuous, and $\Gamma\backslash\Omega$ is a compactification of $\Gamma\backslash\mathcal{V}$.

The fact that $\mathcal{V}$ contains an open G-orbit $\mathcal{U}$ isomorphic to G/H is contained in the following more precise statement. It concludes the proof of Proposition 1.5.

Lemma 5.7. *In cases (iv) and (v) of Table 2 the action of G on $\mathcal{V}$ is transitive. In case (vi) the set $\mathcal{V}$ is the disjoint union of $(m+1)$ open G-orbits isomorphic to $G/\mathrm{U}(p, m-p)$ for p ranging through $\{0, \dots, m\}$.*

Proof. Let $W \in \mathcal{U}$. Since $W \cap \mathbf{K}^N = \{0\}$ there is an $\mathbf{R}$-linear map $J : \mathbf{K}^N \to \mathbf{K}^N$ such that

$$W = \{v + J(v)j \mid v \in \mathbf{K}^N\}. \tag{5.2}$$

The fact that W is an $\mathbf{L}$-subspace is equivalent to J being σ-antilinear (i.e. $J(v\lambda) = J(v)\lambda^\sigma$) and $J^2 = -\mathrm{id}_{\mathbf{K}^N}$. The fact that W is $b_{\mathbf{L}}$-isotropic is equivalent to

$$b_{\mathbf{K}}(J(v), J(w)) = b_{\mathbf{K}}(v, w)^\sigma \quad \text{and} \quad b_{\mathbf{K}}(v, J(w)) = -b_{\mathbf{K}}(J(v), w)^\sigma$$

for all v, w in $\mathbf{K}^N$. Furthermore, for $g \in G$, the linear map corresponding to $g \cdot W$ is gJg^{-1}.

Conversely a linear map J with the above properties defines an element W of $\mathcal{U}$ by the formula (5.2). In cases (iv) and (v) it is easy to see that there is only one conjugacy class of such J whereas in case (vi) there are $(m+1)$ conjugacy classes corresponding to the different signatures of the symmetric form $(v, w) \mapsto \omega_{\mathbf{R}}^{2m}(v, J(w))$. ☐

5.5. **Compactifying more families of (locally) homogeneous spaces.** We now use Remark 1.6 to compactify other reductive homogeneous spaces that are not affine symmetric spaces, together with their Clifford–Klein forms.

Proposition 5.8. *Let (G, H, P, G', P', P'') be as in Table 3.*

(1) *There exists an open G-orbit $\mathcal{U}$ in G'/P'' that is diffeomorphic to G/H; the closure $\overline{\mathcal{U}}$ of $\mathcal{U}$ in G'/P'' provides a compactification of G/H.*
(2) *For any word hyperbolic group Γ and any P-Anosov representation $\rho : \Gamma \to G$, the cocompact domain of discontinuity $\Omega \subset G'/P'$ for $\rho(\Gamma)$ constructed in [GW12] (see Proposition 3.13.(1)) lifts to a cocompact domain of discontinuity $\tilde{\Omega} \subset G'/P''$ that contains $\mathcal{U}$; the quotient $\rho(\Gamma)\backslash(\tilde{\Omega} \cap \overline{\mathcal{U}})$ provides a compactification of $\rho(\Gamma)\backslash G/H$.*

	G	H	P	G'	P'	P''
(vii)	$\mathrm{O}(4p, 4q)$	$\mathrm{Sp}(2p, 2q)$	$\mathrm{Stab}_G(\ell)$	$\mathrm{Sp}(2p+2q, 2p+2q)$	$\mathrm{Stab}_{G'}(W')$	$\mathrm{Stab}_{G'}(W'' \subset W')$
(viii)	$\mathrm{Sp}(4m, \mathbf{R})$	$\mathrm{O}^*(2m)$	$\mathrm{Stab}_G(\ell)$	$\mathrm{O}^*(8m)$	$\mathrm{Stab}_{G'}(W')$	$\mathrm{Stab}_{G'}(W'' \subset W')$

TABLE 3. Cases to which Proposition 5.8 applies. Here m, p, q are any positive integers. We denote by ℓ an isotropic line (over $\mathbf{R}$) and by W' a maximal isotropic subspace (over $\mathbf{H}$), relative to the form b preserved by G or G'. We also denote by $(W'' \subset W')$ a partial flag of isotropic subspaces with W' maximal and $\dim_{\mathbf{R}}(W') = 2\dim_{\mathbf{R}}(W'')$.

Proof of Proposition 5.8 in case (vii) of Table 3. Let us write $\mathbf{H} = \mathbf{R} + \mathbf{R}i + \mathbf{R}j + \mathbf{R}k$ where $i^2 = j^2 = -1$ and $ij = -ji = k$. We identify $\mathbf{H}^{p+q}$ with $\mathbf{R}^{4p+4q}$, and see $H = \mathrm{Aut}_{\mathbf{H}}(b_{\mathbf{H}}^{p,q})$ as the subgroup of $G = \mathrm{Aut}_{\mathbf{R}}(b_{\mathbf{R}}^{4p,4q})$ commuting with the right multiplications by i and by j, which we denote respectively by $I, J \in G$. The tensor product $\mathbf{R}^{4p+4q} \otimes_{\mathbf{R}} \mathbf{H}$ can be realized as the set of "formal" sums

$$\mathbf{R}^{4p+4q} \otimes_{\mathbf{R}} \mathbf{H} = \{v_1 + v_2 i + v_3 j + v_4 k \mid v_1, v_2, v_3, v_4 \in \mathbf{R}^{4p+4q}\}.$$

Consider the real bilinear form b on $\mathbf{R}^{4p+4q} \otimes_{\mathbf{R}} \mathbf{H}$ given by

$$b(v_{\mathbf{H}}, v'_{\mathbf{H}}) = b_{\mathbf{R}}^{4p,4q}(v_1, v'_1) - b_{\mathbf{R}}^{4p,4q}(v_2, v'_2) + b_{\mathbf{R}}^{4p,4q}(v_3, v'_3) - b_{\mathbf{R}}^{4p,4q}(v_4, v'_4)$$

for any $v_{\mathbf{H}} = v_1 + v_2 i + v_3 j + v_4 k$ and $v'_{\mathbf{H}} = v'_1 + v'_2 i + v'_3 j + v'_4 k$ in $\mathbf{R}^{4p+4q} \otimes_{\mathbf{R}} \mathbf{H}$, and let $b_{\mathbf{H}}$ be the $\mathbf{H}$-Hermitian form on $\mathbf{R}^{4p+4q} \otimes_{\mathbf{R}} \mathbf{H}$ with real form b. Then $G' = \mathrm{Sp}(2p+2q, 2p+2q)$ identifies with $\mathrm{Aut}_{\mathbf{H}}(b_{\mathbf{H}})$, and the natural embedding of $G = \mathrm{Aut}_{\mathbf{R}}(b_{\mathbf{R}}^{4p,4q})$ into G' induces a natural embedding of $\mathcal{F}_1(b_{\mathbf{R}}^{4p,4q})$ into $\mathcal{F}_1(b_{\mathbf{H}})$.

Let $\mathcal{F}_{p+q,2p+2q}(b_{\mathbf{H}})$ be the space of partial flags $(W'' \subset W')$ of $\mathbf{R}^{4p+4q} \otimes_{\mathbf{R}} \mathbf{H}$ with $W' \in \mathcal{F}_{2p+2q}(b_{\mathbf{H}})$ and $\dim_{\mathbf{H}}(W') = 2\dim_{\mathbf{H}}(W'')$. (Note that the inclusion $W'' \subset W'$ imposes $b_{\mathbf{H}}|_{W'' \times W''} = 0$, i.e. $W'' \in \mathcal{F}_{p+q}(b_{\mathbf{H}})$.) The space $\mathcal{F}_{p+q,2p+2q}(b_{\mathbf{H}})$ identifies with G'/P'' and fibers G'-equivariantly over $\mathcal{F}_{2p+2q}(b_{\mathbf{H}}) \simeq G'/P'$ with compact fiber. Consider the element $(W''_0 \subset W'_0) \in \mathcal{F}_{p+q,2p+2q}(b_{\mathbf{H}})$ given by

$$\begin{cases} W''_0 & := & \{v + (Iv)i + (Jv)j + (Kv)k \mid v \in \mathbf{R}^{4p+4q}\}, \\ W'_0 & := & \{v + (Iv)i + (Jv')j + (Kv')k \mid v, v' \in \mathbf{R}^{4p+4q}\}. \end{cases}$$

Its stabilizer in $G = \mathrm{Aut}_{\mathbf{R}}(b_{\mathbf{R}}^{4p,4q})$ is the set of elements g commuting with I and J, namely $H = \mathrm{Aut}_{\mathbf{H}}(b_{\mathbf{H}}^{p,q})$. Thus the G-orbit $\mathcal{U}$ of $(W''_0 \subset W'_0)$ in $\mathcal{F}_{p+q,2p+2q}(b_{\mathbf{H}})$ identifies with G/H and the closure $\overline{\mathcal{U}}$ of $\mathcal{U}$ in $\mathcal{F}_{p+q,2p+2q}(b_{\mathbf{H}}) \simeq G/P''$ provides a compactification of G/H.

Let Γ be a word hyperbolic group and $\rho : \Gamma \to G$ a $P_1(b_{\mathbf{R}}^{4p,4q})$-Anosov representation with boundary map $\xi : \partial_\infty \Gamma \to \mathcal{F}_1(b_{\mathbf{R}}^{4p,4q})$. It easily follows from Definition 3.4 (see also [GGKW16, Prop. 3.5]) that the composed representation $\rho' : \Gamma \to G \hookrightarrow G'$ is $P_1(b_{\mathbf{H}})$-Anosov and that its boundary map $\xi' : \partial_\infty \Gamma \to \mathcal{F}_1(b_{\mathbf{H}})$ is the composition of ξ with the natural inclusion $\mathcal{F}_1(b_{\mathbf{R}}^{4p,4q}) \hookrightarrow \mathcal{F}_1(b_{\mathbf{H}})$. By Proposition 3.13.(1), the group Γ acts properly discontinuously and cocompactly, via ρ', on Ω the complement in $\mathcal{F}_{2p+2q}(b_{\mathbf{H}})$ of

$$\mathcal{K}_{\xi'} = \bigcup_{\eta \in \partial_\infty \Gamma} \{W' \in \mathcal{F}_{2p+2q}(b_{\mathbf{H}}) \mid \xi'(\eta) \subset W'\}.$$

Since $\mathcal{F}_{p+q,2p+2q}(b_{\mathbf{H}})$ fibers G'-equivariantly over $\mathcal{F}_{2p+2q}(b_{\mathbf{H}})$ with compact fiber, Γ also acts properly discontinuously and cocompactly, via ρ', on the preimage $\tilde{\Omega}$ of Ω in $\mathcal{F}_{p+q,2p+2q}(b_{\mathbf{H}})$. One checks that $\tilde{\Omega}$ contains the G-invariant open set

$$\mathcal{U}' := \{(W'' \subset W') \in \mathcal{F}_{p+q,2p+2q}(b_{\mathbf{H}}) \mid W' \cap \mathbf{R}^{4p+4q} = \{0\}\},$$

which itself contains $(W''_0 \subset W'_0)$, hence $\mathcal{U}$. Thus Γ acts properly discontinuously on G/H via ρ and the quotient $\rho'(\Gamma)\backslash(\tilde{\Omega} \cap \overline{\mathcal{U}})$ provides a compactification of $\rho(\Gamma)\backslash G/H$. $\square$

Case (viii) of Table 3 is similar to case (vii): just replace the real quadratic form $b_{\mathbf{R}}^{4p,4q}$ on $\mathbf{R}^{4p+4q}$ with the symplectic form $\omega_{\mathbf{R}}^{4m}$ on $\mathbf{R}^{4m}$, and b with the symplectic form $\omega_{\mathbf{R}}^{4m}(v_1, v'_1) - \omega_{\mathbf{R}}^{4m}(v_2, v'_2) + \omega_{\mathbf{R}}^{4m}(v_3, v'_3) - \omega_{\mathbf{R}}^{4m}(v_4, v'_4)$ on $\mathbf{R}^{4m} \otimes_{\mathbf{R}} \mathbf{H}$. The subgroup of $G = \mathrm{Aut}_{\mathbf{R}}(\omega_{\mathbf{R}}^{4m})$ commuting with I and J is $H = \mathrm{O}^*(2m)$.

6. Topological tameness

Lemma 1.10 is a particular case of the following general principle.

Proposition 6.1. *Let X be a real semi-algebraic set and Γ a torsion-free discrete group acting on X by real algebraic homeomorphisms. Suppose Γ acts properly discontinuously and cocompactly on some open subset Ω of X. Let $\mathcal{U}$ be a Γ-invariant real semi-algebraic subset of X contained in Ω (e.g. an orbit of a real algebraic group* *containing* Γ *and acting algebraically on* X*). Then the closure* $\overline{\mathcal{U}}$ *of* $\mathcal{U}$ *in* X *is real semi-algebraic and* $\Gamma\backslash(\overline{\mathcal{U}}\cap\Omega)$ *is compact and has a triangulation such that* $\Gamma\backslash(\partial\mathcal{U}\cap\Omega)$ *is a finite union of simplices. If* $\mathcal{U}$ *is a manifold, then* $\Gamma\backslash\mathcal{U}$ *is topologically tame.*

Here we use the notation $\mathring{D}$ for the interior of a subset D of X and $\partial D = \overline{D} \smallsetminus \mathring{D}$ for its boundary.

6.1. **Real semi-algebraic subsets.** Before proving Proposition 6.1, we first review a few basic definitions on real semi-algebraic sets and maps.

Recall that a *real semi-algebraic subset* of $\mathbf{R}^N$ is a subset defined by polynomial equalities and inequalities. More precisely, the class $\mathcal{S} \subset \mathcal{P}(\mathbf{R}^N)$ of real semi-algebraic subsets is the smallest class stable by finite union, finite intersection, complementary and containing the sets $\{P = 0\}$ and $\{P > 0\}$ for every polynomial P.

A map $f : X \to Y$ between real semi-algebraic subsets is called *semi-algebraic* if its graph is a real semi-algebraic subset of $X \times Y$. Algebraic maps are always semi-algebraic. If f is a semi-algebraic function, then so are $|f|$, $\sqrt{|f|}$, etc. If f' is another semi-algebraic function, then $\max(f, f')$ is semi-algebraic; in particular, $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$ are always semi-algebraic. The inverse of a semi-algebraic homeomorphism is semi-algebraic.

The closure of a real semi-algebraic subset is also real semi-algebraic (see e.g. [Cos00, Cor. 2.5]). The image of a real semi-algebraic subset by a semi-algebraic map is a real semi-algebraic subset [Cos00, Cor. 2.4.(2)].

Definition 6.2. A *locally real semi-algebraic set* is a topological space X which admits an open covering $\mathfrak{U}$ and, for every $U \in \mathfrak{U}$, a continuous map $\phi_U : U \to \mathbf{R}^{N_U}$ such that

- ϕ_U is a homeomorphism onto its image $\phi_U(U)$, which is a real semi-algebraic subset of $\mathbf{R}^{N_U}$,
- for any $U, V \in \mathfrak{U}$ the subset $\phi_U(U \cap V) \subset \mathbf{R}^{N_U}$ is real semi-algebraic;
- for any $U, V \in \mathfrak{U}$, the map $\phi_U \circ \phi_V^{-1} : \phi_V(U \cap V) \to \mathbf{R}^{N_U}$ is semi-algebraic.

Any real semi-algebraic subset is a locally real semi-algebraic set. The notion of semi-algebraic map naturally extends to the setting of locally real semi-algebraic sets.

Remark 6.3. Up to taking a refinement of $\mathfrak{U}$ and composing ϕ_U by an affine transformation of $\mathbf{R}^{N_U}$, we may assume that for every $U \in \mathfrak{U}$ the set $\phi(U) \subset \mathbf{R}^{N_U}$ is contained in the Euclidean ball B_U of radius 1 centered at $0 \in \mathbf{R}^{N_U}$, and that ϕ_U extends to the closure $\overline{U}$ of U in X with $\phi_U : \overline{U} \to \mathbf{R}^{N_U}$ injective and $\phi_U(\partial U) \subset \partial B_U$.

6.2. **Compact locally real semi-algebraic sets.** Proposition 6.1 relies on the following observation.

Proposition 6.4. *If a locally real semi-algebraic set* X *is compact, then it is in fact real semi-algebraic, i.e. there exist an integer* $N \in \mathbf{N}$*, a real semi-algebraic subset* $S \subset \mathbf{R}^N$*, and a semi-algebraic homeomorphism* $\phi : X \to S$*.*

Proof. Let $\mathfrak{U}$ be an open covering and $\phi_U : U \to \mathbf{R}^{N_U}$, for $U \in \mathfrak{U}$, continuous maps defining the locally real semi-algebraic structure of X. We may assume that they are as in Remark 6.3. Since X is compact, we may furthermore assume that $\mathfrak{U}$ is finite.

For any $U \in \mathfrak{U}$, the function $f_U(u) = 1 - \|\phi_U(u)\|_{\mathbf{R}^{N_U}}$ is semi-algebraic on U and zero on ∂U. The map

$$\begin{array}{rccc} \psi_U : & U & \longrightarrow & \mathbf{R} \times \mathbf{R}^{N_U} \\ & u & \longmapsto & (f_U(u), f_U(u)\, \phi_U(u)) \end{array}$$

is continuous, injective, and semi-algebraic. Extending it by zero outside U, we obtain a continuous semi-algebraic map $\psi_U : X \to \mathbf{R}^{N_U+1}$.

The direct sum of the ψ_U, for $U \in \mathfrak{U}$, is a continuous, injective, semi-algebraic map $\phi : X \to \mathbf{R}^N$. Since X is compact, ϕ is a homeomorphism onto its image. This image is the finite union of the real semi-algebraic subsets $\phi(U) \subset \mathbf{R}^N$, hence is real semi-algebraic. □

Proof of Proposition 6.1. Since the closure of a real semi-algebraic subset is real semi-algebraic, $\overline{\mathcal{U}}$ is real semi-algebraic and $\partial\mathcal{U} = \overline{\mathcal{U}} \smallsetminus \mathcal{U}$ is real semi-algebraic.

The quotients $\Gamma\backslash(\overline{\mathcal{U}} \cap \Omega)$ and $\Gamma\backslash(\partial\mathcal{U} \cap \Omega)$ have a natural structure of locally real semi-algebraic sets. Since they are compact, they are real semi-algebraic by Proposition 6.4. Thus the triangulation theorem for real semi-algebraic pairs (see [Cos00, Th. 3.12]) gives the sought-for triangulation.

This triangulation allows us to build a tubular neighborhood of $\Gamma\backslash(\partial\mathcal{U} \cap \Omega)$ such that $\Gamma\backslash\mathcal{U}$ is homeomorphic to the complement of this tubular neighborhood. Thus, if $\mathcal{U}$ is a manifold, then $\Gamma\backslash\mathcal{U}$ is homeomorphic to the interior of a compact manifold with boundary. □

6.3. **Tameness of group manifolds.** From Theorem 4.1 and Lemma 1.10 (particular case of Proposition 6.1), we deduce the following. Theorem 1.12 corresponds to the special case where ρ_R is constant.

Theorem 6.5. *Let Γ be a torsion-free word hyperbolic group, G a real reductive algebraic group, and $\rho_L, \rho_R : \Gamma \to G$ two representations. Let $\alpha \in \Delta$ be a simple restricted root of G. If ρ_L is $P_{\{\alpha\}}$-Anosov and uniformly ω_α-dominates ρ_R, then $(\rho_L, \rho_R)(\Gamma)\backslash(G \times G)/\mathrm{Diag}(G)$ is a topologically tame manifold.*

For $G = \mathrm{SO}(p, 1)$ with $p \geq 2$, this was first proved in [GK16, Th. 1.8 & Prop. 7.2]. In that case, tameness actually still holds when ρ_L is allowed to be geometrically finite instead of convex cocompact.

Recall that any P_θ-Anosov representation is $P_{\{\alpha\}}$-Anosov for all $\alpha \in \theta$ (see Section 3.3).

Proof of Theorem 6.5. By Proposition 3.11, there exist a nondegenerate bilinear form b on a real vector space V and a linear representation $\tau : G \to \mathrm{Aut}_{\mathbf{R}}(b)$ such that $\tau \circ \rho_L : \Gamma \to \mathrm{Aut}_{\mathbf{R}}(b)$ is $P_1(b)$-Anosov and uniformly $\omega_{\alpha_1(b)}$-dominates $\tau \circ \rho_R$. Let Ω be the cocompact domain of discontinuity of $(\tau \circ \rho_L \oplus \tau \circ \rho_R)(\Gamma)$ in $\mathcal{F}_N(b \oplus -b)$ given by Proposition 3.13.(1). By Theorem 4.1, it contains the open $(\mathrm{Aut}_{\mathbf{R}}(b) \times \mathrm{Aut}_{\mathbf{R}}(b))$-orbit $\mathcal{U}_0$ of Theorem 1.1, which identifies with $(\mathrm{Aut}_{\mathbf{R}}(b) \times \mathrm{Aut}_{\mathbf{R}}(b))/\mathrm{Diag}(\mathrm{Aut}_{\mathbf{R}}(b))$. Let u be a point in $\mathcal{U}_0$ with stabilizer equal to $\mathrm{Diag}(\mathrm{Aut}_{\mathbf{R}}(b))$. Applying Lemma 1.10 to the $(\tau \oplus \tau)(G)$-orbit $\mathcal{U}$ of u in $\mathcal{U}_0$, we see that $(\tau \circ \rho_L \oplus \tau \circ \rho_R)(\Gamma)\backslash(\tau(G) \times \tau(G))/\mathrm{Diag}(\tau(G))$ is a topologically tame manifold. If τ has finite kernel, then $(\rho_L \oplus \rho_R)(\Gamma)\backslash(G \times G)/G$ is a topologically tame manifold as well.

However, in general τ might not have finite kernel. To address this issue, we force injectivity by introducing another representation, as follows. Let $\tau' : G \to \mathrm{GL}_{\mathbf{R}}(V')$ be any injective linear representation of G where V' is a real vector space of dimension $N' \in \mathbf{N}$. The Grassmannian $\mathcal{F}_{N'}(V' \oplus V')$ is compact, hence the action of Γ on $\Omega \times \mathcal{F}_{N'}(V' \oplus V')$ via

$$(\tau \circ \rho_L \oplus \tau \circ \rho_R) \times (\tau' \circ \rho_L \oplus \tau' \circ \rho_R)$$

is properly discontinuous and cocompact. By Theorem 2.6, there is an open $(\mathrm{GL}_{\mathbf{R}}(V') \times \mathrm{GL}_{\mathbf{R}}(V'))$-orbit $\mathcal{U}'_0$ in $\mathcal{F}_{N'}(V' \oplus V')$ that identifies with

$$(\mathrm{GL}_{\mathbf{R}}(V') \times \mathrm{GL}_{\mathbf{R}}(V'))/\mathrm{Diag}(\mathrm{GL}_{\mathbf{R}}(V')).$$

Let u' be a point in $\mathcal{U}'_0$ with stabilizer $\mathrm{Diag}(\mathrm{GL}_{\mathbf{R}}(V'))$ in $\mathrm{GL}_{\mathbf{R}}(V') \times \mathrm{GL}_{\mathbf{R}}(V')$. By injectivity of τ', the stabilizer of (u, u') in $G \times G$ for the action of $G \times G$ on $\mathcal{F}_N(b \oplus -b) \times \mathcal{F}_{N'}(V' \oplus V')$ via $(\tau \oplus \tau) \times (\tau' \oplus \tau')$ is $\mathrm{Diag}(G)$. Applying Lemma 1.10 to the $((\tau \oplus \tau) \times (\tau' \oplus \tau'))(G)$-orbit $\mathcal{U}$ of (u, u') and to $\Omega \times \mathcal{F}_{N'}(V' \oplus V')$ instead of Ω, we obtain that $(\rho_L, \rho_R)(\Gamma)\backslash(G \times G)/\mathrm{Diag}(G)$ is a topologically tame manifold. $\square$

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