CONVEX COCOMPACT ACTIONS IN REAL PROJECTIVE GEOMETRY

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Abstract. We study a notion of convex cocompactness for (not necessarily irreducible) discrete subgroups of the projective general linear group acting on real projective space, and give various characterizations. A convex cocompact group in this sense need not be word hyperbolic, but we show that it still has some of the good properties of classical convex cocompact subgroups in rank-one Lie groups. Extending our earlier work [DGK3] from the context of projective orthogonal groups, we show that for word hyperbolic groups preserving a properly convex open set in projective space, the above general notion of convex cocompactness is equivalent to a stronger convex cocompactness condition studied by Crampon–Marquis, and also to the condition that the natural inclusion be a projective Anosov representation. We investigate examples.

Contents

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1. Introduction

In the classical setting of semisimple Lie groups G of real rank one, a discrete subgroup of G is said to be convex cocompact if it acts cocompactly on some nonempty closed convex subset of the Riemannian symmetric space G/K of G. Such subgroups have been abundantly studied, in particular in the context of Kleinian groups and real hyperbolic geometry, where there is a rich world of examples. They are known to display good geometric and dynamical behavior.

On the other hand, in higher-rank semisimple Lie groups G, the condition that a discrete subgroup Γ act cocompactly on some nonempty convex subset of the Riemannian symmetric space G/K turns out to be quite restrictive: Kleiner–Leeb [KL] and Quint [Q] proved, for example, that if G is simple and such a subgroup Γ is Zariski-dense in G, then it is in fact a uniform lattice of G.

The notion of an *Anosov representation* of a word hyperbolic group in a higher-rank semisimple Lie group G, introduced by Labourie [L] and generalized by Guichard–Wienhard [GW3], is a much more flexible notion which has earned a central role in higher Teichmüller–Thurston theory, see e.g. [BIW2, BCLS, KLPc, GGKW, BPS]. Anosov representations are defined, not in terms of convex subsets of the Riemannian symmetric space G/K, but instead in terms of a dynamical condition for the action on a certain flag variety, i.e. on a compact homogeneous space G/P. This dynamical condition guarantees many desirable analogies with convex cocompact subgroups in rank one: see e.g. [L, GW3, KLPa, KLPb, KLPc]. It also allows for the definition of certain interesting geometric structures associated to Anosov representations: see e.g. [GW1, GW3, KLPb, GGKW, CTT]. However, natural *convex* geometric structures associated to Anosov representations have been lacking in general. Such structures could allow geometric intuition to bear more fully on Anosov representations, making them more accessible through familiar geometric constructions such as convex fundamental domains, and potentially unlocking new sources of examples. While there is a rich supply of examples of Anosov representations into higher-rank Lie groups in the case of surface groups or free groups, it has proven difficult to construct examples for more complicated word hyperbolic groups.

One of the goals of this paper is to show that, when $G = \mathrm{PGL}(\mathbb{R}^n)$ is a projective linear group, there are in many cases natural convex cocompact geometric structures modeled on $\mathbb{P}(\mathbb{R}^n)$ associated to Anosov representations into G. In particular, we prove that any Anosov representation into $\mathrm{PGL}(\mathbb{R}^n)$ which preserves a nonempty properly convex open subset of the projective space $\mathbb{P}(\mathbb{R}^n)$ satisfies a strong notion of convex cocompactness introduced by Crampon–Marquis [CM]. Conversely, we show that convex cocompact subgroups of $\mathbb{P}(\mathbb{R}^n)$ in the sense of [CM] always give rise to Anosov representations, which enables us to give new examples of Anosov representations and study their deformation spaces by constructing these geometric structures directly. In [DGK3] we had previously established this close connection between convex cocompactness in projective space and Anosov representations in the case of irreducible representations valued in a projective orthogonal group $\mathrm{PO}(p,q)$.

One context where a connection between Anosov representations and convex projective structures has been known for some time is the deformation theory of real projective surfaces, for $G = \mathrm{PGL}(\mathbb{R}^3)$ [Go, CGo]. More generally, it follows from work of Benoist [B3] that if a discrete subgroup Γ of $G = \mathrm{PGL}(\mathbb{R}^n)$ *divides* (i.e. acts cocompactly on) a strictly convex open subset of $\mathbb{P}(\mathbb{R}^n)$, then Γ is word hyperbolic and the natural inclusion $\Gamma \hookrightarrow G$ is Anosov.

Benoist [B6] also found examples of discrete subgroups of $\mathrm{PGL}(\mathbb{R}^n)$ which divide properly convex open sets that are not strictly convex; these subgroups are not word hyperbolic. In this paper we study a notion of convex cocompactness for discrete subgroups of $\mathrm{PGL}(\mathbb{R}^n)$ acting on $\mathbb{P}(\mathbb{R}^n)$ which simultaneously generalizes Crampon–Marquis's notion and Benoist's convex divisible sets [B3, B4, B5, B6]. In particular, we show that this notion is stable under deformation into larger projective general linear groups, after the model of quasi-Fuchsian deformations of Fuchsian groups. This yields examples of nonhyperbolic irreducible discrete subgroups of $\mathrm{PGL}(\mathbb{R}^n)$ which are convex cocompact in $\mathbb{P}(\mathbb{R}^n)$ but do not divide a properly convex open subset of $\mathbb{P}(\mathbb{R}^n)$.

We now describe our results in more detail.

1.1. Strong convex cocompactness in $\mathbb{P}(V)$ and Anosov representations.

In the whole paper, we fix an integer $n \geq 2$ and set $V := \mathbb{R}^n$. Recall that an open domain Ω in the projective space $\mathbb{P}(V)$ is said to be *properly convex* if it is convex and bounded in some affine chart, and *strictly convex* if in addition its boundary does not contain any nontrivial projective line segment. It is said to have *C^1 boundary* if every point of the boundary of Ω has a unique supporting hyperplane.

In [CM], Crampon–Marquis introduced a notion of *geometrically finite action* of a discrete subgroup Γ of $\mathrm{PGL}(V)$ on a strictly convex open domain of $\mathbb{P}(V)$ with C^1 boundary. If cusps are not allowed, this notion reduces to a natural notion of convex cocompact action on such domains. We will call discrete groups Γ with such actions *strongly convex cocompact*.

Definition 1.1. Let $\Gamma \subset \mathrm{PGL}(V)$ be an infinite discrete subgroup.

- Let Ω be a Γ-invariant properly convex open subset of $\mathbb{P}(V)$ whose boundary is strictly convex and C^1. The action of Γ on Ω is *convex cocompact* if the convex hull in Ω of the *orbital limit set* $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ of

Γ in Ω is nonempty and has compact quotient by Γ. (For convex cocompact actions on non-strictly convex Ω, see Section 1.4.)

- The group Γ is *strongly convex cocompact in* $\mathbb{P}(V)$ if it admits a convex cocompact action on some nonempty properly convex open subset Ω of $\mathbb{P}(V)$ whose boundary is strictly convex and C^1.

Here the orbital limit set $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ of Γ in Ω is defined as the set of accumulation points in $\partial\Omega$ of the Γ-orbit of some point $z \in \Omega$, and the convex hull of $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ in Ω as the intersection of Ω with the convex hull of $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ in the convex set $\overline{\Omega}$. It is easy to see that $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ does not depend on the choice of z since Ω is strictly convex. We deduce the following remark.

Remark 1.2. The action of Γ on the strictly convex open set Ω is convex cocompact in the sense of Definition 1.1 if and only if Γ acts cocompactly on *some* nonempty closed convex subset of Ω.

Example 1.3. For $V = \mathbb{R}^{p,1}$ with $p \geq 2$, any discrete subgroup of $\mathrm{Isom}(\mathbb{H}^p) = \mathrm{PO}(p,1) \subset \mathrm{PGL}(V)$ which is convex cocompact in the usual sense is strongly convex cocompact in $\mathbb{P}(V)$, taking Ω to be the projective model of $\mathbb{H}^p$.

The first main result of this paper is a close connection between strong convex cocompactness in $\mathbb{P}(V)$ and Anosov representations into $\mathrm{PGL}(V)$. Let P_1 (resp. P_{n-1}) be the stabilizer in $G = \mathrm{PGL}(V)$ of a line (resp. hyperplane) of $V = \mathbb{R}^n$; it is a maximal proper parabolic subgroup of G, and G/P_1 (resp. G/P_{n-1}) identifies with $\mathbb{P}(V)$ (resp. with the dual projective space $\mathbb{P}(V^*)$). We shall think of $\mathbb{P}(V^*)$ as the space of projective hyperplanes in $\mathbb{P}(V)$. Let Γ be a word hyperbolic group. A *P_1-Anosov representation* (sometimes also called a *projective Anosov representation*) of Γ into G is a representation $\rho : \Gamma \to G$ for which there exist two continuous, ρ-equivariant boundary maps $\xi : \partial_\infty\Gamma \to \mathbb{P}(V)$ and $\xi^* : \partial_\infty\Gamma \to \mathbb{P}(V^*)$ which

(A1) are compatible, i.e. $\xi(\eta) \in \xi^*(\eta)$ for all $\eta \in \partial_\infty\Gamma$,
(A2) are transverse, i.e. $\xi(\eta) \notin \xi^*(\eta')$ for all $\eta \neq \eta'$ in $\partial_\infty\Gamma$,
(A3) have an associated flow with some uniform contraction/expansion property described in [L, GW3].

We do not state condition (A3) precisely, since we will use in place of it a simple condition on eigenvalues or singular values described in Definition 2.4 and Fact 2.5 below, taken from [GGKW]. Recall that an element $g \in \mathrm{PGL}(V)$ is said to be *proximal in* $\mathbb{P}(V)$ if it admits a unique attracting fixed point in $\mathbb{P}(V)$ (see Section 2.3). A consequence of (A3) is that every infinite-order element of $\rho(\Gamma)$ is proximal in $\mathbb{P}(V)$ and in $\mathbb{P}(V^*)$, and that the image $\xi(\partial_\infty\Gamma)$ (resp. $\xi^*(\partial_\infty\Gamma)$) of the boundary map is the *proximal limit set* of $\rho(\Gamma)$ in $\mathbb{P}(V)$ (resp. $\mathbb{P}(V^*)$), i.e. the closure of the set of attracting fixed points of proximal elements (Definition 2.2). By [GW3, Prop. 4.10], if ρ is irreducible then condition (A3) is automatically satisfied as soon as (A1) and (A2) are, but this is not true in general: see [GGKW, Ex. 7.15].

It is well known (see [GW3, § 6.1 & Th. 4.3]) that a discrete subgroup of $\mathrm{PO}(p,1)$ is convex cocompact in the classical sense if and only if it is word hyperbolic and the natural inclusion $\Gamma \hookrightarrow \mathrm{PO}(p,1) \hookrightarrow \mathrm{PGL}(\mathbb{R}^{p+1})$ is P_1-Anosov. In this paper, we prove the following higher-rank generalization.

Theorem 1.4. *Let Γ be any infinite discrete subgroup of $G = \mathrm{PGL}(V)$ preserving a nonempty properly convex open subset of $\mathbb{P}(V)$. Then the following are equivalent:*

(1) Γ is strongly convex cocompact in $\mathbb{P}(V)$;
(2) Γ is word hyperbolic and the natural inclusion $\Gamma \hookrightarrow G$ is P_1-Anosov.

As mentioned above, for Γ acting cocompactly on a strictly convex open set (which is then a *divisible strictly convex set*), the implication (1) $\Rightarrow$ (2) follows from work of Benoist [B3].

Remarks 1.5. (a) The fact that a strongly convex cocompact group is word hyperbolic is due to Crampon–Marquis [CM, Th. 1.8].

(b) In the case where Γ is irreducible (i.e. does not preserve any nontrivial projective subspace of $\mathbb{P}(V)$) and contained in $\mathrm{PO}(p,q) \subset \mathrm{PGL}(\mathbb{R}^n)$ for some $p,q \in \mathbb{N}^*$ with $p+q=n$, Theorem 1.4 was first proved in our earlier work [DGK3, Th. 1.11 & Prop. 1.17 & Prop. 3.7]. In that case we actually gave a more precise version of Theorem 1.4 involving the notion of negative/positive proximal limit set: see Section 1.8 below. Our proof of Theorem 1.4 in the present paper uses many of the ideas of [DGK3]. One main improvement here is the treatment of duality in the general case where there is no nonzero Γ-invariant quadratic form.

(c) In independent work, Zimmer [Z] also extends [DGK3] by studying a slightly different notion for actions of discrete subgroups Γ of $\mathrm{PGL}(V)$ on properly convex open subsets Ω of $\mathbb{P}(V)$: by definition [Z, Def. 1.8], a subgroup Γ of $\mathrm{Aut}(\Omega)$ is regular convex cocompact if it acts cocompactly on some nonempty, Γ-invariant, closed, properly convex subset $\mathcal{C}$ of Ω such that every extreme point of $\overline{\mathcal{C}}$ in $\partial\Omega$ is a C^1 extreme point of Ω. By [Z, Th. 1.21], if Γ is an irreducible discrete subgroup of $\mathrm{PGL}(V)$ which is regular convex cocompact in $\mathrm{Aut}(\Omega)$ for some $\Omega \subset \mathbb{P}(V)$, then Γ is word hyperbolic and the natural inclusion $\Gamma \hookrightarrow \mathrm{PGL}(V)$ is P_1-Anosov. Conversely, by [Z, Th. 1.10], any irreducible P_1-Anosov representation $\Gamma \hookrightarrow \mathrm{SL}(V)$ can be composed with an irreducible representation $\mathrm{SL}(V) \to \mathrm{SL}(V')$, for some larger vector space V', so that Γ becomes regular convex cocompact in $\mathrm{Aut}(\Omega')$ for some $\Omega' \subset \mathbb{P}(V')$.

(d) In this paper, unlike in [DGK3] or [Z], we do not assume Γ to be irreducible. This makes the notion of Anosov representation slightly more involved (condition (A3) above is not automatic), and also adds some subtleties to the notion of convex cocompactness (see e.g. Remarks 1.22 and 4.6). We note that there exist strongly convex cocompact groups in $\mathbb{P}(V)$ which are not irreducible, and whose Zariski closure is not even reductive (which means that there is an invariant linear subspace W of V which does not admit any invariant complementary subspace): see Proposition 10.9 and Remark 10.10. This contrasts with the case of divisible convex sets described in [Ve].

Remark 1.6. For $n = \dim(V) \geq 3$, there exist P_1-Anosov representations $\rho : \Gamma \to G = \mathrm{PGL}(V)$ which do not preserve any nonempty properly convex subset of $\mathbb{P}(V)$: see [DGK3, Ex. 5.2 & 5.3]. However, by [DGK3, Th. 1.7], if $\partial_\infty\Gamma$ is *connected*, then any P_1-Anosov representation ρ valued in $\mathrm{PO}(p, q) \subset \mathrm{PGL}(\mathbb{R}^{p+q})$ preserves a nonempty properly convex open subset of $\mathbb{P}(\mathbb{R}^{p+q})$, hence $\rho(\Gamma)$ is strongly convex cocompact in $\mathbb{P}(\mathbb{R}^{p+q})$ by Theorem 1.4. Extending this, [Z, Th. 1.24] gives sufficient group-theoretic conditions on Γ for P_1-Anosov representations $\Gamma \hookrightarrow \mathrm{PGL}(V)$ to be regular convex cocompact in $\mathrm{Aut}(\Omega)$ (in the sense of Remark 1.5.(c)) for some $\Omega \subset \mathbb{P}(V)$.

1.2. **Convex projective structures for Anosov representations.** We can apply Theorem 1.4 to show that some well-known families of Anosov representations, such as Hitchin representations in odd dimension, naturally give rise to convex cocompact real projective manifolds.

Proposition 1.7. *Let Γ be a closed surface group of genus ≥ 2 and $\rho : \Gamma \to \mathrm{PSL}(\mathbb{R}^n)$ a Hitchin representation.*

1. *If n is odd, then $\rho(\Gamma)$ is strongly convex cocompact in $\mathbb{P}(\mathbb{R}^n)$.*
2. *If n is even, then $\rho(\Gamma)$ is not strongly convex cocompact in $\mathbb{P}(\mathbb{R}^n)$; in fact it does not even preserve any nonempty properly convex subset of $\mathbb{P}(\mathbb{R}^n)$.*

For statement (1), see also [Z, Cor. 1.31]. This extends [DGK3, Prop. 1.19], about Hitchin representations valued in $\mathrm{SO}(k + 1, k) \subset \mathrm{PSL}(\mathbb{R}^{2k+1})$.

Remarks 1.8. (a) The case $n = 3$ of Proposition 1.7.(1) is due to Choi–Goldman [Go, CGo], who proved that the Hitchin component of $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(\mathbb{R}^3))$ parametrizes the convex projective structures on S for a given closed hyperbolic surface S.

(b) Guichard–Wienhard [GW3, Th. 11.3 & 11.5] associated different geometric structures to Hitchin representations into $\mathrm{PSL}(\mathbb{R}^n)$. For even n, their geometric structures are modeled on $\mathbb{P}(\mathbb{R}^n)$ but can never be convex (see Proposition 1.7.(2)). For odd n, their geometric structures are modeled on the space $\mathcal{F}_{1,n-1} \subset \mathbb{P}(\mathbb{R}^n) \times \mathbb{P}((\mathbb{R}^n)^*)$ of pairs (ℓ, H) where ℓ is a line of $\mathbb{R}^n$ and H a hyperplane containing ℓ; these geometric structures lack convexity but live on a *compact* manifold, unlike the geometric structures of Proposition 1.7.(1) for $n > 3$.

We refer to Proposition 12.1 for a more general statement on convex structures for connected open sets of Anosov representations.

1.3. **New examples of Anosov representations.** We can also use the implication (1) $\Rightarrow$ (2) of Theorem 1.4 to obtain new examples of Anosov representations by constructing explicit strongly convex cocompact groups in $\mathbb{P}(V)$. Following this strategy, in [DGK3, § 8] we showed that any word hyperbolic right-angled Coxeter group W admits reflection-group representations into some $\mathrm{PGL}(V)$ which are P_1-Anosov. Extending this approach, in [DGK4] we give an explicit description of the deformation spaces of such representations.

1.4. **General convex cocompactness in $\mathbb{P}(V)$.** We now discuss generalizations of Definition 1.1 where the properly convex open set Ω is not assumed to have any regularity at the boundary. These cover a larger class of groups, not necessarily word hyperbolic.

Given Remark 1.2, a naive generalization of Definition 1.1 that immediately comes to mind is the following.

Definition 1.9. An infinite discrete subgroup Γ of $\mathrm{PGL}(V)$ is *naively convex cocompact in* $\mathbb{P}(V)$ if it preserves a properly convex open subset Ω of $\mathbb{P}(V)$ and acts cocompactly on some nonempty closed convex subset $\mathcal{C}$ of Ω.

However, the class of naively convex cocompact subgroups of $\mathrm{PGL}(V)$ is not stable under small deformations: see Remark 3.10.(b). This is linked to the fact that if Γ and Ω are as in Definition 1.9 with $\partial\Omega$ not strictly convex, then the set of accumulation points of a Γ-orbit of Ω may depend on the orbit (see Example 3.7).

To address this issue, we introduce a notion of limit set that does not depend on a choice of orbit.

Definition 1.10. Let $\Gamma \subset \mathrm{PGL}(V)$ be an infinite discrete subgroup and let Ω be a properly convex open subset of $\mathbb{P}(V)$ invariant under Γ. The *full orbital limit set* $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ of Γ in Ω is the union of all accumulation points of all Γ-orbits in Ω.

Using this notion, we can introduce another generalization of Definition 1.1 which is slightly stronger and has better properties than Definition 1.9: here is the main definition of the paper.

Definition 1.11. Let Γ be an infinite discrete subgroup of $\mathrm{PGL}(V)$.

- Let Ω be a nonempty Γ-invariant properly convex open subset of $\mathbb{P}(V)$. The action of Γ on Ω is *convex cocompact* if the convex hull $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$ of the full orbital limit set $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ in Ω is nonempty and has compact quotient by Γ.
- Γ is *convex cocompact in* $\mathbb{P}(V)$ if it admits a convex cocompact action on some nonempty properly convex open subset Ω of $\mathbb{P}(V)$.

In the setting of Definition 1.11, the set $\Gamma\backslash\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$ is a compact convex subset of the projective manifold (or orbifold) $\Gamma\backslash\Omega$ which contains all the topology and which is minimal (Lemma 4.1.(2)); we shall call it the *convex core* of $\Gamma\backslash\Omega$. By analogy, we shall also call $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$ the *convex core* of Ω for Γ.

Example 1.12. If Γ is strongly convex cocompact in $\mathbb{P}(V)$, then it is convex cocompact in $\mathbb{P}(V)$. This is immediate from the definitions since when Ω is strictly convex, the full orbital limit set $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ coincides with the accumulation set of a single Γ-orbit of Ω. See Theorem 1.15 below for a refinement.

Example 1.13. If Γ divides (i.e. preserves and acts cocompactly on) a nonempty properly convex open subset Ω of $\mathbb{P}(V)$, then $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma) = \Omega$ and Γ is convex cocompact in $\mathbb{P}(V)$. There are many known examples of divisible convex sets which fail to be strictly convex, yielding convex cocompact groups which are not strongly convex cocompact in $\mathbb{P}(V)$: see Example 3.7 for a baby case. The first indecomposable examples for $\dim(V) = 4$ were constructed by Benoist [B6], and further examples were recently constructed for $\dim(V) = 4$ in [BDL] and for $5 \leq \dim(V) \leq 8$ in [CLM]. See Section 12.2.1.

In Theorem 1.20 we shall give several characterizations of convex cocompactness in $\mathbb{P}(V)$. In particular, we shall prove that the class of subgroups of $\mathrm{PGL}(V)$ that are convex cocompact in $\mathbb{P}(V)$ is precisely the class of holonomy groups of compact properly convex projective orbifolds with strictly convex boundary; this ensures that this class of groups is stable under small deformations, using [CLT1]. Before stating this and other results, let us make the connection with the context of Theorem 1.4 (strong convex cocompactness).

1.5. **Word hyperbolic convex cocompact groups in $\mathbb{P}(V)$.** By [B3, Th. 1.1], if a discrete subgroup $\Gamma \subset \mathrm{PGL}(V)$ divides a properly convex open subset $\Omega \subset \mathbb{P}(V)$, then Ω is strictly convex if and only if Γ is word hyperbolic. A similar statement is not true for convex cocompact actions in general.

Example 1.14 (see Proposition 1.7.(1) and Corollary 8.10)**.** Let $n \geq 5$ be odd, Γ a closed surface group of genus ≥ 2 and $\rho : \Gamma \to \mathrm{PSL}(\mathbb{R}^n)$ a Hitchin representation, with boundary map $\xi^* : \partial_\infty \Gamma \to \mathbb{P}(V^*)$. Then Γ acts convex cocompactly on

$$\Omega := \mathbb{P}(\mathbb{R}^n) \smallsetminus \bigcup_{\eta \in \partial_\infty \Gamma} \xi^*(\eta),$$

which is a properly convex open subset of $\mathbb{P}(\mathbb{R}^n)$ that is *not* strictly convex.

However, we show that for Γ acting convex cocompactly on Ω, the hyperbolicity of Γ is determined by the convexity behavior of $\partial\Omega$ *at the full orbital limit set*: here is an expanded version of Theorem 1.4.

Theorem 1.15. *Let Γ be an infinite discrete subgroup of* $\mathrm{PGL}(V)$*. Then the following are equivalent:*

(i) Γ *is strongly convex cocompact in* $\mathbb{P}(V)$ *(Definition 1.1);*
(ii) Γ *is convex cocompact in* $\mathbb{P}(V)$ *(Definition 1.11) and word hyperbolic;*
(iii) Γ *is convex cocompact in* $\mathbb{P}(V)$ *and for* any *properly convex open set* $\Omega \subset \mathbb{P}(V)$ *on which* Γ *acts convex cocompactly, the full orbital limit set* $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ *does not contain any nontrivial projective line segment;*
(iv) Γ *is convex cocompact in* $\mathbb{P}(V)$ *and for* some *nonempty properly convex open set* $\Omega \subset \mathbb{P}(V)$ *on which* Γ *acts convex cocompactly, the convex core* $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ *does not contain a PET;*
(v) Γ *preserves a properly convex open subset* Ω *of* $\mathbb{P}(V)$ *and acts cocompactly on some closed convex subset* $\mathcal{C}$ *of* Ω *with nonempty interior*

such that $\partial_{\mathrm{i}}\mathcal{C} := \overline{\mathcal{C}} \smallsetminus \mathcal{C} = \overline{\mathcal{C}} \cap \partial\Omega$ *does not contain any nontrivial projective line segment;*

(vi) Γ *is word hyperbolic, the inclusion* $\Gamma \hookrightarrow \mathrm{PGL}(V)$ *is* P_1*-Anosov, and* Γ *preserves some nonempty properly convex open subset of* $\mathbb{P}(V)$*.*

In this case, there is equality between the following four sets:

- *the orbital limit set* $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ *of* Γ *in any* Γ*-invariant properly convex open subset* Ω *of* $\mathbb{P}(V)$ *with strictly convex and* C^1 *boundary as in Definition 1.1 (condition (i));*
- *the full orbital limit set* $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ *for any properly convex open set* Ω *on which* Γ *acts convex cocompactly as in Definition 1.11 (conditions (ii), (iii), (iv));*
- *the segment-free set* $\partial_{\mathrm{i}}\mathcal{C}$ *for any convex subset* $\mathcal{C}$ *on which* Γ *acts cocompactly in a* Γ*-invariant properly convex open set* Ω *(condition (v));*
- *the image of the boundary map* $\xi : \partial_\infty\Gamma \to \mathbb{P}(V)$ *of the Anosov representation* $\Gamma \hookrightarrow \mathrm{PGL}(V)$ *(condition (vi)), which is also the proximal limit set* Λ_Γ *of* Γ *in* $\mathbb{P}(V)$ *(Definition 2.2).*

In condition (iv), we use the following terminology of Benoist [B6].

Definition 1.16. Let $\mathcal{C}$ be a properly convex subset of $\mathbb{P}(V)$. A *PET* (or *properly embedded triangle*) in $\mathcal{C}$ is a nondegenerate planar triangle whose interior is contained in $\mathcal{C}$, but whose edges and vertices are contained in $\partial_{\mathrm{i}}\mathcal{C} = \overline{\mathcal{C}} \smallsetminus \mathcal{C}$.

It is easy to check (see Lemma 6.1) that if Γ acts convex cocompactly on Ω, the existence of a PET in $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$ obstructs the hyperbolicity of Γ.

1.6. **Properties of convex cocompact groups in** $\mathbb{P}(V)$**.** We show that, even in the case of nonhyperbolic discrete groups, the notion of convex cocompactness in $\mathbb{P}(V)$ (Definition 1.11) still has some of the nice properties enjoyed by Anosov representations and convex cocompact subgroups of rank-one Lie groups. In particular, we prove the following.

Theorem 1.17. *Let* Γ *be an infinite discrete subgroup of* $G = \mathrm{PGL}(V)$*.*

(A) *The group* Γ *is convex cocompact in* $\mathbb{P}(V)$ *if and only if it is convex cocompact in* $\mathbb{P}(V^*)$ *(for the dual action).*

(B) *If* Γ *is convex cocompact in* $\mathbb{P}(V)$*, then it is finitely generated and quasi-isometrically embedded in* G*.*

(C) *If* Γ *is convex cocompact in* $\mathbb{P}(V)$*, then* Γ *does not contain any unipotent element.*

(D) *If* Γ *is convex cocompact in* $\mathbb{P}(V)$*, then there is a neighborhood* $\mathcal{U} \subset \mathrm{Hom}(\Gamma, G)$ *of the natural inclusion such that any* $\rho \in \mathcal{U}$ *is injective and discrete with image* $\rho(\Gamma)$ *convex cocompact in* $\mathbb{P}(V)$*.*

(E) *If the semisimplification of* Γ *(Definition 10.5) is convex cocompact in* $\mathbb{P}(V)$*, then so is* Γ*.*

(F) *Let* $V' = \mathbb{R}^{n'}$ *and let* $i : \mathrm{SL}^{\pm}(V) \hookrightarrow \mathrm{SL}^{\pm}(V \oplus V')$ *be the natural inclusion acting trivially on the second factor. If* Γ *is convex cocompact in*

$\mathbb{P}(V)$, then $i(\hat{\Gamma})$ is convex cocompact in $\mathbb{P}(V \oplus V')$, where $\hat{\Gamma}$ is the lift of Γ to $\mathrm{SL}^{\pm}(V)$ that preserves a properly convex cone of V lifting Ω (see Remark 3.1).

Remark 1.18. The equivalence (i) $\Leftrightarrow$ (ii) of Theorem 1.15 shows that Theorem 1.17 still holds if all the occurrences of "convex cocompact" are replaced by "strongly convex cocompact".

While some of the properties of Theorem 1.17 are proved directly from Definition 1.11, others will be most naturally established using alternative characterizations of convex cocompactness in $\mathbb{P}(V)$ (Theorem 1.20). We refer to Proposition 10.8 for a more general version of property (F).

Properties (F) and (D) give a source for many new examples of convex cocompact groups by starting with known examples in $\mathbb{P}(V)$, including them into $\mathbb{P}(V \oplus V')$, and then deforming. This generalizes the picture of Fuchsian groups in $\mathrm{PO}(2,1)$ being deformed into quasi-Fuchsian groups in $\mathrm{PO}(3,1)$. For example, a group that divides a properly convex open subset of $\mathbb{P}(\mathbb{R}^4)$ whose boundary is not strictly convex may always be deformed nontrivially in a larger projective space $\mathbb{P}(\mathbb{R}^n)$ by bending along torus or Klein bottle subgroups. This gives examples of nonhyperbolic irreducible discrete subgroups of $\mathrm{PGL}(\mathbb{R}^n)$ which are convex cocompact in $\mathbb{P}(\mathbb{R}^n)$ but do not divide any properly convex open subset of $\mathbb{P}(\mathbb{R}^n)$. These groups Γ are quasi-isometrically embedded in $\mathrm{PGL}(\mathbb{R}^n)$ and structurally stable (i.e. there is a neighborhood of the natural inclusion in $\mathrm{Hom}(\Gamma, \mathrm{PGL}(\mathbb{R}^n))$ which consists entirely of injective representations, for torsion-free Γ) without being word hyperbolic; compare with Sullivan [Su, Th. A]. We refer to Section 12.2 for more details.

1.7. **Holonomy groups of convex projective orbifolds.** We now give alternative characterizations of convex cocompact subgroups in $\mathbb{P}(V)$. These characterizations are motivated by a familiar picture in rank one: namely, if $\Omega = \mathbb{H}^p$ is the p-dimensional real hyperbolic space and Γ is a convex cocompact torsion-free subgroup of $\mathrm{PO}(p,1) = \mathrm{Isom}(\mathbb{H}^p)$, then any uniform neighborhood $\mathcal{C}_u$ of the convex core $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$ has strictly convex boundary and the quotient $\Gamma\backslash\mathcal{C}_u$ is a compact hyperbolic manifold with strictly convex boundary. Let us fix some terminology and notation in order to discuss the appropriate generalization of this picture to real projective geometry.

Definition 1.19. Let $\mathcal{C}$ be a nonempty convex subset of $\mathbb{P}(V)$ (not necessarily open nor closed).

- The *frontier* of $\mathcal{C}$ is $\mathrm{Fr}(\mathcal{C}) := \overline{\mathcal{C}} \smallsetminus \mathrm{Int}(\mathcal{C})$.
- A *supporting hyperplane* of $\mathcal{C}$ at a point $z \in \mathrm{Fr}(\mathcal{C})$ is a projective hyperplane H such that $z \in H \cap \overline{\mathcal{C}}$ and the lift of H to V bounds a closed halfspace containing a convex cone lifting $\mathcal{C}$.
- The *ideal boundary* of $\mathcal{C}$ is $\partial_{\mathrm{i}}\mathcal{C} := \overline{\mathcal{C}} \smallsetminus \mathcal{C}$.

- The *nonideal boundary* of $\mathcal{C}$ is $\partial_{\mathrm{n}}\mathcal{C} := \mathcal{C} \smallsetminus \mathrm{Int}(\mathcal{C}) = \mathrm{Fr}(\mathcal{C}) \smallsetminus \partial_{\mathrm{i}}\mathcal{C}$. Note that if $\mathcal{C}$ is open, then $\partial_{\mathrm{i}}\mathcal{C} = \mathrm{Fr}(\mathcal{C})$ and $\partial_{\mathrm{n}}\mathcal{C} = \emptyset$; in this case, it is common in the literature to denote $\partial_{\mathrm{i}}\mathcal{C}$ simply by $\partial\mathcal{C}$.
- The convex set $\mathcal{C}$ has *strictly convex nonideal boundary* if every point $z \in \partial_{\mathrm{n}}\mathcal{C}$ is an extreme point of $\overline{\mathcal{C}}$.
- The convex set $\mathcal{C}$ has C^1 *nonideal boundary* if it has a unique supporting hyperplane at each point $z \in \partial_{\mathrm{n}}\mathcal{C}$.
- The convex set $\mathcal{C}$ has *bisaturated boundary* if for any supporting hyperplane H of $\mathcal{C}$, the set $H \cap \overline{\mathcal{C}} \subset \mathrm{Fr}(\mathcal{C})$ is either fully contained in $\partial_{\mathrm{i}}\mathcal{C}$ or fully contained in $\partial_{\mathrm{n}}\mathcal{C}$.

Let $\mathcal{C}$ be a properly convex subset of $\mathbb{P}(V)$ on which the discrete subgroup Γ acts properly discontinuously and cocompactly. To simplify this intuitive discussion, assume Γ torsion-free, so that the quotient $M = \Gamma\backslash\mathcal{C}$ is a compact properly convex projective manifold, possibly with boundary. The group Γ is called the *holonomy group* of M. We show that if the boundary $\partial M = \Gamma\backslash\partial_{\mathrm{n}}\mathcal{C}$ of M is assumed to have some regularity, then Γ is convex cocompact in $\mathbb{P}(V)$ and, conversely, convex cocompact subgroups in $\mathbb{P}(V)$ are holonomy groups of properly convex projective manifolds (or more generally orbifolds, if Γ has torsion) whose boundaries are well-behaved.

Theorem 1.20. *Let Γ be an infinite discrete subgroup of $\mathrm{PGL}(V)$. The following are equivalent:*

(1) Γ is convex cocompact in $\mathbb{P}(V)$ (Definition 1.11): it acts convex cocompactly on a nonempty properly convex open subset Ω of $\mathbb{P}(V)$;
(2) Γ acts properly discontinuously and cocompactly on a nonempty properly convex set $\mathcal{C}_{\mathsf{bisat}} \subset \mathbb{P}(V)$ with bisaturated boundary;
(3) Γ acts properly discontinuously and cocompactly on a nonempty properly convex set $\mathcal{C}_{\mathsf{strict}} \subset \mathbb{P}(V)$ with strictly convex nonideal boundary;
(4) Γ acts properly discontinuously and cocompactly on a nonempty properly convex set $\mathcal{C}_{\mathsf{smooth}} \subset \mathbb{P}(V)$ with strictly convex C^1 nonideal boundary.

In this case $\mathcal{C}_{\mathsf{bisat}}$, $\mathcal{C}_{\mathsf{strict}}$, $\mathcal{C}_{\mathsf{smooth}}$ can be chosen equal, with Ω satisfying $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma) = \partial_{\mathrm{i}}\mathcal{C}_{\mathsf{smooth}}$.

Remark 1.21. Given a properly discontinuous and cocompact action of a group Γ on a properly convex set $\mathcal{C}$, Theorem 1.20 interprets the convex cocompactness of Γ in terms of the regularity of $\mathcal{C}$ at the nonideal boundary $\partial_{\mathrm{n}}\mathcal{C}$, whereas Theorems 1.15 and 1.20 together interpret the *strong* convex cocompactness of Γ in terms of the regularity of $\mathcal{C}$ at both $\partial_{\mathrm{i}}\mathcal{C}$ and $\partial_{\mathrm{n}}\mathcal{C}$.

The equivalence (1) $\Leftrightarrow$ (3) of Theorem 1.20 states that convex cocompact torsion-free subgroups in $\mathbb{P}(V)$ are precisely the holonomy groups of compact properly convex projective manifolds with strictly convex boundary. Cooper–Long–Tillmann [CLT2] studied the deformation theory of such manifolds (allowing in addition certain types of cusps). They established

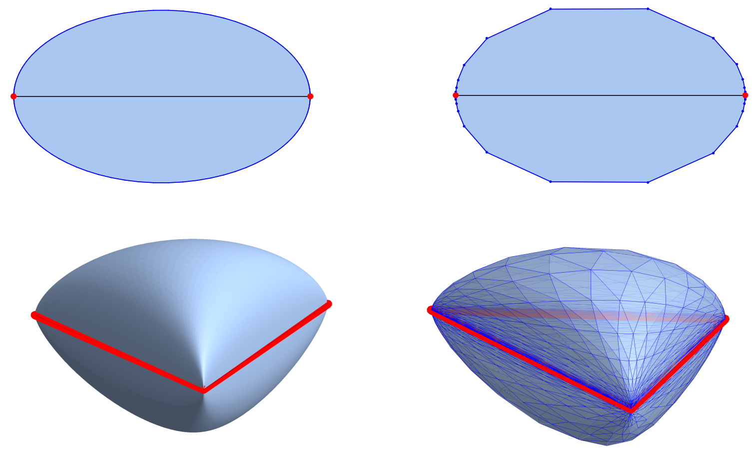

FIGURE 1. The group $\mathbb{Z}^{n-1}$ acts diagonally on $\mathbb{P}(\mathbb{R}^n) \subset \mathbb{P}(\mathbb{R}^{n+1})$, where $n = 2$ (top row) or 3 (bottom row). These actions are convex cocompact; each row shows two convex invariant domains with bisaturated boundary, the domain on the left having also strictly convex C^1 nonideal boundary. We show $\partial_{\mathrm{i}}\Omega$ in red and $\partial_{\mathrm{n}}\Omega$ in blue.

a holonomy principle, which reduces to the following statement in the absence of cusps: the holonomy groups of compact properly convex projective manifolds with strictly convex boundary form an open subset of the representation space of Γ. This result, together with Theorem 1.20, gives the stability property (D) of Theorem 1.17. For the case of divisible convex sets, see [K].

The requirement that $\mathcal{C}_{\mathsf{bisat}}$ have bisaturated boundary in condition (2) can be seen as a "coarse" version of the strict convexity of the nonideal boundary in (4): the prototype situation is that of a convex cocompact real hyperbolic manifold, with $\mathcal{C}_{\mathsf{bisat}}$ a closed polyhedral neighborhood of the convex core, see the right-hand side of Figure 1. Properly convex sets with bisaturated boundary behave well under a natural duality operation generalizing that of open properly convex sets (see Section 5). The equivalence (1) $\Leftrightarrow$ (2) will be used to prove the duality property (A) of Theorem 1.17. When the ideal boundary is closed, both conditions "strictly convex and C^1 nonideal boundary" and "bisaturated boundary" are invariant under duality, but it can be easier to build examples of the latter: condition (2) of Theorem 1.20 is convenient in some key arguments of the paper.

We note that without some form of convexity requirement on the boundary, properly convex projective manifolds have a poorly-behaved deformation theory: see Remark 3.10.(b).

Remark 1.22. If Γ is strongly irreducible (i.e. all finite-index subgroups of Γ are irreducible) and if the equivalent conditions of Theorem 1.20 hold, then

$\Lambda^{\mathsf{orb}}_\Omega(\Gamma) = \partial_{\mathrm{i}}\mathcal{C}_{\mathsf{bisat}} = \partial_{\mathrm{i}}\mathcal{C}_{\mathsf{strict}} = \partial_{\mathrm{i}}\mathcal{C}_{\mathsf{smooth}}$ for *any* $\Omega, \mathcal{C}_{\mathsf{bisat}}, \mathcal{C}_{\mathsf{strict}}, \mathcal{C}_{\mathsf{smooth}}$ as in conditions (1), (2), (3), and (4) respectively: see Section 4.3. In particular, this set then depends only on Γ. This is not necessarily the case if Γ is not strongly irreducible: see Examples 3.7 and 3.8.

1.8. **Convex cocompactness for subgroups of** $\mathrm{PO}(p,q)$**.** As mentioned in Remark 1.5.(b), when $\Gamma \subset \mathrm{PGL}(V)$ is irreducible and contained in the subgroup $\mathrm{PO}(p,q) \subset \mathrm{PGL}(V)$ of projective linear transformations that preserve a nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle_{p,q}$ of some signature (p,q) on $V = \mathbb{R}^n$, a more precise version of Theorem 1.4 was established in [DGK3, Th. 1.11 & Prop. 1.17]. Here we remove the irreducibility assumption on Γ.

For $p,q \in \mathbb{N}^*$, let $\mathbb{R}^{p,q}$ be $\mathbb{R}^{p+q}$ endowed with a symmetric bilinear form $\langle\cdot,\cdot\rangle_{p,q}$ of signature (p,q). The spaces

$$\begin{aligned} \mathbb{H}^{p,q-1} &= \{[x] \in \mathbb{P}(\mathbb{R}^{p,q}) \mid \langle x,x\rangle_{p,q} < 0\} \\ \text{and} \quad \mathbb{S}^{p-1,q} &= \{[x] \in \mathbb{P}(\mathbb{R}^{p,q}) \mid \langle x,x\rangle_{p,q} > 0\} \end{aligned}$$

are the projective models for pseudo-Riemannian hyperbolic and spherical geometry respectively. The geodesics of these two spaces are the intersections of the spaces with the projective lines of $\mathbb{P}(\mathbb{R}^{p,q})$. The group $\mathrm{PO}(p,q)$ acts transitively on $\mathbb{H}^{p,q-1}$ and on $\mathbb{S}^{p-1,q}$. Multiplying the form $\langle\cdot,\cdot\rangle_{p,q}$ by -1 produces a form of signature (q,p) and turns $\mathrm{PO}(p,q)$ into $\mathrm{PO}(q,p)$ and $\mathbb{S}^{p-1,q}$ into a copy of $\mathbb{H}^{q,p-1}$, so we consider only the pseudo-Riemannian hyperbolic spaces $\mathbb{H}^{p,q-1}$. We denote by $\partial\mathbb{H}^{p,q-1}$ the boundary of $\mathbb{H}^{p,q-1}$, namely

$$\partial\mathbb{H}^{p,q-1} = \{[x] \in \mathbb{P}(\mathbb{R}^{p,q}) \mid \langle x,x\rangle_{p,q} = 0\}\,.$$

We call a subset of $\mathbb{H}^{p,q-1}$ *convex* (resp. *properly convex*) if it is convex (resp. properly convex) as a subset of $\mathbb{P}(\mathbb{R}^{p,q})$. Since the straight lines of $\mathbb{P}(\mathbb{R}^{p,q})$ are the geodesics of the pseudo-Riemannian metric on $\mathbb{H}^{p,q-1}$, convexity in $\mathbb{H}^{p,q-1}$ is an intrinsic notion.

Remarks 1.23. Let $\mathcal{C}$ be a closed convex subset of $\mathbb{H}^{p,q-1}$.

(a) If $\mathcal{C}$ has nonempty interior, then $\mathcal{C}$ is properly convex. Indeed, if $\mathcal{C}$ is not properly convex, then it contains a line ℓ of an affine chart $\mathbb{R}^{p+q-1} \supset \mathcal{C}$, and $\mathcal{C}$ is a union of lines parallel to ℓ in that chart. All these lines must be tangent to $\partial\mathbb{H}^{p,q-1}$ at their common endpoint $z \in \partial\mathbb{H}^{p,q-1}$, which implies that $\mathcal{C}$ is contained in the hyperplane $z^\perp$ and has empty interior.
(b) The ideal boundary $\partial_{\mathrm{i}}\mathcal{C}$ is the set of accumulation points of $\mathcal{C}$ in $\partial\mathbb{H}^{p,q-1}$. This set contains no projective line segment if and only if it is *transverse*, i.e. $y \notin z^\perp$ for all $y \neq z$ in $\partial_{\mathrm{i}}\mathcal{C}$.
(c) The nonideal boundary $\partial_{\mathrm{n}}\mathcal{C} = \mathrm{Fr}(\mathcal{C}) \cap \mathbb{H}^{p,q-1}$ is the boundary of $\mathcal{C}$ in $\mathbb{H}^{p,q-1}$ in the usual sense.
(d) It is easy to see (Lemma 11.6) that $\mathcal{C}$ has bisaturated boundary if and only if a segment contained in $\partial_{\mathrm{n}}\mathcal{C} \subset \mathbb{H}^{p,q-1}$ never extends to a full geodesic of $\mathbb{H}^{p,q-1}$ — a form of coarse strict convexity for $\partial_{\mathrm{n}}\mathcal{C}$.

The following definition was studied in [DGK3] for irreducible discrete subgroups Γ.

Definition 1.24 ([DGK3, Def. 1.2])**.** A discrete subgroup Γ of $\mathrm{PO}(p,q)$ is $\mathbb{H}^{p,q-1}$-*convex cocompact* if it acts properly discontinuously with compact quotient on some closed properly convex subset $\mathcal{C}$ of $\mathbb{H}^{p,q-1}$ with nonempty interior whose ideal boundary $\partial_{\mathrm{i}}\mathcal{C} \subset \partial\mathbb{H}^{p,q-1}$ does not contain any nontrivial projective line segment.

If Γ is irreducible, then any nonempty Γ-invariant properly convex subset of $\mathbb{H}^{p,q-1}$ has nonempty interior, and so the nonempty interior requirement in Definition 1.24 may simply be replaced by the requirement that $\mathcal{C}$ be nonempty. See Section 11.9 for examples.

For a word hyperbolic group Γ, we shall say that a representation $\rho : \Gamma \to \mathrm{PO}(p,q)$ is $P_1^{p,q}$-*Anosov* if it is P_1-Anosov as a representation into $\mathrm{PGL}(\mathbb{R}^{p+q})$. We refer to [DGK3] for further discussion of this notion.

If the natural inclusion $\Gamma \hookrightarrow \mathrm{PO}(p,q)$ is P_1^{p+q}-Anosov, then the boundary map ξ takes values in $\partial\mathbb{H}^{p,q-1}$ and the hyperplane-valued boundary map ξ^* satisfies $\xi^*(\cdot) = \xi(\cdot)^\perp$ where $z^\perp$ denotes the orthogonal of z with respect to $\langle\cdot,\cdot\rangle_{p,q}$; the image of ξ is the proximal limit set Λ_Γ of Γ in $\partial\mathbb{H}^{p,q-1}$ (Definition 2.2 and Remark 11.1). Following [DGK3, Def. 1.9], we shall say that Λ_Γ is *negative* (resp. *positive*) if it lifts to a cone of $\mathbb{R}^{p,q} \smallsetminus \{0\}$ on which all inner products $\langle\cdot,\cdot\rangle_{p,q}$ of noncollinear points are negative; equivalently (see [DGK3, Lem. 3.2]), any three distinct points of Λ_Γ span a linear subspace of $\mathbb{R}^{p,q}$ of signature $(2,1)$ (resp. $(1,2)$). With this notation, we prove the following, where Γ is *not* assumed to be irreducible; for the irreducible case, see [DGK3, Th. 1.11 & Prop. 1.17].

Theorem 1.25. *For $p,q \in \mathbb{N}^*$ and for an arbitrary infinite discrete subgroup Γ of $\mathrm{PO}(p,q)$, the following are equivalent:*

(1) Γ is strongly convex cocompact in $\mathbb{P}(\mathbb{R}^{p+q})$ (Definition 1.1);
(2) Γ is $\mathbb{H}^{p,q-1}$-convex cocompact or $\mathbb{H}^{q,p-1}$-convex cocompact (after identifying $\mathrm{PO}(p,q)$ with $\mathrm{PO}(q,p)$ as above).

The following are also equivalent:

(3) Γ is strongly convex cocompact in $\mathbb{P}(\mathbb{R}^{p+q})$ and $\Lambda_\Gamma \subset \partial\mathbb{H}^{p,q-1}$ is negative;
(4) Γ is $\mathbb{H}^{p,q-1}$-convex cocompact;
(5) Γ acts convex cocompactly on some nonempty properly convex open subset Ω of $\mathbb{H}^{p,q-1}$ and Λ_Γ is transverse;
(6) Γ is word hyperbolic, the natural inclusion $\Gamma \hookrightarrow \mathrm{PO}(p,q)$ is $P_1^{p,q}$-Anosov, and $\Lambda_\Gamma \subset \partial\mathbb{H}^{p,q-1}$ is negative.

Similarly, the following are equivalent:

(7) Γ is strongly convex cocompact in $\mathbb{P}(\mathbb{R}^{p+q})$ and $\Lambda_\Gamma \subset \partial\mathbb{H}^{p,q-1}$ is positive;
(8) Γ is $\mathbb{H}^{q,p-1}$-convex cocompact (after identifying $\mathrm{PO}(p,q)$ with $\mathrm{PO}(q,p)$);

(9) *Γ acts convex cocompactly on some nonempty properly convex open subset Ω of $\mathbb{H}^{q,p-1}$ (after identifying $\mathrm{PO}(p,q)$ with $\mathrm{PO}(q,p)$) and Λ_Γ is transverse;*

(10) *Γ is word hyperbolic, the natural inclusion $\Gamma \hookrightarrow \mathrm{PO}(p,q)$ is $P_1^{p,q}$-Anosov, and $\Lambda_\Gamma \subset \partial\mathbb{H}^{p,q-1}$ is positive.*

If $\Gamma < \mathrm{PO}(p,q) < \mathrm{PGL}(\mathbb{R}^{p+q})$ is convex cocompact in $\mathbb{P}(\mathbb{R}^{p+q})$ (Definition 1.11) and irreducible, then it is in fact strongly convex cocompact in $\mathbb{P}(\mathbb{R}^{p+q})$.

It is not difficult to see that if $\mathcal{T}$ is a connected open subset of $\mathrm{Hom}(\Gamma, \mathrm{PO}(p,q))$ consisting entirely of $P_1^{p,q}$-Anosov representations, and if $\Lambda_{\rho(\Gamma)} \subset \partial\mathbb{H}^{p,q-1}$ is negative for some $\rho \in \mathcal{T}$, then it is negative for all $\rho \in \mathcal{T}$ [DGK3, Prop. 3.5]. Therefore Theorem 1.25 implies the following.

Corollary 1.26. *For $p, q \in \mathbb{N}^*$ and for a discrete group Γ, let $\mathcal{T}$ be a connected open subset of $\mathrm{Hom}(\Gamma, \mathrm{PO}(p,q))$ consisting entirely of $P_1^{p,q}$-Anosov representations. If $\rho(\Gamma)$ is $\mathbb{H}^{p,q-1}$-convex cocompact (resp. $\mathbb{H}^{q,p-1}$-convex cocompact) for some $\rho \in \mathcal{T}$, then $\rho(\Gamma)$ is $\mathbb{H}^{p,q-1}$-convex cocompact (resp. $\mathbb{H}^{q,p-1}$-convex cocompact) for all $\rho \in \mathcal{T}$.*

It is also not difficult to see that if a closed subset Λ of $\partial\mathbb{H}^{p,q-1}$ is transverse (i.e. $z^\perp \cap \Lambda = \{z\}$ for all $z \in \Lambda$) and connected, then it is negative or positive [DGK3, Prop. 1.10]. Therefore Theorem 1.25 implies the following.

Corollary 1.27. *Let Γ be a word hyperbolic group with* connected *boundary $\partial_\infty\Gamma$, and let $p, q \in \mathbb{N}^*$. For any $P_1^{p,q}$-Anosov representation $\rho : \Gamma \to \mathrm{PO}(p,q)$, the group $\rho(\Gamma)$ is $\mathbb{H}^{p,q-1}$-convex cocompact or $\mathbb{H}^{q,p-1}$-convex cocompact (after identifying $\mathrm{PO}(p,q)$ with $\mathrm{PO}(q,p)$).*

Remark 1.28. In the special case when $q = 2$ (i.e. $\mathbb{H}^{p,q-1} = \mathrm{AdS}^{p+1}$ is the Lorentzian *anti-de Sitter space*) and Γ is the fundamental group of a closed hyperbolic p-manifold, the equivalence (4) $\Leftrightarrow$ (6) of Theorem 1.25 follows from work of Mess [Me] for $p = 2$ and Barbot–Mérigot [BM] for $p \geq 3$.

As a consequence of Theorem 1.20, we obtain characterizations of $\mathbb{H}^{p,q-1}$-convex cocompactness where the assumption on the ideal boundary $\partial_{\mathrm{i}}\mathcal{C} \subset \partial\mathbb{H}^{p,q-1}$ in Definition 1.24 is replaced by various regularity conditions on the nonideal boundary $\partial_{\mathrm{n}}\mathcal{C} \subset \mathbb{H}^{p,q-1}$.

Theorem 1.29. *For any $p, q \in \mathbb{N}^*$ and any arbitrary infinite discrete subgroup Γ of $\mathrm{PO}(p,q)$, the following are equivalent:*

(1) *Γ is $\mathbb{H}^{p,q-1}$-convex cocompact: it acts properly discontinuously and cocompactly on a closed convex subset $\mathcal{C}$ of $\mathbb{H}^{p,q-1}$ with nonempty interior whose ideal boundary $\partial_{\mathrm{i}}\mathcal{C}$ does not contain any nontrivial projective line segment;*

(2) *Γ acts properly discontinuously and cocompactly on a nonempty closed convex subset $\mathcal{C}_{\mathsf{bisat}}$ of $\mathbb{H}^{p,q-1}$ whose boundary $\partial_{\mathrm{n}}\mathcal{C}_{\mathsf{bisat}}$ in $\mathbb{H}^{p,q-1}$ does not contain any infinite geodesic line of $\mathbb{H}^{p,q-1}$;*

(3) Γ *acts properly discontinuously and cocompactly on a nonempty closed convex subset* $\mathcal{C}_{\mathsf{strict}}$ *of* $\mathbb{H}^{p,q-1}$ *whose boundary* $\partial_{\mathrm{n}}\mathcal{C}_{\mathsf{strict}}$ *in* $\mathbb{H}^{p,q-1}$ *is strictly convex;*

(4) Γ *acts properly discontinuously and cocompactly on a nonempty closed convex set* $\mathcal{C}_{\mathsf{smooth}}$ *of* $\mathbb{H}^{p,q-1}$ *whose boundary* $\partial_{\mathrm{n}}\mathcal{C}_{\mathsf{smooth}}$ *in* $\mathbb{H}^{p,q-1}$ *is strictly convex and* C^1*.*

In this case, $\mathcal{C}$*,* $\mathcal{C}_{\mathsf{bisat}}$*,* $\mathcal{C}_{\mathsf{strict}}$*, and* $\mathcal{C}_{\mathsf{smooth}}$ *may be taken equal.*

1.9. **Organization of the paper.** Section 2 contains reminders about properly convex domains in projective space, the Cartan decomposition, and Anosov representations. In Section 3 we establish some facts about actions on convex subsets of $\mathbb{P}(V)$, and give a few basic (non-)examples for convex cocompactness. In Section 4 we prove the equivalence (1) $\Leftrightarrow$ (2) of Theorem 1.20. In Section 5 we study a notion of duality and establish Theorem 1.17.(A). Sections 6 to 9 are devoted to the proofs of the main Theorems 1.15 and 1.20, which contain Theorem 1.4. In Section 10 we establish properties (B)–(F) of Theorem 1.17. In Section 11 we prove Theorems 1.25 and 1.29 on $\mathbb{H}^{p,q-1}$-convex cocompactness. Section 12 is devoted to examples, in particular Proposition 1.7. Finally, in Appendix A we collect a few open questions on convex cocompact groups in $\mathbb{P}(V)$.

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2. Reminders

2.1. **Properly convex domains in projective space.** Let Ω be a properly convex open subset of $\mathbb{P}(V)$, with boundary $\partial\Omega$. Recall the *Hilbert metric* d_Ω on Ω:

$$d_\Omega(y,z) := \frac{1}{2}\log\,[a,y,z,b]$$

for all distinct $y,z\in\Omega$, where $[\cdot,\cdot,\cdot,\cdot]$ is the cross-ratio on $\mathbb{P}^1(\mathbb{R})$, normalized so that $[0,1,z,\infty]=z$, and where a,b are the intersection points of $\partial\Omega$ with the projective line through y and z, with a,y,z,b in this order. The metric space (Ω,d_Ω) is proper (closed balls are compact) and complete, and the group

$$\mathrm{Aut}(\Omega) := \{g\in\mathrm{PGL}(V)\mid g\cdot\Omega=\Omega\}$$

acts on Ω by isometries for d_Ω. As a consequence, any discrete subgroup of $\mathrm{Aut}(\Omega)$ acts properly discontinuously on Ω.

Let V^* be the dual vector space of V. By definition, the *dual convex set* of Ω is

$$\Omega^* := \mathbb{P}\big(\{\ell\in V^*\mid \ell(x)<0\quad\forall x\in\overline{\widetilde{\Omega}}\}\big),$$

where $\overline{\widetilde{\Omega}}$ is the closure in $V \smallsetminus \{0\}$ of an open convex cone of V lifting Ω. The set Ω^* is a properly convex open subset of $\mathbb{P}(V^*)$ which is preserved by the dual action of $\mathrm{Aut}(\Omega)$ on $\mathbb{P}(V^*)$.

Straight lines (contained in projective lines) are always geodesics for the Hilbert metric d_Ω. When Ω is not strictly convex, there may be other geodesics as well. However, a biinfinite geodesic of (Ω, d_Ω) always has well-defined, distinct endpoints in $\partial\Omega$, see [DGK3, Lem. 2.6].

2.2. **Cartan decomposition.** The group $\tilde{G} = \mathrm{GL}(V)$ admits the *Cartan decomposition* $\tilde{G} = \tilde{K}\exp(\tilde{\mathfrak{a}}^+)\tilde{K}$ where $\tilde{K} = \mathrm{O}(n)$ and

$$\tilde{\mathfrak{a}}^+ := \{\mathrm{diag}(t_1, \dots, t_n) \mid t_1 \geq \dots \geq t_n\}.$$

This means that any $g \in \tilde{G}$ may be written $g = k_1 \exp(a) k_2$ for some $k_1, k_2 \in \tilde{K}$ and a unique $a = \mathrm{diag}(t_1, \dots, t_n) \in \tilde{\mathfrak{a}}^+$; for any $1 \leq i \leq n$, the real number t_i is the logarithm $\mu_i(g)$ of the i-th largest *singular value* of g. This induces a Cartan decomposition $G = K\exp(\mathfrak{a}^+)K$ of $G = \mathrm{PGL}(V)$, with $K = \mathrm{PO}(n)$ and $\mathfrak{a}^+ = \tilde{\mathfrak{a}}^+/\mathbb{R}$, and for any $1 \leq i < j \leq n$ a map

$$\mu_i - \mu_j : G \longrightarrow \mathbb{R}^+. \tag{2.1}$$

If $\|\cdot\|_V$ denotes the operator norm associated with the standard Euclidean norm on $V = \mathbb{R}^n$ invariant under $\tilde{K} = \mathrm{O}(n)$, then for all $g \in G$ we have

$$(\mu_1 - \mu_n)(g) = \log\big(\|g\|_V \, \|g^{-1}\|_V\big). \tag{2.2}$$

2.3. **Proximality in projective space.** We shall use the following classical terminology.

Definition 2.1. An element $g \in \mathrm{PGL}(V)$ is *proximal in* $\mathbb{P}(V)$ (resp. $\mathbb{P}(V^*)$) if it admits a unique attracting fixed point in $\mathbb{P}(V)$ (resp. $\mathbb{P}(V^*)$). Equivalently, g has a unique complex eigenvalue of maximal (resp. minimal) modulus. This eigenvalue is necessarily real.

For any $g \in \mathrm{GL}(V)$, we denote by $\lambda_1(g) \geq \lambda_2(g) \geq \dots \geq \lambda_n(g)$ the logarithms of the moduli of the complex eigenvalues of g. For any $1 \leq i < j \leq n$, this induces a function

$$\lambda_i - \lambda_j : \mathrm{PGL}(V) \longrightarrow \mathbb{R}^+. \tag{2.3}$$

Thus, an element $g \in \mathrm{PGL}(V)$ is proximal in $\mathbb{P}(V)$ (resp. $\mathbb{P}(V^*)$) if and only if $(\lambda_1 - \lambda_2)(g) > 0$ (resp. $(\lambda_{n-1} - \lambda_n)(g) > 0$). We shall use the following terminology.

Definition 2.2. Let Γ be a discrete subgroup of $\mathrm{PGL}(V)$. The *proximal limit set* of Γ in $\mathbb{P}(V)$ is the closure Λ_Γ of the set of attracting fixed points of elements of Γ which are proximal in $\mathbb{P}(V)$.

Remark 2.3. When Γ is an *irreducible* discrete subgroup of $\mathrm{PGL}(V)$ containing at least one proximal element, the proximal limit set Λ_Γ was first studied in [Gu, B1, B2]. In that setting, the action of Γ on Λ_Γ is minimal (i.e. any orbit is dense), and Λ_Γ is contained in any nonempty, closed, Γ-invariant subset of $\mathbb{P}(V)$, by [B2, Lem. 2.5].

2.4. **Anosov representations.** Let P_1 (resp. P_{n-1}) be the stabilizer in $G = \mathrm{PGL}(V)$ of a line (resp. hyperplane) of V; it is a maximal proper parabolic subgroup of G, and G/P_1 (resp. G/P_{n-1}) identifies with $\mathbb{P}(V)$ (resp. with the dual projective space $\mathbb{P}(V^*)$). As in the introduction, we shall think of $\mathbb{P}(V^*)$ as the space of projective hyperplanes in $\mathbb{P}(V)$. The following is not the original definition from [L, GW3], but an equivalent characterization taken from [GGKW, Th. 1.7 & Rem. 4.3.(c)].

Definition 2.4. Let Γ be a word hyperbolic group. A representation $\rho : \Gamma \to G = \mathrm{PGL}(V)$ is P_1-*Anosov* if there exist two continuous, ρ-equivariant boundary maps $\xi : \partial_\infty\Gamma \to \mathbb{P}(V)$ and $\xi^* : \partial_\infty\Gamma \to \mathbb{P}(V^*)$ such that

- (A1) ξ and ξ^* are compatible, i.e. $\xi(\eta) \in \xi^*(\eta)$ for all $\eta \in \partial_\infty\Gamma$;
- (A2) ξ and ξ^* are transverse, i.e. $\xi(\eta) \notin \xi^*(\eta')$ for all $\eta \neq \eta'$ in $\partial_\infty\Gamma$;
- (A3)' ξ and ξ^* are dynamics-preserving and there exist $c, C > 0$ such that for any $\gamma \in \Gamma$,
$$(\lambda_1 - \lambda_2)(\rho(\gamma)) \geq c\,\ell_\Gamma(\gamma) - C,$$
where $\ell_\Gamma : \Gamma \to \mathbb{N}$ is the translation length function of Γ in its Cayley graph (for some fixed choice of finite generating subset).

In condition (A3)' we use the notation $\lambda_1 - \lambda_2$ from (2.3). By *dynamics-preserving* we mean that for any $\gamma \in \Gamma$ of infinite order, the element $\rho(\gamma) \in G$ is proximal and ξ (resp. ξ^*) sends the attracting fixed point of γ in $\partial_\infty\Gamma$ to the attracting fixed point of $\rho(\gamma)$ in $\mathbb{P}(V)$ (resp. $\mathbb{P}(V^*)$). In particular, the set $\xi(\partial_\infty\Gamma)$ (resp. $\xi^*(\partial_\infty\Gamma)$) is the proximal limit set (Definition 2.2) of $\rho(\Gamma)$ in $\mathbb{P}(V)$ (resp. $\mathbb{P}(V^*)$). By [GW3, Prop. 4.10], if $\rho(\Gamma)$ is irreducible then condition (A3)' is automatically satisfied as soon as (A1) and (A2) are, but this is not true in general (see [GGKW, Ex. 7.15]).

If Γ is not elementary (i.e. not cyclic up to finite index), then the action of Γ on $\partial_\infty\Gamma$ is minimal, i.e. every orbit is dense; therefore the action of Γ on $\xi(\partial_\infty\Gamma)$ and $\xi^*(\partial_\infty\Gamma)$ is also minimal.

We shall use a related characterization of Anosov representations, which we take from [GGKW, Th. 1.3]; it also follows from [KLPa].

Fact 2.5. *Let Γ be a word hyperbolic group. A representation $\rho : \Gamma \to G = \mathrm{PGL}(V)$ is* P_1-Anosov *if there exist two continuous, ρ-equivariant boundary maps $\xi : \partial_\infty\Gamma \to \mathbb{P}(V)$ and $\xi^* : \partial_\infty\Gamma \to \mathbb{P}(V^*)$ satisfying conditions (A1) – (A2) of Definition 2.4, as well as*

(A3)" ξ and ξ^* are dynamics-preserving and
$$(\mu_1 - \mu_2)(\rho(\gamma)) \underset{|\gamma|\to+\infty}{\longrightarrow} +\infty,$$
where $|\cdot|_\Gamma : \Gamma \to \mathbb{N}$ is the word length function of Γ (for some fixed choice of finite generating subset).

In condition (A3)" we use the notation $\mu_1 - \mu_2$ from (2.1).

By construction, the image of the boundary map $\xi : \partial_\infty\Gamma \to \mathbb{P}(V)$ (resp. $\xi^* : \partial_\infty\Gamma \to \mathbb{P}(V^*)$) of a P_1-Anosov representation $\rho : \Gamma \to \mathrm{PGL}(V)$ is the closure of the set of attracting fixed points of proximal elements of $\rho(\Gamma)$ in $\mathbb{P}(V)$ (resp. $\mathbb{P}(V^*)$. Here is a useful alternative description.

Fact 2.6 ([GGKW, Th. 1.3 & 5.3])**.** *Let Γ be a word hyperbolic group and $\rho : \Gamma \to \mathrm{PGL}(V)$ a P_1-Anosov representation with boundary maps $\xi : \partial_\infty\Gamma \to \mathbb{P}(V)$ and $\xi^* : \partial_\infty\Gamma \to \mathbb{P}(V^*)$. Let $(\gamma_m)_{m\in\mathbb{N}}$ be a sequence of elements of Γ converging to some $\eta \in \partial_\infty\Gamma$. For any m, choose $k_m \in K$ such that $\rho(\gamma_m) \in k_m \exp(\mathfrak{a}^+)K$ (see Section 2.2). Then, writing $[e_n^*] := \mathbb{P}(\mathrm{span}(e_1, \dots, e_{n-1}))$,*

$$\begin{cases} \xi(\eta) &= \lim_{m\to+\infty} k_m \cdot [e_1], \\ \xi^*(\eta) &= \lim_{m\to+\infty} k_m \cdot [e_n^*]. \end{cases}$$

In particular, the image of ξ is the set of accumulation points in $\mathbb{P}(V)$ of the set $\{k_{\rho(\gamma)} \cdot [e_1] \,|\, \gamma \in \Gamma\}$ where $\gamma \in k_{\rho(\gamma)} \exp(\mathfrak{a}^+)K$; the image of ξ^* is the set of accumulation points in $\mathbb{P}(V^*)$ of $\{k_{\rho(\gamma)} \cdot [e_n^*] \,|\, \gamma \in \Gamma\}$.

Here is an easy consequence of Fact 2.5.

Remark 2.7. Let Γ be a word hyperbolic group, Γ' a subgroup of Γ, and $\rho : \Gamma \to \mathrm{PGL}(V)$ a representation.

- If Γ' has finite index in Γ, then ρ is P_1-Anosov if and only if its restriction to Γ' is P_1-Anosov.
- If Γ' is quasi-isometrically embedded in Γ and if ρ is P_1-Anosov, then the restriction of ρ to Γ' is P_1-Anosov.

3. Basic facts: actions on convex subsets of $\mathbb{P}(V)$

In this section we collect a few useful facts, and give some basic examples and non-examples of convex cocompact actions.

3.1. Divergence for actions on properly convex cones.

We shall often use the following observation.

Remark 3.1. Let Γ be a discrete subgroup of $\mathrm{PGL}(V)$ preserving a nonempty properly convex open subset Ω of $\mathbb{P}(V)$. There is a unique lift $\hat{\Gamma}$ of Γ to $\mathrm{SL}^{\pm}(V)$ that preserves a properly convex cone $\widetilde{\Omega}$ of V lifting Ω.

Here we say that a convex open cone of V is *properly convex* if its projection to $\mathbb{P}(V)$ is properly convex in the sense of Section 1.1.

The following observation will be useful in Sections 7 and 10.

Lemma 3.2. *Let $\hat{\Gamma}$ be a discrete subgroup of $\mathrm{SL}^{\pm}(V)$ preserving a properly convex open cone $\widetilde{\Omega}$ in V. For any sequence $(\gamma_m)_{m\in\mathbb{N}}$ of pairwise distinct elements of $\hat{\Gamma}$ and any nonzero vector $x \in \widetilde{\Omega}$, the sequence $(\gamma_m \cdot x)_{m\in\mathbb{N}}$ goes to infinity as $m \to +\infty$. This divergence is uniform as x varies in a compact set $\mathcal{K} \subset \widetilde{\Omega}$.*

Proof. Fix a compact subset $\mathcal{K}$ of $\widetilde{\Omega}$. Let $\ell \in V^*$ be a linear form which takes positive values on the closure of $\widetilde{\Omega}$. The set $\widetilde{\Omega} \cap \{\ell = 1\}$ is bounded, with compact boundary $\mathcal{B}$ in V. By compactness of $\mathcal{K}$ and $\mathcal{B}$, we can find $0 < \varepsilon < 1$ such that for any $x \in \mathcal{K}$ and $y \in \mathcal{B}$ the line through $x/\ell(x)$ and y intersects $\mathcal{B}$ in a point $y' \neq y$ such that $x/\ell(x) = ty + (1-t)y'$ for some $t \geq \varepsilon$. For any $m \in \mathbb{N}$, we then have $\ell(\gamma_m \cdot x)/\ell(x) \geq \varepsilon\, \ell(\gamma_m \cdot y)$, and since this holds for any $y \in \mathcal{B}$ we obtain

$$\ell(\gamma_m \cdot x) \geq \kappa \max_{\mathcal{B}}(\ell \circ \gamma_m)$$

where $\kappa := \varepsilon \min_{\mathcal{K}}(\ell) > 0$. Thus it is sufficient to see that the maximum of $\ell \circ \gamma_m$ over $\mathcal{B}$ tends to infinity with m. By convexity, it is in fact sufficient to see that the maximum of $\ell \circ \gamma_m$ over $\widetilde{\Omega} \cap \{\ell < 1\}$ tends to infinity with m. This follows from the fact that the set $\widetilde{\Omega} \cap \{\ell < 1\}$ is open, that the operator norm of $\gamma_m \in \hat{\Gamma} \subset \mathrm{End}(V)$ goes to $+\infty$, and that ℓ is equivalent to any norm of V in restriction to $\widetilde{\Omega}$. ☐

The following immediate corollary will be used in Section 11.

Corollary 3.3. *For $p, q \in \mathbb{N}^*$, let Γ be an infinite discrete subgroup of $\mathrm{PO}(p,q)$ preserving a properly convex open subset Ω of $\mathbb{H}^{p,q-1}$. Then the full orbital limit set $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ is contained in $\partial\mathbb{H}^{p,q-1}$.*

3.2. **Comparison between the Hilbert and Euclidean metrics.** The following will be used later in this section and in the proofs of Lemmas 4.1 and 10.6.

Lemma 3.4. *Let $\Omega \subset \mathbb{P}(V)$ be open, properly convex, contained in a Euclidean affine chart $(\mathbb{R}^{n-1}, d_{\mathrm{Euc}})$ of $\mathbb{P}(V)$. If $R > 0$ is the diameter of Ω in $(\mathbb{R}^{n-1}, d_{\mathrm{Euc}})$, then the natural inclusion defines an $R/2$-Lipschitz map*

$$(\Omega, d_\Omega) \longrightarrow (\mathbb{R}^{n-1}, d_{\mathrm{Euc}}).$$

Proof. We may assume $n = 2$ up to restricting to one line of Ω, and $R = 2$ up to scaling. The arclength parametrization of $\Omega \simeq \mathbb{H}^1 \simeq \mathbb{R}$ is then given by the 1-Lipschitz map $\mathbb{R} \to (-1, 1)$ sending t to $\tanh(t)$. ☐

Corollary 3.5. *Let Ω be a properly convex open subset of $\mathbb{P}(V)$. Let $(y_m)_{m\in\mathbb{N}}$ and $(z_m)_{m\in\mathbb{N}}$ be two sequences of points of Ω, and $y \in \partial\Omega$. If $y_m \to y$ and if $d_\Omega(y_m, z_m) \to 0$, then $z_m \to y$.*

3.3. **Closed ideal boundary.** The following observation will be used in Sections 4, 5, and 9.

Lemma 3.6. *Let Γ be a discrete subgroup of $\mathrm{PGL}(V)$ and $\mathcal{C}$ a Γ-invariant convex subset of $\mathbb{P}(V)$. Suppose the action of Γ on $\mathcal{C}$ is properly discontinuous and cocompact. Then $\partial_{\mathrm{i}}\mathcal{C}$ is closed in $\mathbb{P}(V)$.*

Proof. Let $(z_m)_{m\in\mathbb{N}}$ be a sequence of points of $\partial_{\mathrm{i}}\mathcal{C}$ converging to some $z_\infty \in \mathrm{Fr}(\mathcal{C})$. Suppose for contradiction that $z_\infty \notin \partial_{\mathrm{i}}\mathcal{C}$, so that $z_\infty \in \partial_{\mathrm{n}}\mathcal{C}$. For each m, let $(z_{m,k})_{k\in\mathbb{N}}$ be a sequence of points of $\mathcal{C}$ converging to z_m as $k \to +\infty$.

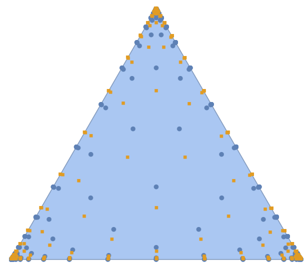


FIGURE 2. The subgroup $\Gamma \subset \mathrm{PGL}(\mathbb{R}^3)$ of all diagonal matrices with each entry a power of $t = 2$ divides a triangle Ω in $\mathbb{P}(\mathbb{R}^3)$ with vertices the standard basis. Shown are two Γ-orbits with different accumulation sets in $\partial\Omega$.

Let $\mathcal{D} \subset \mathcal{C}$ be a compact fundamental domain for the action of Γ. For any $m, k \in \mathbb{N}$, there exists $\gamma_{m,k} \in \Gamma$ such that $z_{m,k} \in \gamma_{m,k} \cdot \mathcal{D}$. Let $(z_{m,k_m})_{m\in\mathbb{N}}$ be a sequence converging to z_∞. Then the set $\{z_\infty\} \cup \{z_{m,k_m}\}_{m\in\mathbb{N}}$ is compact and intersects all $\gamma_{m,k_m} \cdot \mathcal{D}$, contradicting properness. $\square$

3.4. **Examples and non-examples.** The following basic examples are designed to make the notion of convex cocompactness in $\mathbb{P}(V)$ more concrete, and to point out some subtleties.

Example 3.7. For $V = \mathbb{R}^n$ with $n \geq 2$, let $\Gamma \simeq \mathbb{Z}^{n-1}$ be the discrete subgroup of $\mathrm{PGL}(V)$ of diagonal matrices whose entries are powers of some fixed $t > 1$; it is not word hyperbolic. The hyperplanes

$$H_k = \mathbb{P}(\mathrm{span}(e_1, \dots, e_{k-1}, e_{k+1}, \dots, e_n))$$

for $1 \leq k \leq n$ cut $\mathbb{P}(V)$ into 2^{n-1} properly convex open connected components Ω, which are not strictly convex (see Figure 2). The group Γ acts properly discontinuously and cocompactly on each of them, hence is convex cocompact in $\mathbb{P}(V)$ (see Example 1.13).

We note that in Example 3.7 the set of accumulation points of one Γ-orbit of Ω depends on the choice of orbit (see Figure 2), but the full orbital limit set $\Lambda^{\mathsf{orb}}_\Omega(\Gamma) = \partial\Omega$ does not; the proximal limit set Λ_Γ (Definition 2.2) is the set of extremal points of the full orbital limit set $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$.

Here is a noncommutative version of Example 3.7, containing it as a subgroup of finite index.

Example 3.8. Let $V = \mathbb{R}^n$ with $n \geq 2$. For any $1 \leq i \leq n-1$, let $s_i \in \mathrm{GL}(V)$ switch e_i and e_{i+1} and keep all other e_j fixed. Let $s_n \in \mathrm{GL}(V)$ send (e_1, e_n) to $(te_1, t^{-1}e_n)$ for some $t > 1$, and keep all other e_j fixed. The discrete subgroup of $\mathrm{PGL}(V)$ generated by $s_1, \dots, s_n$ (an affine Coxeter group of type A_n) is irreducible but not strongly irreducible. It still preserves and acts cocompactly on each of the 2^{n-1} properly convex connected components of Example 3.7, hence it is convex cocompact in $\mathbb{P}(V)$.

Examples 3.9. Let $V = \mathbb{R}^3$, with standard basis (e_1, e_2, e_3). Let Γ be a cyclic group generated by an element $\gamma \in \mathrm{PGL}(V)$.

(1) Suppose $\gamma = \left(\begin{smallmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{smallmatrix}\right)$ where $a > b > c > 0$. Then γ has attracting fixed point at $[e_1]$ and repelling fixed point at $[e_3]$. There is a Γ-invariant open properly convex neighborhood Ω of an open segment $([e_1], [e_3])$ connecting $[e_1]$ to $[e_3]$. The full orbital limit set $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ is just $\{[e_1], [e_3]\}$ and its convex hull $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma) = ([e_1], [e_3])$ has compact quotient by Γ (a circle). Thus Γ is convex cocompact in $\mathbb{P}(V)$.

(2) Suppose $\gamma = \left(\begin{smallmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{smallmatrix}\right)$ where $a > b > 0$. Any Γ-invariant properly convex open set Ω is a triangle with tip $[e_1]$ and base a segment I of the line $\mathbb{P}(\mathrm{span}(e_2, e_3))$. For any $z \in \Omega$, the Γ-orbit of z has two accumulation points, namely $[e_1]$ and the intersection of I with the line through $[e_1], z$. Therefore, the full orbital limit set $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ consists of the point $[e_1]$ and the interior of I, hence the convex hull $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$ of $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ is the whole of Ω, and Γ does not act cocompactly on it. Thus Γ is not convex cocompact in $\mathbb{P}(V)$. However, Γ is naively convex cocompact in $\mathbb{P}(V)$ (Definition 1.9): it acts cocompactly on the convex hull $\mathcal{C}$ in Ω of $\{[e_1]\}$ and of any closed segment I' contained in the interior of I. The quotient $\Gamma\backslash\mathcal{C}$ is a closed convex projective annulus (or a circle if I' is reduced to a singleton).

(3) Suppose $\gamma = \left(\begin{smallmatrix} 2 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{smallmatrix}\right)$ where $t > 0$. Then there exist nonempty Γ-invariant properly convex open subsets of $\mathbb{P}(V)$, for instance $\Omega_s = \mathbb{P}(\{(x_1, x_2, 1) \,|\, x_1 > s\, 2^{x_2/t}\})$ for any $s > 0$. However, for any such set Ω, we have $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma) = \{[e_1], [e_2]\}$, and the convex hull of $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ in $\overline{\Omega}$ is contained in $\partial\Omega$ (see [Ma, Prop. 2.13]). Hence $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$ is empty and Γ is not convex cocompact in $\mathbb{P}(V)$. It is an easy exercise to check that Γ is not even naively convex cocompact in $\mathbb{P}(V)$ (Definition 1.9).

(4) Suppose $\gamma = \left(\begin{smallmatrix} a & 0 & 0 \\ 0 & b\cos\theta & -b\sin\theta \\ 0 & b\sin\theta & b\cos\theta \end{smallmatrix}\right)$ where $a > b > 0$ and $0 < \theta \leq \pi$. Then Γ does not preserve any nonempty properly convex open subset of $\mathbb{P}(V)$ (see [Ma, Prop. 2.4]).

From these examples we observe the following.

Remarks 3.10. (a) Convex cocompactness in $\mathbb{P}(V)$ is not a closed condition in general. Indeed, Example 3.9.(2), which is not convex cocompact, is a limit of Example 3.9.(1), which is convex cocompact.

(b) Naive convex cocompactness (Definition 1.9) is not an open condition (even if we require the cocompact convex subset $\mathcal{C} \subset \Omega$ to have nonempty interior). Indeed, Example 3.9.(2), which satisfies the condition, is a limit of both 3.9.(3) and 3.9.(4), which do not.

Here is a slightly more complicated example of a discrete subgroup of $\mathrm{PGL}(V)$ which is naively convex cocompact but not convex cocompact in $\mathbb{P}(V)$.

Example 3.11. Let Γ_1 be a convex cocompact subgroup of $\mathrm{SO}(2,1) \subset \mathrm{GL}(\mathbb{R}^3)$, let $\Gamma_2 \simeq \mathbb{Z}$ be a discrete subgroup of $\mathrm{GL}(\mathbb{R}^1) \simeq \mathbb{R}^*$ acting on $\mathbb{R}^1$ by scaling, and let $\Gamma = \Gamma_1 \times \Gamma_2 \subset \mathrm{GL}(\mathbb{R}^3 \oplus \mathbb{R}^1)$. Any Γ-invariant properly convex open subset Ω of $\mathbb{P}(\mathbb{R}^3 \oplus \mathbb{R}^1)$ is a cone $\Omega_1 \star z$ with base some Γ_1-invariant properly convex open subset Ω_1 of $\mathbb{P}(\mathbb{R}^3) \subset \mathbb{P}(\mathbb{R}^3 \oplus \mathbb{R}^1)$ and tip $z := \mathbb{P}(\mathbb{R}^1) \subset \mathbb{P}(\mathbb{R}^3 \oplus \mathbb{R}^1)$. The full orbital limit set $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ contains both $\{z\}$ and the full base Ω_1, hence $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ is equal to Ω. Therefore Γ acts convex cocompactly on Ω if and only if Γ_1 acts cocompactly on Ω_1. If Γ_1 is not cocompact in $\mathrm{SO}(2,1)$, then Γ_1 does not act cocompactly on any nonempty properly convex open subset of $\mathbb{P}(\mathbb{R}^3)$, hence Γ is not convex cocompact in $\mathbb{P}(\mathbb{R}^3 \oplus \mathbb{R}^1)$. Nevertheless, in the case that $\Omega_1 = \mathbb{H}^2$ is the round disc, Γ acts cocompactly on the closed convex subcone $\mathcal{C} \subset \Omega$ with base the convex hull in $\mathbb{H}^2$ of the limit set of Γ_1.

In further work [DGK5], we shall describe an example of an *irreducible* discrete subgroup of $\mathrm{PGL}(\mathbb{R}^4)$ which is naively convex cocompact but not convex cocompact in $\mathbb{P}(V)$, namely a free group containing Example 3.9.(2) as a free factor.

4. Convex sets with bisaturated boundary

In this section we prove the equivalence (1) $\Leftrightarrow$ (2) of Theorem 1.20. In Sections 7 and 8, we will use these conditions interchangeably for Definition 1.11. We prove (Corollaries 4.2 and 4.4) that when these conditions hold, sets Ω as in condition (1) and $\mathcal{C}_{\mathsf{bisat}}$ as in condition (2) may be chosen so that the full orbital limit set $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ of Γ in Ω coincides with the ideal boundary $\partial_{\mathsf{i}}\mathcal{C}_{\mathsf{bisat}}$ of $\mathcal{C}_{\mathsf{bisat}}$. In Section 4.3, we show that if Γ is strongly irreducible and satisfies conditions (1) and (2), then $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ and $\partial_{\mathsf{i}}\mathcal{C}_{\mathsf{bisat}}$ are uniquely determined (Fact 4.7 and Proposition 4.8).

Recall that $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ denotes the convex core of Ω for Γ, i.e. the convex hull of $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ in Ω.

4.1. **Ideal boundaries of invariant convex sets.** Let us prove the implication (1) $\Rightarrow$ (2) of Theorem 1.20. It relies on the following lemma, which will be used many times in the paper.

Lemma 4.1. *Let Γ be a discrete subgroup of* $\mathrm{PGL}(V)$ *and Ω a nonempty Γ-invariant properly convex open subset of* $\mathbb{P}(V)$.

1. *If Γ acts cocompactly on some closed convex subset $\mathcal{C}$ of Ω, then for any $z \in \partial_{\mathsf{i}}\mathcal{C}$ the* open stratum *of $\partial\Omega$ at z is contained in $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$; in particular, $\partial_{\mathsf{i}}\mathcal{C} \subset \Lambda^{\mathsf{orb}}_\Omega(\Gamma)$. If moreover $\mathcal{C}$ contains $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$, then*
$$\partial_{\mathsf{i}}\mathcal{C} = \partial_{\mathsf{i}}\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma) = \Lambda^{\mathsf{orb}}_\Omega(\Gamma)$$
and the set $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ is closed in $\mathbb{P}(V)$, the set $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ is closed in Ω, and the action of Γ on Ω is convex cocompact (Definition 1.11).
2. *If the action of Γ on Ω is convex cocompact, then $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ is a minimal nonempty Γ-invariant closed properly convex subset of Ω.*

(3) *Let $\mathcal{C}_0 \subset \mathcal{C}_1$ be two nonempty closed properly convex subsets of Ω on which Γ acts cocompactly. If $\partial_{\mathrm{i}}\mathcal{C}_1 = \partial_{\mathrm{i}}\mathcal{C}_0$ and if $\mathcal{C}_1$ contains a neighborhood of $\mathcal{C}_0$, then $\mathcal{C}_1$ has bisaturated boundary.*

In (1), we call *open stratum of $\partial\Omega$ at y* the union F_y of $\{y\}$ and of all open segments of $\partial\Omega$ containing y. It is a properly convex subset of $\partial\Omega$ which is open in the projective subspace $\mathbb{P}(W_y)$ that it spans. In particular, we can consider the Hilbert metric d_{F_y} on F_y seen as a properly convex open subset of $\mathbb{P}(W_y)$. The set $\partial\Omega$ is the disjoint union of its open strata.

Proof. (1) Suppose that Γ acts cocompactly on a closed convex subset $\mathcal{C}$ of Ω. Let $z' \in \partial\Omega$ belong to the stratum of some $z \in \partial_{\mathrm{i}}\mathcal{C}$. Let us show that $z' \in \Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$. Let (a,b) be a maximal open interval of $\partial\Omega$ containing z and z'. Choose $y \in \mathcal{C}$. The rays (y,z) and (y,z') remain at bounded distance in (Ω, d_Ω). Therefore we can find points $z_m \in (y,z)$ and $z'_m \in (y,z')$ such that $z_m \to z$ and $z'_m \to z'$ and $d_\Omega(z_m, z'_m)$ remains bounded. Since Γ acts cocompactly on $\mathcal{C}$, there exists $(\gamma_m) \in \Gamma^{\mathbb{N}}$ such that $\gamma_m^{-1} \cdot z_m$ remains in some compact subset of $\mathcal{C} \subset \Omega$. Since $d_\Omega(\gamma_m^{-1} \cdot z_m, \gamma_m^{-1} \cdot z'_m) = d_\Omega(z_m, z'_m)$ remains bounded, $\gamma_m^{-1} \cdot z'_m$ also remains in some compact subset of Ω. Up to passing to a subsequence, we may assume that $\gamma_m^{-1} \cdot z'_m$ converges to some $y' \in \Omega$. Then $d_\Omega(\gamma_m \cdot y', z'_m) = d_\Omega(y', \gamma_m^{-1} \cdot z'_m) \to 0$, and so $\gamma_m \cdot y' \to z'$ by Corollary 3.5. Thus $z' \in \Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$. In particular, $\partial_{\mathrm{i}}\mathcal{C} \subset \Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$. If $\mathcal{C}$ contains $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$, then $\partial_{\mathrm{i}}\mathcal{C} \supset \partial_{\mathrm{i}}\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma) \supset \Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$, and so $\partial_{\mathrm{i}}\mathcal{C} = \partial_{\mathrm{i}}\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma) = \Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$. In particular, $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ is closed in $\mathbb{P}(V)$ by Lemma 3.6, and so its convex hull $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$ is closed in Ω, and compact modulo Γ because $\mathcal{C}$ is.

(2) Suppose the action of Γ on Ω is convex cocompact. Consider a nonempty closed convex Γ-invariant subset $\mathcal{C}$ of Ω contained in $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$; let us prove that $\mathcal{C} = \mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$. By definition of $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$, it is sufficient to prove that $\partial_{\mathrm{i}}\mathcal{C} = \Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$, or equivalently that $\partial_{\mathrm{i}}\mathcal{C} \cap F = \Lambda^{\mathsf{orb}}_{\Omega}(\Gamma) \cap F$ for all open strata F of $\partial\Omega$. We now fix such a stratum F. Since the action of Γ on $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$ is cocompact, there exists $R > 0$ such that any point of $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$ lies at distance $\leq R$ from a point of $\mathcal{C}$ for the Hilbert metric d_Ω. It is sufficient to prove that the set $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma) \cap F$ is contained in the uniform R-neighborhood of $\partial_{\mathrm{i}}\mathcal{C} \cap F$ for the Hilbert metric d_F on F; indeed, since $\partial_{\mathrm{i}}\mathcal{C} \cap F$ is a closed convex subset of F and since $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma) \cap F = F$ by (1), we then obtain $\partial_{\mathrm{i}}\mathcal{C} \cap F = \Lambda^{\mathsf{orb}}_{\Omega}(\Gamma) \cap F$. Consider a point $y \in \Lambda^{\mathsf{orb}}_{\Omega}(\Gamma) \cap F$. Let us find a point $z \in \partial_{\mathrm{i}}\mathcal{C} \cap F$ such that $d_F(y,z) \leq R$. Since $y \in \Lambda^{\mathsf{orb}}_{\Omega}(\Gamma) \subset \overline{\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)}$, we can find a sequence (y_m) of points of $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$ converging to y. For any m, there exists $z_m \in \mathcal{C}$ such that $d_\Omega(y_m, z_m) \leq R$; let $a_m, b_m \in \partial\Omega$ be such that a_m, y_m, z_m, b_m are aligned in this order. Up to passing to a subsequence, we may assume that a_m, y_m, z_m, b_m converge respectively to $a, y, z, b \in \partial\Omega$, with $z \in \partial_{\mathrm{i}}\mathcal{C}$. If $y = z$, then the proof is finished. Suppose $y \neq z$, so that a, y, z, b lie in $\overline{F}$. The cross ratios $[a_m, y_m, z_m, b_m] \leq e^{2R}$ converge to the cross ratio $[a, y, z, b]$, hence $[a, y, z, b] \leq e^{2R}$ and a, y, z, b are pairwise distinct and

$d_F(y,z) \leq d_{(a,b)}(y,z) \leq R$. Thus $\partial_{\mathrm{i}}\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma) \cap F$ is contained in the uniform R-neighborhood of $\partial_{\mathrm{i}}\mathcal{C} \cap F$, which completes the proof of (2).

(3) Suppose that $\partial_{\mathrm{i}}\mathcal{C}_1 = \partial_{\mathrm{i}}\mathcal{C}_0$ and that $\mathcal{C}_1$ contains a neighborhood of $\mathcal{C}_0$. By cocompactness, there exists $\varepsilon > 0$ such that any point of $\partial_{\mathrm{n}}\mathcal{C}_1$ is at d_Ω-distance $\geq \varepsilon$ from $\mathcal{C}_0$. Let H be a supporting hyperplane of $\mathcal{C}_1$ and suppose for contradiction that H contains both a point $w \in \partial_{\mathrm{n}}\mathcal{C}_1$ and a point $y \in \partial_{\mathrm{i}}\mathcal{C}_1$. Since $\partial_{\mathrm{i}}\mathcal{C}_1$ is closed in $\mathbb{P}(V)$ by Lemma 3.6, we may assume without loss of generality that the interval $[w,y)$ is contained in $\partial_{\mathrm{n}}\mathcal{C}_1$. Consider a sequence $z_m \in [w,y)$ converging to y. Since Γ acts cocompactly on $\mathcal{C}_1$, there is a sequence $(\gamma_m) \in \Gamma^{\mathbb{N}}$ such that $\gamma_m \cdot z_m$ remains in some compact subset of $\mathcal{C}_1$. Up to taking a subsequence, we may assume that $\gamma_m \cdot w \to w_\infty$ and $\gamma_m \cdot y \to y_\infty$ and $\gamma_m \cdot z_m \to z_\infty$ for some $w_\infty, y_\infty, z_\infty \in \overline{\mathcal{C}_1}$. We have $y_\infty \in \partial_{\mathrm{i}}\mathcal{C}_1$ since $\partial_{\mathrm{i}}\mathcal{C}_1$ is closed and $w_\infty \in \partial_{\mathrm{i}}\mathcal{C}_1$ since the action of Γ on $\mathcal{C}_1$ is properly discontinuous, and $z_\infty \in \partial_{\mathrm{n}}\mathcal{C}_1$ since $\gamma_m \cdot z_m$ remains in some compact subset of $\mathcal{C}_1$. The point $z_\infty \in \partial_{\mathrm{n}}\mathcal{C}_1$ belongs to the convex hull of $\{w_\infty, y_\infty\} \subset \partial_{\mathrm{i}}\mathcal{C}_0$, hence $z_\infty \in \overline{\mathcal{C}_0}$: contradiction since $z_\infty \in \partial_{\mathrm{n}}\mathcal{C}_1$ is at distance $\geq \varepsilon$ from $\overline{\mathcal{C}_0}$. ☐

The implication (1) ⇒ (2) of Theorem 1.20 is contained in the following consequence of Lemma 4.1.

Corollary 4.2. *Let Γ be an infinite discrete subgroup of* $\mathrm{PGL}(V)$ *acting convex cocompactly (Definition 1.11) on some nonempty properly convex open subset Ω of $\mathbb{P}(V)$. Let $\mathcal{C}$ be a closed uniform neighborhood of $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$ in (Ω, d_Ω). Then $\mathcal{C}$ is a properly convex subset of $\mathbb{P}(V)$ with bisaturated boundary on which Γ acts properly discontinuously and cocompactly. Moreover, $\partial_{\mathrm{i}}\mathcal{C} = \partial_{\mathrm{i}}\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma) = \Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$.*

Proof. The set $\mathcal{C}$ is properly convex by [Bu, (18.12)]. The group Γ acts properly discontinuously and cocompactly on $\mathcal{C}$ since it acts properly discontinuously on Ω and cocompactly on $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$. By Lemma 4.1.(1), we have $\partial_{\mathrm{i}}\mathcal{C} = \Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$. By Lemma 4.1.(3), the set $\mathcal{C}$ has bisaturated boundary. ☐

4.2. **Proper cocompact actions on convex sets with bisaturated boundary.** In this section we prove the implication (2) ⇒ (1) of Theorem 1.20. We first establish the following general observations.

Lemma 4.3. *Let $\mathcal{C}$ be a properly convex subset of $\mathbb{P}(V)$ with bisaturated boundary. Assume $\mathcal{C}$ is not closed in $\mathbb{P}(V)$. Then*

1. *$\mathcal{C}$ has nonempty interior* $\mathrm{Int}(\mathcal{C})$*;*
2. *the convex hull of $\partial_{\mathrm{i}}\mathcal{C}$ in $\mathcal{C}$ is contained in* $\mathrm{Int}(\mathcal{C})$ *whenever $\partial_{\mathrm{i}}\mathcal{C}$ is closed in $\mathbb{P}(V)$.*

Proof. (1) If the interior of $\mathcal{C}$ were empty, then $\mathcal{C}$ would be contained in a hyperplane. Since $\mathcal{C}$ has bisaturated boundary, $\mathcal{C}$ would be equal to its ideal boundary (hence empty) or to its nonideal boundary (hence closed).

(2) Since $\mathcal{C}$ has bisaturated boundary, every supporting hyperplane of $\mathcal{C}$ at a point $z \in \partial_{\mathrm{n}}\mathcal{C}$ misses $\partial_{\mathrm{i}}\mathcal{C}$, hence by the compactness assumption on $\partial_{\mathrm{i}}\mathcal{C}$ there is a hyperplane strictly separating z from $\partial_{\mathrm{i}}\mathcal{C}$ (in an affine chart

containing $\overline{\mathcal{C}}$). It follows that the convex hull of $\partial_{\mathrm{i}}\mathcal{C}$ in $\mathcal{C}$ does not meet $\partial_{\mathrm{n}}\mathcal{C}$, hence is contained in $\mathrm{Int}(\mathcal{C})$. □

The implication (2) ⇒ (1) of Theorem 1.20 is contained in the following consequence of Lemmas 4.1 and 4.3.

Corollary 4.4. *Let Γ be an infinite discrete subgroup of* $\mathrm{PGL}(V)$ *acting properly discontinuously and cocompactly on a nonempty properly convex subset $\mathcal{C}$ of $\mathbb{P}(V)$ with bisaturated boundary. Then $\Omega := \mathrm{Int}(\mathcal{C})$ is nonempty and Γ acts convex cocompactly (Definition 1.11) on Ω. Moreover, $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma) = \partial_{\mathrm{i}}\mathcal{C}$.*

Proof. Since Γ is infinite, $\mathcal{C}$ is not closed in $\mathbb{P}(V)$, and so $\Omega = \mathrm{Int}(\mathcal{C})$ is nonempty by Lemma 4.3.(1). The set $\partial_{\mathrm{i}}\mathcal{C}$ is closed in $\mathbb{P}(V)$ by Lemma 3.6, and so the convex hull $\mathcal{C}_0$ of $\partial_{\mathrm{i}}\mathcal{C}$ in $\mathcal{C}$ is closed in $\mathcal{C}$ and contained in Ω by Lemma 4.3.(2). The action of Γ on $\mathcal{C}_0$ is still cocompact. Since Γ acts properly discontinuously on $\mathcal{C}$, the set $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ is contained in $\partial_{\mathrm{i}}\mathcal{C}$, and $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$ is contained in $\mathcal{C}$. By Lemma 4.1.(1), the group Γ acts convex cocompactly on Ω and $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma) = \partial_{\mathrm{i}}\mathcal{C}_0 = \partial_{\mathrm{i}}\mathcal{C}$. □

4.3. **Maximal invariant convex sets.** The following was first observed by Benoist [B2, Prop. 3.1] for irreducible discrete subgroups of $\mathrm{PGL}(\mathbb{R}^n)$. Here we do not make any irreducibility assumption.

Proposition 4.5. *Let Γ be a discrete subgroup of* $\mathrm{PGL}(V)$ *preserving a nonempty properly convex open subset Ω of $\mathbb{P}(V)$ and containing a proximal element. Let Λ_Γ (resp. Λ^*_Γ) be the proximal limit set of Γ in $\mathbb{P}(V)$ (resp. $\mathbb{P}(V^*)$) (Definition 2.2). Then*

1. *Λ_Γ (resp. Λ^*_Γ) is contained in the boundary of Ω (resp. its dual Ω^*);*
2. *more specifically, Ω and Λ_Γ lift to cones $\widetilde{\Omega}$ and $\widetilde{\Lambda}_\Gamma$ of $V \smallsetminus \{0\}$ with $\widetilde{\Omega}$ properly convex containing $\widetilde{\Lambda}_\Gamma$ in its boundary, and Ω^* and Λ^*_Γ lift to cones $\widetilde{\Omega}^*$ and $\widetilde{\Lambda}^*_\Gamma$ of $V^* \smallsetminus \{0\}$ with $\widetilde{\Omega}^*$ properly convex containing $\widetilde{\Lambda}^*_\Gamma$ in its boundary, such that $\ell(x) \geq 0$ for all $x \in \widetilde{\Lambda}_\Gamma$ and $\ell \in \widetilde{\Lambda}^*_\Gamma$;*
3. *for $\widetilde{\Lambda}^*_\Gamma$ as in (2), the set*
$$\Omega_{\max} = \mathbb{P}(\{x \in V \mid \ell(x) > 0 \quad \forall \ell \in \widetilde{\Lambda}^*_\Gamma\})$$
is the unique connected component of $\mathbb{P}(V) \smallsetminus \bigcup_{z^* \in \Lambda^*_\Gamma} z^*$ containing Ω; it is Γ-invariant, convex, and open in $\mathbb{P}(V)$; any Γ-invariant properly convex open subset Ω' of $\mathbb{P}(V)$ containing Ω is contained in $\Omega_{\max}$.

Proof. (1) Let $\gamma \in \Gamma$ be proximal in $\mathbb{P}(V)$, with attracting fixed point z^+_γ and complementary γ-invariant hyperplane H^-_γ. Since Ω is open, there exists $y \in \Omega \smallsetminus H^-_\gamma$. We then have $\gamma^m \cdot y \to z^+_\gamma$, and so $z^+_\gamma \in \partial\Omega$ since the action of Γ on Ω is properly discontinuous. Thus $\Lambda_\Gamma \subset \partial\Omega$. Similarly, $\Lambda^*_\Gamma \subset \partial\Omega^*$.

(2) The set Ω lifts to a properly convex cone $\widetilde{\Omega}$ of $V \smallsetminus \{0\}$, unique up to global sign. This determines a cone $\widetilde{\Lambda}_\Gamma$ of $V \smallsetminus \{0\}$ lifting $\Lambda_\Gamma \subset \partial\Omega$ and contained in the boundary of $\widetilde{\Omega}$. By definition, Ω^* is the projection to $\mathbb{P}(V^*)$

of the dual cone
$$\widetilde{\Omega}^* := \big\{\ell \in V^* \smallsetminus \{0\} \mid \ell(x) > 0 \ \forall x \in \overline{\widetilde{\Omega}} \smallsetminus \{0\}\big\}.$$
This cone determines a cone $\widetilde{\Lambda}^*_\Gamma$ of $V^* \smallsetminus \{0\}$ lifting $\Lambda^*_\Gamma \subset \partial\Omega^*$ and contained in the boundary of $\widetilde{\Omega}^*$. By construction, $\ell(x) \geq 0$ for all $x \in \widetilde{\Lambda}_\Gamma$ and $\ell \in \widetilde{\Lambda}^*_\Gamma$.

(3) The set $\Omega_{\max} := \mathbb{P}(\{x \in V \mid \ell(x) > 0 \ \forall \ell \in \widetilde{\Lambda}^*_\Gamma\})$ is a connected component of $\mathbb{P}(V) \smallsetminus \bigcup_{z^* \in \Lambda^*_\Gamma} z^*$. It is convex and it contains Ω. It is Γ-invariant because Ω is. It is open in $\mathbb{P}(V)$ by compactness of Λ^*_Γ. Moreover, any Γ-invariant properly convex open subset Ω' of $\mathbb{P}(V)$ containing Ω is contained in $\Omega_{\max}$: indeed, Ω' cannot meet z^* for $z^* \in \Lambda^*_\Gamma$ since $\Lambda^*_\Gamma \subset \partial\Omega'^*$ by (1). ☐

Remark 4.6. In the context of Proposition 4.5, when Γ preserves a nonempty properly convex open subset Ω of $\mathbb{P}(V)$ but Γ is not irreducible, the following may happen:

(a) Γ may not contain any proximal element in $\mathbb{P}(V)$: this is the case if $\Gamma \subset \mathrm{PGL}(V' \oplus V')$ is the image of a diagonal embedding of a discrete group $\hat{\Gamma}' \subset \mathrm{SL}^{\pm}(V')$ preserving a properly convex open set in $\mathbb{P}(V')$;

(b) assuming that Γ contains a proximal element in $\mathbb{P}(V)$ (so $\Lambda_\Gamma, \Lambda^*_\Gamma \neq \emptyset$), the set $\Omega_{\max}$ of Proposition 4.5.(3) may fail to be properly convex: this is the case if Γ is a convex cocompact subgroup of $\mathrm{SO}(2,1)_0$, embedded into $\mathrm{PGL}(\mathbb{R}^4)$ where it preserves the properly convex open set $\mathbb{H}^3 \subset \mathbb{P}(\mathbb{R}^4)$;

(c) even if $\Omega_{\max}$ is properly convex, it may not be the unique maximal Γ-invariant properly convex open set in $\mathbb{P}(V)$: indeed, there may be multiple components of $\mathbb{P}(V) \smallsetminus \bigcup_{z^* \in \Lambda^*_\Gamma} z^*$ that are properly convex and Γ-invariant, as in Example 3.7.

However, if Γ is irreducible, then the following holds.

Fact 4.7 ([B2, Prop. 3.1])**.** *Let Γ be an irreducible discrete subgroup of $\mathrm{PGL}(V)$ preserving a nonempty properly convex open subset Ω of $\mathbb{P}(V)$. Then*

- *Γ always contains a proximal element and the set $\Omega_{\max}$ of Proposition 4.5.*(3) *is always properly convex (see Figure 3); it is a maximal Γ-invariant properly convex open subset of $\mathbb{P}(V)$ containing Ω;*
- *if moreover Γ is strongly irreducible (i.e. all finite-index subgroups of Γ are irreducible), then $\Omega_{\max}$ is the unique maximal Γ-invariant properly convex open set in $\mathbb{P}(V)$; it contains all other invariant properly convex open subsets;*
- *in general, there is a smallest nonempty Γ-invariant convex open subset $\Omega_{\min}$ of $\Omega_{\max}$, namely the interior of the convex hull of Λ_Γ in $\Omega_{\max}$.*

In the irreducible case, using Lemma 4.1, we can describe the full orbital limit set $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ as follows.

Proposition 4.8. *Let Γ be an infinite irreducible discrete subgroup of $\mathrm{PGL}(V)$ acting convex cocompactly on some nonempty properly convex open subset Ω*

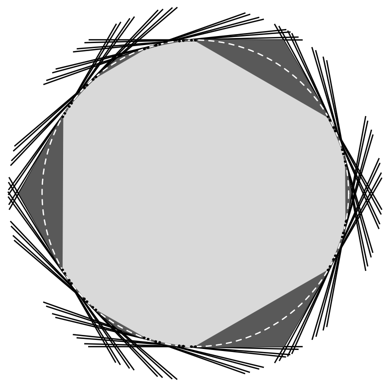


FIGURE 3. The sets $\Omega_{\min}$ (light gray) and $\Omega_{\max}$ (dark gray) for a convex cocompact subgroup of $\mathrm{PO}(2,1)$. Here $\mathbb{H}^2$ is the disc bounded by the dashed circle. Note: $\Gamma\backslash\mathcal{C}^{\mathsf{cor}}_{\Omega_{\max}}(\Gamma)$ is compact (equal to $\Gamma\backslash\mathcal{C}^{\mathsf{cor}}_{\mathbb{H}^2}(\Gamma)$) but $\Gamma\backslash\mathcal{C}^{\mathsf{cor}}_{\Omega_{\min}}(\Gamma)$ is not.

of $\mathbb{P}(V)$. Let $\Omega_{\min}\subset\Omega\subset\Omega_{\max}$ be nonempty properly convex open subsets of $\mathbb{P}(V)$ with $\Omega_{\min}$ minimal and $\Omega_{\max}$ maximal for inclusion. Then

$$\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)=\partial\Omega_{\min}\cap\partial\Omega_{\max},$$

and the convex hull $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$ of $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ in Ω is the convex hull of the proximal limit set Λ_Γ in $\Omega_{\max}$, namely

$$\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)=\overline{\Omega_{\min}}\cap\Omega_{\max}.$$

Proof. By Lemma 4.1.(1), the set $\mathcal{C}:=\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$ is closed in Ω. Since the action of Γ on $\Omega_{\max}$ is properly discontinuous (see Section 2.1), we have $\partial_{\mathrm{i}}\mathcal{C}=\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)\subset\partial\Omega_{\max}$, hence $\mathcal{C}$ is also closed in $\Omega_{\max}$. The interior $\mathrm{Int}(\mathcal{C})$ is nonempty since Γ is irreducible, and $\mathrm{Int}(\mathcal{C})$ contains $\Omega_{\min}$. In fact $\mathrm{Int}(\mathcal{C})=\Omega_{\min}$: indeed, if $\Omega_{\min}$ were strictly smaller than $\mathrm{Int}(\mathcal{C})$, then the closure of $\Omega_{\min}$ in $\mathcal{C}$ would be a nonempty strict closed subset of $\mathcal{C}$ on which Γ acts properly discontinuously and cocompactly, contradicting Lemma 4.1.(2). Thus $\mathcal{C}$ is the closure of $\Omega_{\min}$ in $\Omega_{\max}$ and $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)=\overline{\Omega_{\min}}\cap\partial\Omega_{\max}=\partial\Omega_{\min}\cap\partial\Omega_{\max}$. ☐

For an arbitrary irreducible discrete subgroup Γ of $\mathrm{PGL}(V)$ preserving a nonempty properly convex open subset Ω of $\mathbb{P}(V)$, the convex hull $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$ of the full orbital limit set $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ may be larger than the convex hull $\overline{\Omega_{\min}}\cap\Omega_{\max}$ of the proximal limit set Λ_Γ in Ω. This happens for instance if Γ is naively convex cocompact in $\mathbb{P}(V)$ (Definition 1.9) but not convex cocompact in $\mathbb{P}(V)$. Examples of such behavior will be given in the forthcoming paper [DGK5].

4.4. **The case of a connected proximal limit set.** We make the following observation.

Lemma 4.9. *Let Γ be an infinite discrete subgroup of* $\mathrm{PGL}(V)$ *preserving a nonempty properly convex open subset Ω of $\mathbb{P}(V)$. Suppose the proximal limit set Λ^*_Γ of Γ in $\mathbb{P}(V^*)$ (Definition 2.2) is* connected. *Then the set $\mathbb{P}(V) \smallsetminus \bigcup_{z^* \in \Lambda^*_\Gamma} z^*$ is a nonempty Γ-invariant convex open subset of $\mathbb{P}(V)$, not necessarily properly convex, but containing all Γ-invariant properly convex open subsets of $\mathbb{P}(V)$; it is equal to the set $\Omega_{\max}$ of Proposition 4.5.*

Proof. By Proposition 4.5.(3), it is enough to observe the following basic fact: if L^* is a nonempty, closed, connected subset of $\mathbb{P}(V^*)$, then $\mathbb{P}(V) \smallsetminus \bigcup_{z^* \in L^*} z^*$ has at most one connected component, necessarily convex. To see this, regard the points of $\mathbb{P}(V) = \mathbb{P}((V^*)^*)$ as projective hyperplanes in $\mathbb{P}(V^*)$. The points of $\mathbb{P}(V) \smallsetminus \bigcup_{z^* \in L^*} z^*$ are precisely the hyperplanes in $\mathbb{P}(V^*)$ that miss L^*. Suppose there is such a hyperplane; then its complement in $\mathbb{P}(V^*)$ is an affine chart $\mathbb{A}^*$ containing L^*. Let $\mathrm{CH}(L^*)$ be the convex hull of L^* in the affine chart $\mathbb{A}^*$; it is a closed properly convex set. Since L^* is connected and closed, $\mathrm{CH}(L^*)$ is well-defined independent of the affine chart $\mathbb{A}^*$: the convex hull of L^* in any affine chart containing L^* yields the same properly convex set $\mathrm{CH}(L^*)$. Further, a hyperplane in $\mathbb{P}(V^*)$ misses L^* if and only if it misses the convex hull $\mathrm{CH}(L^*)$, again because L^* is closed and connected. It follows that $\mathbb{P}(V) \smallsetminus \bigcup_{z^* \in L^*} z^*$ coincides with $\mathbb{P}(V) \smallsetminus \bigcup_{z^* \in \mathrm{CH}(L^*)} z^*$, which is a connected convex open set in $\mathbb{P}(V)$. ☐

4.5. **Conical limit points.** As a consequence of the proof of Lemma 4.1.(1), we observe that the dynamical behavior of orbits for general convex cocompact actions in $\mathbb{P}(V)$ partially extends the dynamical behavior for classical convex cocompact subgroups of $\mathrm{PO}(d,1)$. The contents of this subsection are not needed anywhere else in the paper.

Definition 4.10. Let Γ be an infinite discrete subgroup of $\mathrm{PGL}(V)$ and Ω a nonempty Γ-invariant properly convex open subset of $\mathbb{P}(V)$. For $y \in \Omega$ and $z \in \partial\Omega$, we say that z is a *conical limit point of the orbit* $\Gamma \cdot y$ if there exist a sequence $(\gamma_m) \in \Gamma^{\mathbb{N}}$ and a projective line L in Ω such that $\gamma_m \cdot y \to z$ and $d_\Omega(\gamma_m \cdot y, L)$ is bounded. We say that z is a *conical limit point of* Γ *in* $\partial\Omega$ if it is a conical limit point of the orbit $\Gamma \cdot y$ for some $y \in \Omega$.

Proposition 4.11. *Let Γ be an infinite discrete subgroup of* $\mathrm{PGL}(V)$ *and Ω a nonempty Γ-invariant properly convex open subset of $\mathbb{P}(V)$.*

(1) *If Γ acts cocompactly on some nonempty closed convex subset $\mathcal{C}$ of Ω, then the ideal boundary $\partial_{\mathrm{i}}\mathcal{C}$ consists entirely of conical limit points of Γ in $\partial\Omega$.*
(2) *In particular, if the action of Γ on Ω is convex cocompact (Definition 1.11), then the full orbital limit set $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ consists entirely of conical limit points of Γ in $\partial\Omega$.*

Proof. (1) We proceed similarly to the proof of Lemma 4.1.(1). Let $y \in \partial_{\mathrm{i}}\mathcal{C}$ and let $y_m \to y$ be a sequence in a ray $[y_0, y)$ in $\mathcal{C}$. By cocompactness, there is a sequence (γ_m) in Γ such that $y'_m := \gamma_m^{-1} \cdot y_m$ stays in a compact subset of $\mathcal{C} \subset \Omega$. Up to passing to a subsequence, we may assume $y'_m \to y' \in \mathcal{C}$. We have $d_\Omega(y_m, \gamma_m \cdot y'_m) \to 0$, and so $(\gamma_m \cdot y')_m$ remains within bounded distance (in fact is asymptotic to) $[y_0, y)$ and converges to y by Corollary 3.5.

(2) Apply (1) to $\mathcal{C} = \mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ and use Lemma 4.1.(1). □

When the boundary of Ω is strictly convex with C^1 boundary, the property that $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ consist entirely of conical limit points implies that the action of Γ on Ω is convex cocompact [CM, Cor. 8.6]. The converse is false in general when $\partial\Omega$ is not strictly convex: indeed, in Example 3.9.(2), the action of Γ on Ω is not convex cocompact, but every point of $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ is conical.

5. Duality for convex cocompact actions

In this section we study a notion of duality for properly convex sets which are not necessarily open. We prove Theorem 1.17.(A), and in fact a more precise version of it, namely Proposition 5.6.

5.1. The dual of a properly convex set.

Given an open properly convex subset $\Omega \subset \mathbb{P}(V)$, there is a notion of dual convex set $\Omega^* \subset \mathbb{P}(V^*)$ which is very useful in the study of divisible convex sets: see Section 2.1. We generalize this notion here to properly convex sets with possibly nonempty nonideal boundary.

Definition 5.1. Let $\mathcal{C}$ be a properly convex subset of $\mathbb{P}(V)$ with nonempty interior, but not necessarily open nor closed. The *dual* $\mathcal{C}^* \subset \mathbb{P}(V^*)$ of $\mathcal{C}$ is the set of projective hyperplanes in $\mathbb{P}(V)$ that do *not* meet $\mathrm{Int}(\mathcal{C}) \cup \partial_{\mathrm{i}}\mathcal{C}$, i.e. those hyperplanes that meet $\overline{\mathcal{C}}$ in a (possibly empty) subset of $\partial_{\mathrm{n}}\mathcal{C}$.

Remark 5.2. The interior of $\mathcal{C}^*$ is the dual of the interior of $\mathcal{C}$, in the usual sense for open convex subsets of $\mathbb{P}(V)$. In fact $\mathcal{C}^*$ is convex in $\mathbb{P}(V^*)$: if hyperplanes $H, H'', H' \in \overline{\mathcal{C}^*} \subset \mathbb{P}(V^*)$ are aligned in this order and if $H, H' \in \mathcal{C}^*$, then $H'' \cap \overline{\mathcal{C}} \subset (H \cap H') \cap \overline{\mathcal{C}} \subset \partial_{\mathrm{n}}\mathcal{C}$ so $H'' \in \mathcal{C}^*$. By construction,

- $\mathrm{Int}(\mathcal{C}^*)$ is the set of projective hyperplanes in $\mathbb{P}(V)$ that miss $\overline{\mathcal{C}}$;
- $\partial_{\mathrm{n}}\mathcal{C}^*$ is the set of projective hyperplanes in $\mathbb{P}(V)$ whose intersection with $\overline{\mathcal{C}}$ is a nonempty subset of $\partial_{\mathrm{n}}\mathcal{C}$;
- $\partial_{\mathrm{i}}\mathcal{C}^*$ is the set of supporting projective hyperplanes of $\mathcal{C}$ at points of $\partial_{\mathrm{i}}\mathcal{C}$ (such a hyperplane misses $\partial_{\mathrm{n}}\mathcal{C}$ if $\mathcal{C}$ has bisaturated boundary).

Lemma 5.3. *Let $\mathcal{C}$ be a properly convex subset of $\mathbb{P}(V)$, not necessarily open nor closed, but with bisaturated boundary.*

(1) The dual $\mathcal{C}^*$ has bisaturated boundary.
(2) The bidual $(\mathcal{C}^*)^*$ coincides with $\mathcal{C}$ (after identifying $(V^*)^*$ with V).
(3) The dual $\mathcal{C}^*$ has a PET (Definition 1.16) if and only if $\mathcal{C}$ does.

Proof. (1): By Remark 5.2, since $\mathcal{C}$ has bisaturated boundary, $\partial_{\mathrm{n}}\mathcal{C}^*$ (resp. $\partial_{\mathrm{i}}\mathcal{C}^*$) is the set of projective hyperplanes in $\mathbb{P}(V)$ whose intersection with $\overline{\mathcal{C}}$ is a nonempty subset of $\partial_{\mathrm{n}}\mathcal{C}$ (resp. $\partial_{\mathrm{i}}\mathcal{C}$). In particular, a point of $\partial_{\mathrm{n}}\mathcal{C}^*$ and a point of $\partial_{\mathrm{i}}\mathcal{C}^*$, seen as projective hyperplanes of $\mathbb{P}(V)$, can only meet outside of $\overline{\mathcal{C}}$. This means exactly that a supporting hyperplane of $\mathcal{C}^*$ in $\mathbb{P}(V^*)$ cannot meet both $\partial_{\mathrm{n}}\mathcal{C}^*$ and $\partial_{\mathrm{i}}\mathcal{C}^*$.

(2): By definition, $(\mathcal{C}^*)^*$ is the set of hyperplanes of $\mathbb{P}(V^*)$ missing $\mathrm{Int}(\mathcal{C}^*)\cup \partial_{\mathrm{i}}\mathcal{C}^*$. Viewing hyperplanes of $\mathbb{P}(V^*)$ as points of $\mathbb{P}(V)$, by Remark 5.2 the set $(\mathcal{C}^*)^*$ consists of those points of $\mathbb{P}(V)$ not belonging to any hyperplane that misses $\mathcal{C}$, or any supporting hyperplane at a point of $\partial_{\mathrm{i}}\mathcal{C}$. Since $\mathcal{C}$ has bisaturated boundary, this is $\overline{\mathcal{C}} \smallsetminus \partial_{\mathrm{i}}\mathcal{C}$, namely $\mathcal{C}$.

(3): By (1) and (2) it is enough to prove one implication. Suppose $\mathcal{C}$ has a PET contained in a two-dimensional projective plane P, i.e. $\mathcal{C}\cap P = T$ is an open triangle. Let H_1, H_2, H_3 be projective hyperplanes of $\mathbb{P}(V)$ supporting $\mathcal{C}$ and containing the edges $E_1, E_2, E_3 \subset \partial_{\mathrm{i}}\mathcal{C}$ of T. For $1 \leq k \leq 3$, the supporting hyperplane H_k intersects $\partial_{\mathrm{i}}\mathcal{C}$, hence lies in the ideal boundary $\partial_{\mathrm{i}}\mathcal{C}^*$ of the dual convex. Since $\mathcal{C}^*$ has bisaturated boundary by (1), the whole edge $[H_k, H_{k'}] \subset \overline{\mathcal{C}^*}$ is contained in $\partial_{\mathrm{i}}\mathcal{C}^*$. Hence H_1, H_2, H_3 span a 2-plane $Q \subset \mathbb{P}(V^*)$ whose intersection with $\mathcal{C}^*$ is a PET of $\mathcal{C}^*$. □

5.2. **Proper and cocompact actions on the dual.** The following is the key ingredient in Theorem 1.17.(A).

Proposition 5.4. *Let Γ be a discrete subgroup of $\mathrm{PGL}(V)$ and $\mathcal{C}$ a Γ-invariant convex subset of $\mathbb{P}(V)$ with bisaturated boundary. The action of Γ on $\mathcal{C}$ is properly discontinuous and cocompact if and only if the action of Γ on $\mathcal{C}^*$ is.*

In order to prove Proposition 5.4, we assume $n = \dim(V) \geq 2$ and first make some definitions. Consider a properly convex open subset Ω in $\mathbb{P}(V)$. For any distinct points $y, z \in \mathbb{P}(V)\smallsetminus\partial\Omega$, the line L through y and z intersects $\partial\Omega$ in two points a, b and we set

$$\delta_\Omega(y,z) := \max\big\{[a,y,z,b],[b,y,z,a]\big\}.$$

If $y, z \in \Omega$, then $\delta(y,z) = \exp(2d_\Omega(y,z)) > 1$ where d_Ω is the Hilbert metric on Ω (see Section 2.1). However, if $y \in \Omega$ and $z \in \mathbb{P}(V) \smallsetminus \overline{\Omega}$, then we have $-1 \leq \delta_\Omega(y,z) < 0$. For any point $y \in \Omega$ and any projective hyperplane $H \in \Omega^*$ (i.e. disjoint from $\overline{\Omega}$), we set

$$\delta_\Omega(y,H) := \max_{z\in H} \delta_\Omega(y,z).$$

Then $\delta(y,H) \in [-1,0)$ is close to 0 when H is "close" to $\partial\Omega$ as seen from y.

Lemma 5.5. *Let Ω be a nonempty properly convex open subset of $\mathbb{P}(V) = \mathbb{P}(\mathbb{R}^n)$. For any $H \in \Omega^*$, there exists $y \in \Omega$ such that*

$$\delta_\Omega(y,H) \leq \frac{-1}{n-1}.$$

Proof. Fix $H \in \Omega^*$ and consider an affine chart $\mathbb{R}^{n-1}$ of $\mathbb{P}(V)$ for which H is at infinity, endowed with a Euclidean norm $\|\cdot\|$. We take for y the center of mass of Ω in this affine chart with respect to the Lebesgue measure. It is enough to show that if $a, b \in \partial\Omega$ satisfy $y \in [a,b]$, then $\|y-a\|/\|b-a\| \leq (n-1)/n$. Up to translation, we may assume $a = 0 \in \mathbb{R}^{n-1}$. Let ℓ be a linear form on $\mathbb{R}^{n-1}$ such that $\ell(b) = 1 = \sup_\Omega \ell$. Let $h := \ell(y)$, so that $\|y-a\|/\|b-a\| = h$, and let $\Omega' := \mathbb{R}_{>0} \cdot (\Omega \cap \ell^{-1}(h)) \cap \{\ell < 1\}$ (see Figure 4). The average value of ℓ on Ω' for the Lebesgue measure is at least that on Ω,

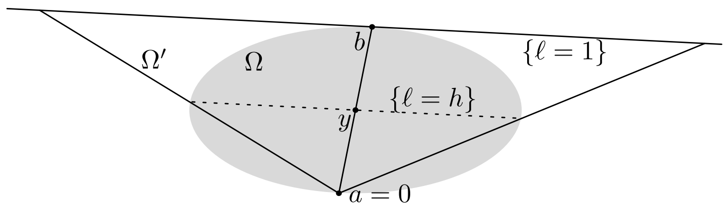


FIGURE 4. Illustration for the proof of Lemma 5.5

since by convexity

$$\begin{aligned} \Omega' \cap \ell^{-1}(-\infty, h] \quad &\subset \quad \Omega \cap \ell^{-1}(-\infty, h], \\ \Omega' \cap \ell^{-1}[h, 1) \quad &\supset \quad \Omega \cap \ell^{-1}[h, 1). \end{aligned}$$

The average value of ℓ on Ω is $\ell(y) = h$ since y is the center of mass of Ω. The average value of ℓ on Ω' is $(n-1)/n$ since Ω' is a truncated open cone in $\mathbb{R}^{n-1}$. ☐

Proof of Proposition 5.4. By Lemma 5.3, the set $\mathcal{C}^*$ has bisaturated boundary and $(\mathcal{C}^*)^* = \mathcal{C}$. Thus it is enough to prove that if the action on $\mathcal{C}$ is properly discontinuous and cocompact, then so is the action on $\mathcal{C}^*$.

Let us begin with properness. Recall from Lemma 3.6 that the set $\partial_{\mathrm{i}}\mathcal{C}$ is closed in $\mathbb{P}(V)$. Let $\mathcal{C}_0$ be the convex hull of $\partial_{\mathrm{i}}\mathcal{C}$ in $\mathcal{C}$. Note that any supporting hyperplane of $\mathcal{C}_0$ contains a point of $\overline{\partial_{\mathrm{i}}\mathcal{C}}$: otherwise it would be separated from $\partial_{\mathrm{i}}\mathcal{C}$ by a hyperplane, contradicting the definition of $\mathcal{C}_0$. By Lemma 4.3.(2), we have $\mathcal{C}_0 \subset \Omega := \mathrm{Int}(\mathcal{C})$. Let $\mathcal{C}_1$ be the closed uniform 1-neighborhood of $\mathcal{C}_0$ in (Ω, d_Ω). It is properly convex by [Bu, (18.12)], with nonempty interior, and $\partial_{\mathrm{n}}\mathcal{C}_1 \cap \partial_{\mathrm{n}}\mathcal{C} = \emptyset$. Taking the dual, we obtain that $\mathrm{Int}(\mathcal{C}_1)^*$ is a Γ-invariant properly convex open set containing $\mathcal{C}^*$. In particular, the action of Γ on $\mathcal{C}^* \subset \mathrm{Int}(\mathcal{C}_1)^*$ is properly discontinuous (see Section 2.1).

Let us show that the action of Γ on $\mathcal{C}^*$ is cocompact. Let $\Omega = \mathrm{Int}(\mathcal{C})$ and $\Omega^* = \mathrm{Int}(\mathcal{C}^*)$. Let $\mathcal{D} \subset \mathcal{C}$ be a compact fundamental domain for the action of Γ on $\mathcal{C}$. Consider the following subset of $\mathcal{C}^*$, where $n := \dim(V)$:

$$\mathcal{U}^* = \bigcup_{y \in \mathcal{D} \cap \Omega} \left\{ H \in \Omega^* \mid \delta_\Omega(y, H) \leq \tfrac{-1}{n-1} \right\}.$$

It follows from Lemma 5.5 that $\Gamma \cdot \mathcal{U}^* = \Omega^*$. We claim that $\overline{\mathcal{U}^*} \subset \mathcal{C}^*$. To see this, suppose a sequence of elements $H_m \in \mathcal{U}^*$ converges to some $H \in \overline{\mathcal{U}^*}$; let us show that $H \in \mathcal{C}^*$. If $H \in \Omega^*$ there is nothing to prove, so we may assume that $H \in \partial\Omega^* = \mathrm{Fr}(\mathcal{C}^*)$ is a supporting hyperplane of $\mathcal{C}$ at a point $z \in \mathrm{Fr}(\mathcal{C})$. For every m, let $y_m \in \mathcal{D} \cap \Omega$ satisfy $\delta_\Omega(y_m, H_m) \leq \frac{-1}{n-1}$. Up to passing to a subsequence, we may assume $y_m \to y \in \mathcal{D} \subset \mathcal{C}$. Let $z_m \in H_m$

such that $z_m \to z$. For every m, let $a_m, b_m \in \partial\Omega$ be such that a_m, y_m, b_m, z_m are aligned in this order. Then

$$\frac{-1}{n-1} \geq \delta_\Omega(y_m, z_m) \geq [b_m, y_m, z_m, a_m] \geq -\frac{\|b_m - z_m\|}{\|b_m - y_m\|},$$

where $\|\cdot\|$ is a fixed Euclidean norm on an affine chart containing $\overline{\Omega}$. Since $\|b_m - z_m\| \to 0$, we deduce $\|b_m - y_m\| \to 0$, and so $y = z \in \mathcal{C} \cap H \subset \partial_{\mathrm{n}}\mathcal{C}$. Since $\mathcal{C}$ has bisaturated boundary, we must have $H \in \mathcal{C}^*$, by definition of $\mathcal{C}^*$. Therefore $\overline{\mathcal{U}^*} \subset \mathcal{C}^*$. Since $\overline{\mathcal{U}^*}$ is compact and the action on $\mathcal{C}^*$ is properly discontinuous, the fact that $\Gamma \cdot \mathcal{U}^* = \Omega^*$ yields $\Gamma \cdot \overline{\mathcal{U}^*} = \mathcal{C}^*$. □

5.3. **Proof of Theorem 1.17.(A).** We prove in fact the following more precise result, which implies Theorem 1.17.(A).

Proposition 5.6. *An infinite discrete subgroup Γ of $\mathrm{PGL}(V)$ is convex cocompact in $\mathbb{P}(V)$ if and only if it is convex cocompact in $\mathbb{P}(V^*)$. In this case, there is a nonempty Γ-invariant properly convex open subset Ω of $\mathbb{P}(V)$ such that the actions of Γ on Ω and on its dual Ω^* are both convex cocompact, and such that $\overline{\Omega} \smallsetminus \Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ and $\overline{\Omega^*} \smallsetminus \Lambda^{\mathsf{orb}}_{\Omega^*}(\Gamma)$ both have bisaturated boundary.*

Proof. Suppose Γ is convex cocompact in $\mathbb{P}(V)$. By the implication (1) ⇒ (2) of Theorem 1.20 (proved in Section 4.1), the group Γ acts properly discontinuously and cocompactly on a nonempty properly convex set $\mathcal{C} \subset \mathbb{P}(V)$ with bisaturated boundary. By Lemma 5.3.(1), the dual $\mathcal{C}^* \subset \mathbb{P}(V^*)$ has bisaturated boundary, and Γ acts properly discontinuously and cocompactly on $\mathcal{C}^*$ by Proposition 5.4. By the implication (2) ⇒ (1) of Theorem 1.20 (proved in Section 4.2), the group Γ is convex cocompact in $\mathbb{P}(V^*)$.

Let Ω be the interior of $\mathcal{C}$. By construction (see Remark 5.2), the dual Ω^* of Ω is the interior of the dual $\mathcal{C}^*$ of $\mathcal{C}$. By Corollary 4.4, the actions of Γ on Ω and on its dual Ω^* are both convex cocompact, and we have $\overline{\Omega} \smallsetminus \Lambda^{\mathsf{orb}}_\Omega(\Gamma) = \mathcal{C}$ and $\overline{\Omega^*} \smallsetminus \Lambda^{\mathsf{orb}}_{\Omega^*}(\Gamma) = \mathcal{C}^*$; these convex sets have bisaturated boundary. □

Note that in the example of Figure 3, the action of Γ on $\Omega := \Omega_{\max}$ is convex cocompact while the action on its dual $\Omega^* = \Omega_{\min}$ is not.

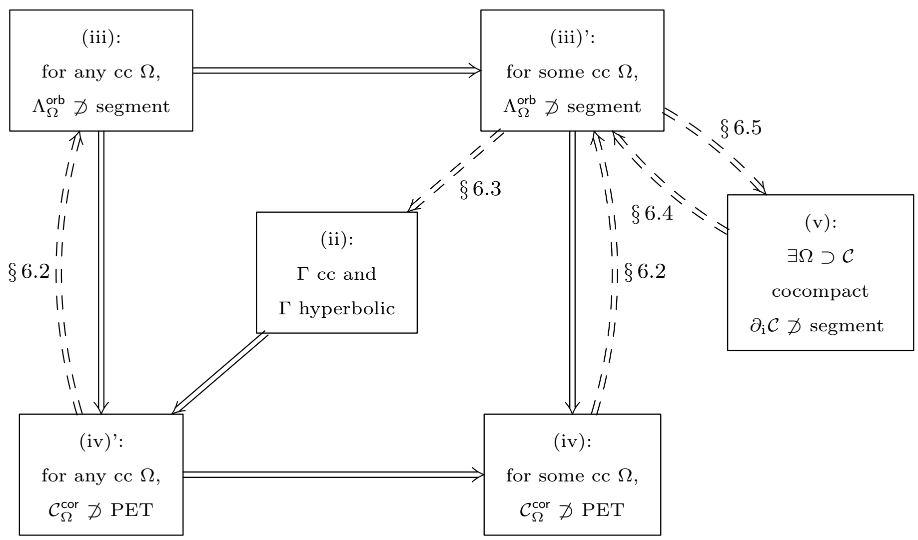
6. Segments in the full orbital limit set

In this section we establish the equivalences (ii) ⇔ (iii) ⇔ (iv) ⇔ (v) in Theorem 1.15.

For this it will be helpful to introduce two intermediate conditions, weaker than (iv) but stronger than (iii):

- (iii)' Γ is convex cocompact in $\mathbb{P}(V)$ and for *some* nonempty properly convex open set Ω on which Γ acts convex cocompactly, $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ does not contain a nontrivial projective line segment;
- (iv)' Γ is convex cocompact in $\mathbb{P}(V)$ and for *any* nonempty properly convex open set Ω on which Γ acts convex cocompactly, $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ does not contain a PET.

The implications (iii) ⇒ (iii)', (iv)' ⇒ (iv), (iii) ⇒ (iv)', and (iii)' ⇒ (iv) are trivial. The implication (ii) ⇒ (iv)' holds by Lemma 6.1 below. In Section 6.2 we simultaneously prove (iv)' ⇒ (iii) and (iv) ⇒ (iii)'. In Section 6.3 we prove (iii)' ⇒ (ii). In Section 6.4 we prove (v) ⇒ (iii)', and in Section 6.5 we prove (iii)' ⇒ (v). We include the following diagram of implications for the reader's convenience:



6.1. **PETs obstruct hyperbolicity.** We start with an elementary remark.

Lemma 6.1. *Let Γ be an infinite discrete subgroup of* $\mathrm{PGL}(V)$ *preserving a properly convex open subset Ω and acting cocompactly on a closed convex subset $\mathcal{C}$ of Ω. If $\mathcal{C}$ contains a PET (Definition 1.16), then Γ is not word hyperbolic.*

Proof. A PET in Ω is totally geodesic for the Hilbert metric d_Ω and is quasi-isometric to the Euclidean plane. Thus, if $\mathcal{C}$ contains a PET, then $(\mathcal{C}, d_\Omega)$ cannot be Gromov hyperbolic, hence Γ cannot be word hyperbolic by the Švarc–Milnor lemma. □

6.2. **Segments in the full orbital limit set yield PETs in the convex core.** The implications (iv)' ⇒ (iii) and (iv) ⇒ (iii)' in Theorem 1.15 will be a consequence of the following lemma, which is similar to [B6, Prop. 3.8.(a)] but without the divisibility assumption nor the restriction to dimension 3; our proof is different, close to [B3].

Lemma 6.2. *Let Γ be an infinite discrete subgroup of* $\mathrm{PGL}(V)$*. Let Ω be a nonempty Γ-invariant properly convex open subset of $\mathbb{P}(V)$. Suppose Γ acts cocompactly on some nonempty closed convex subset $\mathcal{C}$ of Ω. If $\partial_{\mathrm{i}}\mathcal{C}$ contains a nontrivial segment which is inextendable in $\partial\Omega$, then $\mathcal{C}$ contains a PET.*

Proof. The ideal boundary $\partial_{\mathrm{i}}\mathcal{C}$ is closed in $\mathbb{P}(V)$ by Lemma 3.6. Suppose $\partial_{\mathrm{i}}\mathcal{C}$ contains a nontrivial segment $[a, b]$ which is inextendable in $\partial\Omega$. Let

$c \in \mathcal{C}$ and consider a sequence of points $y_m \in \mathcal{C}$ lying inside the triangle with vertices a, b, c and converging to a point in $y \in (a, b)$.

We claim that the d_Ω-distance from y_m to either projective interval $(a, c]$ or $(b, c]$ tends to infinity with m. Indeed, consider a sequence $(z_m)_m$ of points of $(a, c]$ converging to $z \in [a, c]$ and let us check that $d_\Omega(y_m, z_m) \to +\infty$ (the proof for $(b, c]$ is the same). If $z \in (a, c]$, then $z \in \mathcal{C}$ and so $d_\Omega(y_m, z_m) \to +\infty$ by properness of the Hilbert metric. Otherwise $z = a$. In that case, for each m, consider $y'_m, z'_m \in \partial\Omega$ such that y'_m, y_m, z_m, z'_m are aligned in that order. Up to taking a subsequence, we may assume $y'_m \to y'$ and $z'_m \to z'$ for some $y', z' \in \partial\Omega$, with y', y, a, z' aligned in that order. By inextendability of $[a, b]$ in $\partial\Omega$, we must have $z = a = z'$, hence $d_\Omega(y_m, z_m) \to +\infty$ in this case as well, proving the claim.

Since the action of Γ on $\mathcal{C}$ is cocompact, there is a sequence $(\gamma_m) \in \Gamma^{\mathbb{N}}$ such that $\gamma_m \cdot y_m$ remains in a fixed compact subset of $\mathcal{C}$. Up to passing to a subsequence, we may assume that $(\gamma_m \cdot y_m)_m$ converges to some $y_\infty \in \mathcal{C}$, and $(\gamma_m \cdot a)_m$ and $(\gamma_m \cdot b)_m$ and $(\gamma_m \cdot c)_m$ converge respectively to some $a_\infty, b_\infty, c_\infty \in \partial_{\mathrm{i}}\mathcal{C}$, with $[a_\infty, b_\infty] \subset \partial_{\mathrm{i}}\mathcal{C}$ since $\partial_{\mathrm{i}}\mathcal{C}$ is closed (Lemma 4.1.(1)). The triangle with vertices $a_\infty, b_\infty, c_\infty$ is nondegenerate since it contains $y_\infty \in \mathcal{C}$. Further, y_∞ is infinitely far (for the Hilbert metric on Ω) from the edges $[a_\infty, c_\infty]$ and $[b_\infty, c_\infty]$, and so these edges are fully contained in $\partial_{\mathrm{i}}\mathcal{C}$. Thus the triangle with vertices $a_\infty, b_\infty, c_\infty$ is a PET of $\mathcal{C}$. ☐

Simultaneous proof of (iv)' ⇒ (iii) and (iv) ⇒ (iii)'. We prove the contrapositive. Suppose $\Gamma \subset \mathrm{PGL}(V)$ acts convex cocompactly on the properly convex open set $\Omega \subset \mathbb{P}(V)$ and $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ contains a nontrivial segment. By Lemma 4.1.(1), the set $\partial_{\mathrm{i}}\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ is equal to $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ and closed in $\mathbb{P}(V)$, and it contains the open stratum of $\partial\Omega$ at any interior point of that segment; in particular, $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ contains a nontrivial segment which is inextendable in $\partial\Omega$. By Lemma 6.2 with $\mathcal{C} = \mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$, the set $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ contains a PET. ☐

6.3. **Word hyperbolicity of the group in the absence of segments.** In this section we prove the implication (iii)' ⇒ (ii) in Theorem 1.15. We proceed exactly as in [DGK3, § 4.3], with arguments inspired from [B3]. It is sufficient to apply the following general result to $\mathcal{C} = \mathcal{C}^{\mathsf{cor}}_\Omega$. Recall that any geodesic ray of $(\mathcal{C}, d)$ has a well-defined endpoint in $\partial_{\mathrm{i}}\mathcal{C}$ (see [FK, Th. 3] or [DGK3, Lem. 2.6.(1)]).

Lemma 6.3. *Let Γ be a discrete subgroup of $\mathrm{PGL}(V)$. Let Ω be a nonempty Γ-invariant properly convex open subset of $\mathbb{P}(V)$. Suppose Γ acts cocompactly on some nonempty closed convex subset $\mathcal{C}$ of Ω such that $\partial_{\mathrm{i}}\mathcal{C}$ does not contain any nontrivial projective line segment. Then*

(1) there exists $R > 0$ such that any geodesic ray of $(\mathcal{C}, d_\Omega)$ lies at Hausdorff distance $\leq R$ from the projective interval with the same endpoints;

(2) the metric space $(\mathcal{C}, d_\Omega)$ is Gromov hyperbolic with Gromov boundary Γ-equivariantly homeomorphic to $\partial_{\mathrm{i}}\mathcal{C}$;

(3) *the group* Γ *is word hyperbolic and any orbit map* $\Gamma \to (\mathcal{C}, d_\Omega)$ *is a quasi-isometric embedding which extends to a* Γ*-equivariant homeomorphism* $\xi : \partial_\infty \Gamma \to \partial_{\mathrm{i}} \mathcal{C}$*.*

(4) *if* $\mathcal{C}$ *contains* $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ *(hence* Γ *acts convex cocompactly on* Ω*), then any orbit map* $\Gamma \to (\Omega, d_\Omega)$ *is a quasi-isometric embedding and extends to a* Γ*-equivariant homeomorphism* $\xi : \partial_\infty \Gamma \to \Lambda^{\mathsf{orb}}_\Omega(\Gamma) = \partial_{\mathrm{i}} \mathcal{C}$ *which is independent of the orbit.*

As usual, we denote by d_Ω the Hilbert metric on Ω.

Proof. (1) Suppose by contradiction that for any $m \in \mathbb{N}$ there is a geodesic ray $\mathcal{G}_m$ of $(\mathcal{C}, d_\Omega)$ with endpoints $a_m \in \mathcal{C}$ and $b_m \in \partial_{\mathrm{i}} \mathcal{C}$ and a point $y_m \in \mathcal{C}$ on that geodesic which lies at distance $\geq m$ from the projective interval $[a_m, b_m)$. By cocompactness of the action of Γ on $\mathcal{C}$, for any $m \in \mathbb{N}$ there exists $\gamma_m \in \Gamma$ such that $\gamma_m \cdot y_m$ belongs to a fixed compact set of $\mathcal{C}$. Up to taking a subsequence, $(\gamma_m \cdot y_m)_m$ converges to some $y_\infty \in \mathcal{C}$, and $(\gamma_m \cdot a_m)_m$ and $(\gamma_m \cdot b_m)_m$ converge respectively to some $a_\infty \in \overline{\mathcal{C}}$ and $b_\infty \in \partial_{\mathrm{i}} \mathcal{C}$. Since the distance from y_m to $[a_m, b_m)$ goes to infinity, we have $[a_\infty, b_\infty] \subset \partial_{\mathrm{i}} \mathcal{C}$, hence $a_\infty = b_\infty$ since $\partial_{\mathrm{i}} \mathcal{C}$ does not contain a segment. Therefore, up to extracting, the geodesics $\mathcal{G}_m$ converge to a biinfinite geodesic of (Ω, d_Ω) with both endpoints equal. But such a geodesic does not exist (see [FK, Th. 3] or [DGK3, Lem. 2.6.(2)]): contradiction.

(2) Suppose by contradiction that triangles of $(\mathcal{C}, d_\Omega)$ are not uniformly thin. By (1), triangles of $(\mathcal{C}, d_\Omega)$ whose sides are projective line segments are not uniformly thin: namely, there exist $a_m, b_m, c_m \in \mathcal{C}$ and $y_m \in [a_m, b_m]$ such that

$$d_\Omega(y_m, [a_m, c_m] \cup [c_m, b_m]) \underset{m \to +\infty}{\longrightarrow} +\infty. \tag{6.1}$$

By cocompactness, for any m there exists $\gamma_m \in \Gamma$ such that $\gamma_m \cdot y_m$ belongs to a fixed compact set of $\mathcal{C}$. Up to taking a subsequence, $(\gamma_m \cdot y_m)_m$ converges to some $y_\infty \in \mathcal{C}$, and $(\gamma_m \cdot a_m)_m$ and $(\gamma_m \cdot b_m)_m$ and $(\gamma_m \cdot c_m)_m$ converge respectively to some $a_\infty, b_\infty, c_\infty \in \overline{\mathcal{C}}$. By (6.1) we have $[a_\infty, c_\infty] \cup [c_\infty, b_\infty] \subset \partial_{\mathrm{i}} \mathcal{C}$, hence $a_\infty = b_\infty = c_\infty$ since $\partial_{\mathrm{i}} \mathcal{C}$ does not contain any nontrivial projective line segment. This contradicts the fact that $y_\infty \in (a_\infty, b_\infty)$. Therefore $(\mathcal{C}, d_\Omega)$ is Gromov hyperbolic.

Fix a basepoint $y \in \mathcal{C}$. The Gromov boundary of $(\mathcal{C}, d_\Omega)$ is the set of equivalence classes of infinite geodesic rays in $\mathcal{C}$ starting at y, for the equivalence relation "to remain at bounded distance for d_Ω". Consider the Γ-equivariant continuous map φ from $\partial_{\mathrm{i}} \mathcal{C}$ to this Gromov boundary sending $z \in \partial_{\mathrm{i}} \mathcal{C}$ to the class of the geodesic ray from y to z. This map is clearly surjective, since any infinite geodesic ray in $\mathcal{C}$ terminates at the ideal boundary $\partial_{\mathrm{i}} \mathcal{C}$. Moreover, it is injective, since the non-existence of line segments in $\partial_{\mathrm{i}} \mathcal{C}$ means that no two points of $\partial_{\mathrm{i}} \mathcal{C}$ lie in a common face of $\partial \Omega$, hence the Hilbert distance between rays going out to two different points of $\partial_{\mathrm{i}} \mathcal{C}$ goes to infinity.

(3) The group Γ acts properly discontinuously and cocompactly, by isometries, on the metric space $(\mathcal{C}, d_\Omega)$, which is Gromov hyperbolic with Gromov boundary $\partial_{\mathrm{i}}\mathcal{C}$ by (2) above. We apply the Švarc–Milnor lemma.

(4) Suppose $\mathcal{C}$ contains $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$. Consider a point $y \in \Omega$. It lies in a uniform neighborhood $\mathcal{C}_y$ of $\mathcal{C}$ in (Ω, d_Ω), on which the action of Γ is also cocompact. By Lemma 4.1.(1)–(2), we have $\partial_{\mathrm{i}}\mathcal{C}_y = \partial_{\mathrm{i}}\mathcal{C} = \Lambda^{\mathsf{orb}}_\Omega(\Gamma)$, and this set contains no nontrivial segment by assumption. By (3), the orbit map $\Gamma \to (\mathcal{C}_y, d_\Omega)$ associated with y is a quasi-isometry which extends to a Γ-equivariant homeomorphism $\partial_\infty\Gamma \to \partial_{\mathrm{i}}\mathcal{C}_y = \partial_{\mathrm{i}}\mathcal{C}$, and this extension is clearly independent of y. □

6.4. **Proof of (v) ⇒ (iii)' in Theorem 1.15.** It is sufficient to prove the following lemma: then Γ is convex cocompact in $\mathbb{P}(V)$ and $\partial_{\mathrm{i}}\mathcal{C} = \Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ by Lemma 4.1.(1).

Lemma 6.4. *Let Γ be an infinite discrete subgroup of $\mathrm{PGL}(V)$ preserving a properly convex open subset Ω of $\mathbb{P}(V)$ and acting cocompactly on some closed convex subset $\mathcal{C}$ of Ω with nonempty interior. If $\partial_{\mathrm{i}}\mathcal{C}$ does not contain any nontrivial segment, then $\mathcal{C}$ contains $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$.*

In particular, if $\mathcal{C}$ is as in Lemma 6.3 and has nonempty interior, then the conclusion of Lemma 6.3.(4) holds.

The proof of Lemma 6.4 is similar to [DGK3, Lem. 4.3].

Proof. Suppose that $\partial_{\mathrm{i}}\mathcal{C}$ does not contain any nontrivial segment. It is sufficient to prove that $\Lambda^{\mathsf{orb}}_\Omega(\Gamma) \subset \partial_{\mathrm{i}}\mathcal{C}$. Suppose by contradiction that there exists a point z_∞ in $\Lambda^{\mathsf{orb}}_\Omega(\Gamma) \smallsetminus \partial_{\mathrm{i}}\mathcal{C}$. We can write $z_\infty = \lim_m \gamma_m \cdot z$ for some $(\gamma_m) \in \Gamma^{\mathbb{N}}$ and $z \in \Omega \smallsetminus \mathcal{C}$. Let $y \in \mathrm{Int}(\mathcal{C})$. The segment $[y, z]$ contains a point $w \in \partial_{\mathrm{n}}\mathcal{C}$ in its interior. Up to passing to a subsequence, we may assume that $(\gamma_m \cdot y)_{m\in\mathbb{N}}$ and $(\gamma_m \cdot w)_{m\in\mathbb{N}}$ converge respectively to y_∞ and w_∞, with $[y_\infty, w_\infty] \subset \partial_{\mathrm{i}}\mathcal{C}$. Since $\partial_{\mathrm{i}}\mathcal{C}$ does not contain any nontrivial segment, we have $y_\infty = w_\infty$. Let (a, b) be the maximal interval of Ω containing a, y, w, z, b in that order, used to compute the Hilbert distances between the points y, w, z. Since $d_\Omega(\gamma_m \cdot y, \gamma_m \cdot w) = d_\Omega(y, w) > 0$ and $y_\infty = w_\infty$, it follows that at least one of $a_\infty = \lim_m \gamma_m \cdot a_m$ or $b_\infty = \lim_m \gamma_m \cdot b_m$ is equal to $y_\infty = w_\infty$. We cannot have $b_\infty = y_\infty$, since $y_\infty \neq z_\infty$ and $y_\infty, z_\infty, b_\infty$ are aligned in this order. Therefore $a_\infty = y_\infty$. But then $z_\infty = y_\infty$ since $d_\Omega(\gamma_m \cdot y, \gamma_m \cdot z) = d_\Omega(y, z)$ is bounded: contradiction. □

6.5. **Proof of (iii)' ⇒ (v) in Theorem 1.15.** Suppose that Γ acts convex cocompactly on some nonempty properly convex open subset Ω of $\mathbb{P}(V)$ and that $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ does not contain any nontrivial projective line segment. The group Γ acts cocompactly on the closed uniform 1-neighborhood $\mathcal{C}$ of $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ in (Ω, d_Ω), which is properly convex with nonempty interior. By Lemma 4.1.(1), we have $\partial_{\mathrm{i}}\mathcal{C} = \Lambda^{\mathsf{orb}}_\Omega(\Gamma)$.

7. Convex cocompactness and no segment implies P_1-Anosov

In this section we continue with the proof of Theorem 1.15. We have already established the equivalences (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) in Section 6. On the other hand, the implication (i) $\Rightarrow$ (iii) is trivial.

We now prove the implication (ii) $\Rightarrow$ (vi) in Theorem 1.15. By the above, this yields the implication (i) $\Rightarrow$ (vi) in Theorem 1.15, which is also the implication (1) $\Rightarrow$ (2) in Theorem 1.4. We build on Lemma 6.3.(3).

7.1. Compatible, transverse, dynamics-preserving boundary maps.

Let Γ be an infinite discrete subgroup of $\mathrm{PGL}(V)$. Suppose Γ is word hyperbolic and convex cocompact in $\mathbb{P}(V)$. By Proposition 5.6, there is a nonempty Γ-invariant properly convex open subset Ω of $\mathbb{P}(V)$ such that the actions of Γ on Ω and on its dual Ω^* are both convex cocompact. Our goal is to show that the natural inclusion $\Gamma \hookrightarrow \mathrm{PGL}(V)$ is P_1-Anosov.

By the implication (ii) $\Rightarrow$ (iii) in Theorem 1.15 (which we have proved in Section 6) and Theorem 1.17.(A) (which we have proved in Section 5), the full orbital limit sets $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma) \subset \mathbb{P}(V)$ and $\Lambda^{\mathsf{orb}}_{\Omega^*}(\Gamma) \subset \mathbb{P}(V^*)$ do not contain any nontrivial projective line segment. Let $\mathcal{C}^{\mathsf{cor}}_{\Omega} \subset \mathbb{P}(V)$ (resp. $\mathcal{C}^{\mathsf{cor}}_{\Omega^*} \subset \mathbb{P}(V^*)$) be the convex hull of $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ in Ω (resp. of $\Lambda^{\mathsf{orb}}_{\Omega^*}(\Gamma)$ in Ω^*). By Lemma 6.3.(3), any orbit map $\Gamma \to (\mathcal{C}^{\mathsf{cor}}_{\Omega}, d_{\Omega})$ (resp. $\Gamma \to (\mathcal{C}^{\mathsf{cor}}_{\Omega^*}, d_{\Omega^*})$) is a quasi-isometry which extends to a Γ-equivariant homeomorphism

$$\xi : \partial_\infty\Gamma \longrightarrow \Lambda^{\mathsf{orb}}_{\Omega}(\Gamma) \subset \mathbb{P}(V)$$

(resp.

$$\xi^* : \partial_\infty\Gamma \longrightarrow \Lambda^{\mathsf{orb}}_{\Omega^*}(\Gamma) \subset \mathbb{P}(V^*)).$$

We see $\mathbb{P}(V^*)$ as the space of projective hyperplanes of $\mathbb{P}(V)$.

Lemma 7.1. *The boundary maps ξ and ξ^* are compatible, i.e. for any $\eta \in \partial_\infty\Gamma$ we have $\xi(\eta) \in \xi^*(\eta)$; more precisely, $\xi^*(\eta)$ is a supporting hyperplane of Ω at $\xi(\eta)$.*

Proof. Let $(\gamma_m)_{m\in\mathbb{N}}$ be a quasi-geodesic ray in Γ with limit $\eta \in \partial_\infty\Gamma$. For any $y \in \Omega$ and any $H \in \Omega^*$, we have $\gamma_m \cdot y \to \xi(\eta)$ and $\gamma_m \cdot H \to \xi^*(\eta)$. Lift y to a vector $x \in V$ and lift H to a linear form $\varphi \in V^*$. Lift the sequence γ_m to a sequence $\hat{\gamma}_m \in \mathrm{SL}^{\pm}(V)$. By Lemma 3.2, the sequences $(\hat{\gamma}_m \cdot x)_{m\in\mathbb{N}}$ and $(\hat{\gamma}_m \cdot \varphi)_{m\in\mathbb{N}}$ go to infinity in V and V^* respectively. Since $(\hat{\gamma}_m \cdot \varphi)(\hat{\gamma}_m \cdot x) = \varphi(x)$ is independent of m, we obtain $\xi(\eta) \in \xi^*(\eta)$ by passing to the limit. Since $\xi^*(\eta)$ belongs to $\partial\Omega^*$, it is a supporting hyperplane of Ω. ☐

Lemma 7.2. *The maps ξ and ξ^* are transverse, i.e. for any $\eta \neq \eta'$ in $\partial_\infty\Gamma$ we have $\xi(\eta) \notin \xi^*(\eta')$.*

Proof. Consider $\eta, \eta' \in \partial_\infty\Gamma$ such that $\xi(\eta) \in \xi^*(\eta')$. Let us check that $\eta = \eta'$. By Lemma 4.1.(1), we have $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma) = \partial_{\mathrm{i}}\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$, and so the projective line segment $[\xi(\eta), \xi(\eta')]$, contained in the supporting hyperplane $\xi^*(\eta')$, is contained in $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$. Since $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ contains no nontrivial projective line segment by condition (iii) of Theorem 1.15, we deduce $\eta = \eta'$. $\square$

For any $\gamma \in \Gamma$ of infinite order, we denote by η^+_γ (resp. η^-_γ) the attracting (resp. repelling) fixed point of γ in $\partial_\infty\Gamma$.

Lemma 7.3. *The maps ξ and ξ^* are dynamics-preserving.*

Proof. We only prove it for ξ; the argument is similar for ξ^*. We fix a norm $\|\cdot\|_V$ on V. Let $\gamma \in \Gamma$ be an element of infinite order; we lift it to an element $\hat{\gamma} \in \mathrm{SL}^\pm(V)$ that preserves a properly convex cone of $V \smallsetminus \{0\}$ lifting Ω. Let L^+ be the line of V corresponding to $\xi(\eta^+_\gamma)$ and H^- the hyperplane of V corresponding to $\xi^*(\eta^-_\gamma)$. By transversality, we have $V = L^+ \oplus H^-$, and this decomposition is preserved by $\hat{\gamma}$. Let $[x] \in \Omega$ and write $x = \ell^+ + h^-$ with $\ell^+ \in L^+$ and $h^- \in H^-$. Since Ω is open, we may choose x so that $\ell^+ \neq 0$ and h^- satisfies $\|\hat{\gamma}^m \cdot h^-\|_V \geq \delta t^m \|h^-\|_V > 0$ for all $m \in \mathbb{N}$, where $\delta > 0$ and where $t > 0$ is the spectral radius of the restriction of $\hat{\gamma}$ to H^-. On the other hand, $\hat{\gamma}^m \cdot \ell^+ = s^m \ell^+$ where s is the eigenvalue of $\hat{\gamma}$ on L^+. By Lemma 6.3.(4), we have $\hat{\gamma}^m \cdot [x] \to \xi(\eta^+_\gamma)$ as $m \to +\infty$, hence

$$\frac{\|\hat{\gamma}^m \cdot h^-\|_V}{\|\hat{\gamma}^m \cdot \ell^+\|_V} \underset{m\to+\infty}{\longrightarrow} 0.$$

Necessarily $s > t$, and so $\xi(\eta^+_\gamma)$ is an attracting fixed point for the action of γ on $\mathbb{P}(V)$. $\square$

As an immediate consequence of Lemma 6.3.(3) and Lemma 7.3, we obtain the following.

Corollary 7.4. *For any infinite-order element $\gamma \in \Gamma$, the element $\rho(\gamma) \in \mathrm{PGL}(V)$ is proximal in $\mathbb{P}(V)$, and the full orbital limit set $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ is the closure of the set of attracting fixed points of such elements.*

7.2. The natural inclusion $\Gamma \hookrightarrow \mathrm{PGL}(V)$ is P_1-Anosov.

Lemma 7.5. *We have $(\mu_1 - \mu_2)(\gamma) \to +\infty$ as $\gamma \to \infty$ in Γ.*

Proof. Consider a sequence $(\gamma_m) \in \Gamma^{\mathbb{N}}$ going to infinity in Γ. Up to extracting we can assume that there exists $\eta \in \partial_\infty\Gamma$ such that $\gamma_m \to \eta$. Then $\gamma_m \cdot y \to \xi(\eta)$ for all $y \in \Omega$ by Lemma 6.3.(3). For any m we can write $\gamma_m = k_m a_m k'_m \in K(\exp \mathfrak{a}^+)K$ where $a_m = \mathrm{diag}(a_{i,m})_{1\leq i\leq n}$ with $a_{i,m} \geq a_{i+1,m}$ (see Section 2.2). Up to extracting, we may assume that $(k_m)_{m\in\mathbb{N}}$ converges to some $k \in K$ and $(k'_m)_{m\in\mathbb{N}}$ to some $k' \in K$. Since Ω is open, we can find points y, z of Ω lifting respectively to $w, x \in V$ with

$$k' \cdot w = \sum_{i=1}^n s_i e_i \qquad \text{and} \qquad k' \cdot x = \sum_{i=1}^n t_i e_i$$

such that $s_1 = t_1$ but $s_2 \neq t_2$. For any m, let us write

$$k'_m \cdot w = \sum_{i=1}^n s_{i,m} e_i \qquad \text{and} \qquad k'_m \cdot x = \sum_{i=1}^n t_{i,m} e_i$$

where $s_{i,m} \to s_i$ and $t_{i,m} \to t_i$. Then

$$a_m k'_m \cdot y = \left[s_{1,m} e_1 + \sum_{i=2}^{n} \frac{a_{i,m}}{a_{1,m}} \, s_{i,m} e_i \right], \quad a_m k'_m \cdot z = \left[t_{1,m} e_1 + \sum_{i=2}^{n} \frac{a_{i,m}}{a_{1,m}} \, t_{i,m} e_i \right].$$

The sequences $(\gamma_m \cdot y)_{m\in\mathbb{N}}$ and $(\gamma_m \cdot z)_{m\in\mathbb{N}}$ both converge to $\xi(\eta)$. Hence the sequences $(a_m k'_m \cdot y)_{m\in\mathbb{N}}$ and $(a_m k'_m \cdot z)_{m\in\mathbb{N}}$ both converge to the same point $k^{-1} \cdot \xi(\eta) \in \mathbb{P}(V)$. Since $s_1 = t_1$ and $s_2 \neq t_2$, we must have $a_{2,m}/a_{1,m} \to 0$, i.e. $(\mu_1 - \mu_2)(\gamma_m) \to +\infty$. ☐

By Fact 2.5, the natural inclusion $\Gamma \hookrightarrow \mathrm{PGL}(V)$ is P_1-Anosov. This completes the proof of the implication (ii) ⇒ (vi) of Theorem 1.15.

8. P_1-Anosov implies convex cocompactness

In this section we prove the implication (vi) ⇒ (ii) in Theorem 1.15.

Let Γ be an infinite discrete subgroup of $\mathrm{PGL}(V)$. Suppose that Γ is word hyperbolic, that the inclusion $\Gamma \hookrightarrow \mathrm{PGL}(V)$ is P_1-Anosov, and that Γ preserves some nonempty properly convex open subset $\mathcal{O}$ of $\mathbb{P}(V)$. We wish to prove that Γ acts convex cocompactly on some nonempty properly convex open subset Ω of $\mathbb{P}(V)$. We note that Ω cannot always be taken to be equal to $\mathcal{O}$: see Remark 8.8 below.

Recall that, by Proposition 4.5, when Γ preserves a nonempty properly convex open set $\mathcal{O}$, the proximal limit set Λ^*_Γ of Γ in $\mathbb{P}(V^*)$ always lifts to a cone $\widetilde{\Lambda}^*_\Gamma$ of $V^* \smallsetminus \{0\}$ such that the convex open set

$$\Omega_{\max} := \mathbb{P}(\{x \in V \mid \ell(x) > 0 \quad \forall \ell \in \widetilde{\Lambda}^*_\Gamma\})$$

contains $\mathcal{O}$ and is Γ-invariant. Thus it is sufficient to prove the following.

Proposition 8.1. *Let Γ be an infinite discrete subgroup of $\mathrm{PGL}(V)$ which is word hyperbolic, such that the natural inclusion $\Gamma \hookrightarrow \mathrm{PGL}(V)$ is P_1-Anosov, with boundary maps $\xi : \partial_\infty\Gamma \to \mathbb{P}(V)$ and $\xi^* : \partial_\infty\Gamma \to \mathbb{P}(V^*)$. Suppose that the proximal limit set $\Lambda^*_\Gamma = \xi^*(\partial_\infty\Gamma)$ lifts to a cone $\widetilde{\Lambda}^*_\Gamma$ of $V^* \smallsetminus \{0\}$ such that the convex open set*

$$\Omega_{\max} := \mathbb{P}(\{x \in V \mid \ell(x) > 0 \quad \forall \ell \in \widetilde{\Lambda}^*_\Gamma\})$$

is nonempty and Γ-invariant. Then Γ acts convex cocompactly (Definition 1.11) on some nonempty properly convex open set $\Omega \subset \Omega_{\max}$ (which can be taken to be $\Omega_{\max}$ if $\Omega_{\max}$ is properly convex). Moreover, the full orbital limit set $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ for any such Ω is equal to the proximal limit set $\Lambda_\Gamma = \xi(\partial_\infty\Gamma)$.

The rest of the section is devoted to the proof of Proposition 8.1.

8.1. **Convergence for Anosov representations.** We first make the following general observation.

Lemma 8.2. *Let Γ be an infinite discrete subgroup of $\mathrm{PGL}(V)$ which is word hyperbolic, such that the natural inclusion $\Gamma \hookrightarrow \mathrm{PGL}(V)$ is P_1-Anosov, with* *boundary maps $\xi : \partial_\infty\Gamma \to \mathbb{P}(V)$ and $\xi^* : \partial_\infty\Gamma \to \mathbb{P}(V^*)$. Let $(\gamma_m)_{m\in\mathbb{N}}$ be a sequence of elements of Γ converging to some $\eta \in \partial_\infty\Gamma$, such that $(\gamma_m^{-1})_{m\in\mathbb{N}}$ converges to some $\eta' \in \partial_\infty\Gamma$. Then*

(1) for any $y \in \mathbb{P}(V)$ with $y \notin \xi^*(\eta')$ we have $\gamma_m \cdot y \to \xi(\eta)$;

(2) for any $y^* \in \mathbb{P}(V^*)$ with $\xi(\eta') \notin y^*$ we have $\gamma_m \cdot y^* \to \xi^*(\eta)$.

Moreover, convergence is uniform for y, y^* ranging over compact sets.

Proof. The two statements are dual to each other, so we only need to prove (1). For any $m \in \mathbb{N}$, we write $\gamma_m = k_m a_m k'_m \in K \exp(\mathfrak{a}^+)K$ (see Section 2.2). Up to extracting, (k_m) converges to some $k \in K$, and (k'_m) converges to some $k' \in K$. Note that

$$\gamma_m^{-1} = ({k'_m}^{-1} w_0)(w_0 a_m^{-1} w_0)(w_0 k_m^{-1}) \in KA^+K,$$

where $w_0 \in \mathrm{PGL}(V) \simeq \mathrm{PGL}(\mathbb{R}^n)$ is the image of the permutation matrix exchanging e_i and e_{n+1-i}. Therefore, by Fact 2.6, we have

$$\xi^*(\eta') = {k'}^{-1} w_0 \cdot \mathbb{P}(\mathrm{span}(e_1, \dots, e_{n-1})).$$

In particular, for any $y \in \mathbb{P}(V)$ with $y \notin \xi^*(\eta')$ we have

$$k' \cdot y \notin w_0 \cdot \mathbb{P}(\mathrm{span}(e_1, \dots, e_{n-1})) = \mathbb{P}(\mathrm{span}(e_2, \dots, e_n)).$$

On the other hand, writing $a_m = \mathrm{diag}(a_{m,i})_{1\leq i\leq n}$, we have $a_{m,1}/a_{m,2} = e^{(\mu_1-\mu_2)(\gamma_m)} \to +\infty$ by Fact 2.5. Therefore $a_m k'_m \cdot y \to [e_1]$, and so $\gamma_m \cdot y = k_m a_m k'_m \cdot y \to k \cdot [e_1]$. By Fact 2.6, we have $k \cdot [e_1] = \xi(\eta)$. This proves (1). Uniformity follows from the fact $k' \cdot y = [e_1 + \sum_{i=2}^n u_i e_i]$ with *bounded* u_i, when y ranges over a compact set disjoint from the hyperplane $\xi^*(\eta')$. $\square$

Corollary 8.3. *In the setting of Lemma 8.2, for any nonempty Γ-invariant properly convex open subset Ω of $\mathbb{P}(V)$, the full orbital limit set $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ is equal to the proximal limit set $\Lambda_\Gamma = \xi(\partial_\infty\Gamma)$.*

Proof. By Proposition 4.5, we have $\Lambda^*_\Gamma = \xi^*(\partial_\infty\Gamma) \subset \partial\Omega^*$, hence $y \notin \xi^*(\eta')$ for all $y \in \Omega$ and $\eta' \in \partial_\infty\Gamma$. We then apply Lemma 8.2.(1) to get both $\Lambda_\Gamma \subset \Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ and $\Lambda^{\mathsf{orb}}_\Omega(\Gamma) \subset \Lambda_\Gamma$. $\square$

8.2. **Proof of Proposition 8.1.** Let Γ be an infinite discrete subgroup of $\mathrm{PGL}(V)$. Suppose that Γ is word hyperbolic, that the natural inclusion $\Gamma \hookrightarrow \mathrm{PGL}(V)$ is P_1-Anosov, and that the proximal limit set $\Lambda^*_\Gamma = \xi^*(\partial_\infty\Gamma)$ lifts to a cone $\widetilde{\Lambda}^*_\Gamma$ of $V^* \smallsetminus \{0\}$ such that the convex open set

$$\Omega_{\max} := \mathbb{P}(\{x \in V \mid \ell(x) > 0 \quad \forall \ell \in \widetilde{\Lambda}^*_\Gamma\})$$

is nonempty and Γ-invariant (see Figure 3).

Let $\mathcal{C}_0$ be the convex hull of Λ_Γ in $\Omega_{\max}$; it is a nonempty closed convex subset of $\Omega_{\max}$. In fact, $\mathcal{C}_0$ is properly convex because any point of $\partial\Omega_{\max}$ that is contained in a full projective line of $\overline{\Omega}_{\max}$ must also be contained in every hyperplane of Λ^*_Γ hence, by transversality of Λ_Γ and Λ^*_Γ, it must not be contained in Λ_Γ.

Lemma 8.4. *We have $\partial_{\mathrm{i}}\mathcal{C}_0 = \Lambda_\Gamma$.*

Proof. By construction, $\partial_{\mathrm{i}}\mathcal{C}_0 = \overline{\mathcal{C}_0} \cap \partial\Omega_{\max} \supset \Lambda_\Gamma$. Suppose a point $y \in \overline{\mathcal{C}_0}$ lies in $\partial\Omega_{\max}$. Then y lies in a hyperplane $\xi^*(\eta)$ for some $\eta \in \partial_\infty\Gamma$. Then any minimal subset S of Λ_Γ for which y lies in the convex hull of S must also be contained in $\xi^*(\eta)$. By the transversality of ξ and ξ^*, this implies $y \in \Lambda_\Gamma$. □

The set $\Omega_{\max}$ is convex, but it may fail to be properly convex when Γ is not irreducible (see Example 10.11). However, we observe the following.

Lemma 8.5. *There exists a Γ-invariant properly convex open subset Ω containing $\mathcal{C}_0$.*

Proof. If $\Omega_{\max}$ is properly convex, then we take $\Omega = \Omega_{\max}$. Suppose that $\Omega_{\max}$ fails to be properly convex. Choose projective hyperplanes $H_0, \dots, H_n$ bounding an open simplex Δ containing $\overline{\mathcal{C}_0}$. Define $\mathcal{V} := \Omega_{\max} \cap \Delta$ and

$$\Omega := \bigcap_{\gamma\in\Gamma} \gamma \cdot \mathcal{V}.$$

Then Ω contains $\mathcal{C}_0$ and is properly convex. We must show Ω is open. Suppose not and let $z \in \partial_{\mathrm{n}}\Omega$. Then there is a sequence $(\gamma_m) \in \Gamma^{\mathbb{N}}$ and $i \in \{1, \dots, \ell\}$ such that $\gamma_m \cdot H_i$ converges to a hyperplane H_∞ containing z. By Lemma 8.2.(2), the hyperplane H_∞ lies in Λ_Γ^*, contradicting the fact that the hyperplanes of Λ_Γ^* are disjoint from $\Omega_{\max}$. □

Lemma 8.6. *For any Γ-invariant properly convex open subset Ω of $\mathbb{P}(V)$ containing $\mathcal{C}_0$, we have $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma) = \mathcal{C}_0$.*

Proof. By Corollary 8.3, we have $\Lambda^{\mathsf{orb}}_\Omega(\Gamma) = \Lambda_\Gamma$, hence $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ is the convex hull of Λ_Γ in Ω, i.e. $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma) = \mathcal{C}_0 \cap \Omega$. But $\mathcal{C}_0$ is contained in Ω by construction (see Lemma 8.5), hence $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma) = \mathcal{C}_0$. □

In order to conclude the proof of Proposition 8.1, it only remains to check the following.

Lemma 8.7. *The action of Γ on $\mathcal{C}_0$ is cocompact.*

Proof. By [KLPa, Th. 1.7] (see also [GGKW, Rem. 5.15]), the action of Γ on $\mathbb{P}(V)$ at any point $z \in \Lambda_\Gamma$ is expanding: there exist an element $\gamma \in \Gamma$, a neighborhood $\mathcal{U}$ of z in $\mathbb{P}(V)$, and a constant $c > 1$ such that γ is c-expanding on $\mathcal{U}$ for the metric

$$d_{\mathbb{P}}([x], [x']) := |\sin \measuredangle(x, x')|$$

on $\mathbb{P}(V)$. We now use a version of the argument of [KLPa, Prop. 2.5], inspired by Sullivan's dynamical characterization [Su] of convex cocompactness in the real hyperbolic space. (The argument in [KLPa] is a little more technical because it deals with bundles, whereas we work directly in $\mathbb{P}(V)$.)

Suppose by contradiction that the action of Γ on $\mathcal{C}_0$ is *not* cocompact, and let $(\varepsilon_m)_{m\in\mathbb{N}}$ be a sequence of positive reals converging to 0. For any m, the set $K_m := \{z \in \mathcal{C}_0 \,|\, d_{\mathbb{P}}(z, \Lambda_\Gamma) \geq \varepsilon_m\}$ is compact (Lemma 8.4), hence there exists a Γ-orbit contained in $\mathcal{C}_0 \smallsetminus K_m$. By proper discontinuity of the action on $\mathcal{C}_0$, the supremum of $d_{\mathbb{P}}(\cdot, \Lambda_\Gamma)$ on this orbit is achieved at some point $z_m \in \mathcal{C}_0$, and by construction $0 < d_{\mathbb{P}}(z_m, \Lambda_\Gamma) \leq \varepsilon_m$. Then, for all $\gamma \in \Gamma$,

$$d_{\mathbb{P}}(\gamma \cdot z_m, \Lambda_\Gamma) \leq d_{\mathbb{P}}(z_m, \Lambda_\Gamma).$$

Up to extracting, we may assume that $(z_m)_{m\in\mathbb{N}}$ converges to some $z \in \Lambda_\Gamma$. Consider an element $\gamma \in \Gamma$, a neighborhood $\mathcal{U}$ of z in $\mathbb{P}(V)$, and a constant $c > 1$ such that γ is c-expanding on $\mathcal{U}$. For any $m \in \mathbb{N}$, there exists $z'_m \in \Lambda_\Gamma$ such that $d_{\mathbb{P}}(\gamma \cdot z_m, \Lambda_\Gamma) = d_{\mathbb{P}}(\gamma \cdot z_m, \gamma \cdot z'_m)$. For large enough m we have $z_m, z'_m \in \mathcal{U}$, and so

$$d_{\mathbb{P}}(\gamma \cdot z_m, \Lambda_\Gamma) \geq c\, d_{\mathbb{P}}(z_m, z'_m) \geq c\, d_{\mathbb{P}}(z_m, \Lambda_\Gamma) \geq c\, d_{\mathbb{P}}(\gamma \cdot z_m, \Lambda_\Gamma) > 0.$$

This is impossible since $c > 1$. □

This shows that the word hyperbolic group Γ acts convex cocompactly on any Γ-invariant properly convex open subset Ω of $\mathbb{P}(V)$ containing $\mathcal{C}_0$, hence condition (ii) of Theorem 1.15 is satisfied.

8.3. **Two remarks on Proposition 8.1 and its proof.**

Remark 8.8. In the proof of Proposition 8.1 it is important, in order to obtain convex cocompactness, to consider only properly convex open sets Ω that contain $\mathcal{C}_0$. For instance, here would be two bad choices for Ω:

(1) if we took Ω to be the interior $\mathrm{Int}(\mathcal{C}_0)$ of $\mathcal{C}_0$, then Γ would not act convex cocompactly on Ω: we would have $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma) = \Omega = \mathrm{Int}(\mathcal{C}_0)$, which does not have compact quotient by Γ;
(2) if we took Ω such that $\mathcal{C}_0 \subset \partial\Omega$, then Γ would not act convex cocompactly on Ω, as $\mathcal{C}^{\mathsf{cor}}_\Omega = \emptyset$.

Here is a concrete example for Remark 8.8.(2): suppose Γ is a convex cocompact subgroup (in the classical sense) of $\mathrm{PO}(d-1,1) \subset \mathrm{PO}(d,1) \subset \mathrm{PGL}(\mathbb{R}^{d+1})$. Then the set $\Lambda_\Gamma = \partial_{\mathrm{i}}\mathcal{C}_0$ is contained in the equatorial sphere $\partial\mathbb{H}^{d-1}$ of $\partial\mathbb{H}^d \subset \mathbb{P}(\mathbb{R}^{d+1})$. If we took Ω to be a Γ-invariant hyperbolic half-space of $\mathbb{H}^d$, then we would have $\mathcal{C}_0 \subset \partial\Omega$.

Remark 8.9. Suppose that Γ is word hyperbolic, that the natural inclusion $\Gamma \hookrightarrow G = \mathrm{PGL}(V)$ is P_1-Anosov, and that Γ preserves a nonempty properly convex open subset of $\mathbb{P}(V)$. If the set $\Omega_{\max}$ above is properly convex (which is always the case e.g. if Γ is irreducible), then we may take $\Omega = \Omega_{\max}$ in Lemmas 8.5 and 8.6, and so Γ acts convex cocompactly on $\Omega = \Omega_{\max}$. However, if $\partial\Omega \neq \Lambda^{\mathsf{orb}}_\Omega(\Gamma)$, then the convex set $\overline{\Omega} \smallsetminus \Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ does not have bisaturated boundary, and it is possible to show that the action of Γ on $\overline{\Omega} \smallsetminus \Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ is not properly discontinuous.

8.4. **The case of groups with connected boundary.**

Here is an immediate consequence of Propositions 4.5 and 8.1 and Lemma 4.9.

Corollary 8.10. *Let Γ be an infinite discrete subgroup of* $\mathrm{PGL}(V)$ *which is word hyperbolic with* connected *boundary $\partial_\infty\Gamma$, which preserves a properly* *convex open subset of* $\mathbb{P}(V)$*, and such that the natural inclusion* $\Gamma \hookrightarrow \mathrm{PGL}(V)$ *is* P_1*-Anosov. Then the set*

$$\Omega_{\max} := \mathbb{P}(V) \smallsetminus \bigcup_{z^* \in \Lambda^*_\Gamma} z^*$$

is a nonempty Γ*-invariant convex open subset of* $\mathbb{P}(V)$*, containing all other such sets. If* $\Omega_{\max}$ *is properly convex (e.g. if* Γ *is irreducible), then* Γ *acts convex cocompactly on* $\Omega_{\max}$*.*

9. Smoothing out the nonideal boundary

We now make the connection with strong projective convex cocompactness and prove the remaining implications of Theorems 1.15 and 1.20.

Concerning Theorem 1.15, the equivalences (ii) ⇔ (iii) ⇔ (iv) ⇔ (v) have been proved in Section 6, the implication (ii) ⇒ (vi) in Section 7, and the implication (vi) ⇒ (ii) in Section 8. The implication (i) ⇒ (iii) is trivial. We shall prove (iii) ⇒ (i) in Section 9.2, which will complete the proof of Theorem 1.15. It will also complete the proof of Theorem 1.4, since the implication (2) ⇒ (1) of Theorem 1.4 is the implication (vi) ⇒ (i) of Theorem 1.15.

Concerning Theorem 1.20, the equivalence (1) ⇔ (2) has been proved in Section 4. The implication (4) ⇒ (3) is immediate. We shall prove the implication (1) ⇒ (4) in Section 9.2. The implication (3) ⇒ (2) is an immediate consequence of Lemma 3.6 and of the following lemma, and completes the proof of Theorem 1.20.

Lemma 9.1. *Let* $\mathcal{C}_{\mathsf{strict}}$ *be a nonempty convex subset of* $\mathbb{P}(V)$ *with strictly convex nonideal boundary. If* $\partial_{\mathrm{i}}\mathcal{C}_{\mathsf{strict}}$ *is closed in* $\mathbb{P}(V)$*, then* $\mathcal{C}_{\mathsf{strict}}$ *has bisaturated boundary.*

Proof. If $\partial_{\mathrm{i}}\mathcal{C}_{\mathsf{strict}}$ is closed in $\mathbb{P}(V)$, then a supporting hyperplane H of $\mathcal{C}_{\mathsf{strict}}$ at a point $y \in \partial_{\mathrm{n}}\mathcal{C}_{\mathsf{strict}}$ cannot meet $\partial_{\mathrm{i}}\mathcal{C}_{\mathsf{strict}}$, otherwise it would need to contain a nontrivial segment of $\partial_{\mathrm{n}}\mathcal{C}_{\mathsf{strict}}$, which does not exist since $\mathcal{C}_{\mathsf{strict}}$ has strictly convex nonideal boundary. This shows that $\mathcal{C}_{\mathsf{strict}}$ has bisaturated boundary. □

9.1. **Smoothing out the nonideal boundary.** Here is the main result of this section.

Lemma 9.2. *Let* Γ *be an infinite discrete subgroup of* $\mathrm{PGL}(V)$ *and* Ω *a nonempty* Γ*-invariant properly convex open subset of* $\mathbb{P}(V)$*. Suppose* Γ *acts convex cocompactly on* Ω*. Fix a uniform neighborhood* $\mathcal{C}_u$ *of* $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ *in* (Ω, d_Ω)*. Then the convex core* $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ *admits a* Γ*-invariant, properly convex, closed neighborhood* $\mathcal{C} \subset \mathcal{C}_u$ *in* Ω *which has* C^1*, strictly convex nonideal boundary.*

Constructing a neighborhood $\mathcal{C}$ as in the lemma clearly involves arbitrary choices; here is one of many possible constructions, taken from [DGK3, Lem. 6.4]. Cooper–Long–Tillmann [CLT2, Prop. 8.3] give a different construction yielding, in the case Γ is torsion-free, a convex set $\mathcal{C}$ as in the lemma whose nonideal boundary has the slightly stronger property that it is locally the graph of a smooth function with positive definite Hessian.

Proof of Lemma 9.2. In this proof, we fix a finite-index subgroup Γ_0 of Γ which is torsion-free; such a subgroup exists by the Selberg lemma [Se, Lem. 8].

We proceed in three steps. Firstly, we construct a Γ-invariant closed neighborhood $\mathcal{C}_{\clubsuit}$ of $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$ in Ω which is contained in $\mathcal{C}_u$ and whose nonideal boundary is C^1 but not necessarily strictly convex. Secondly, we construct a small deformation $\mathcal{C}_{\diamond} \subset \mathcal{C}_u$ of $\mathcal{C}_{\clubsuit}$ which has C^1 and strictly convex nonideal boundary, but which is only Γ_0-invariant, not necessarily Γ-invariant. Finally, we use an averaging procedure over translates $\gamma \cdot \mathcal{C}_{\diamond}$ of $\mathcal{C}_{\diamond}$, for $\gamma\Gamma_0$ ranging over the Γ_0-cosets of Γ, to construct a Γ-invariant closed neighborhood $\mathcal{C} \subset \mathcal{C}_u$ of $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$ in Ω which has C^1 and strictly convex nonideal boundary.

• **Construction of $\mathcal{C}_{\clubsuit}$:** Consider a compact fundamental domain $\mathcal{D}$ for the action of Γ on $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$. The convex hull of $\mathcal{D}$ in Ω is still contained in $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma)$. Let $\mathcal{D}' \subset \mathcal{C}_u$ be a closed neighborhood of this convex hull in Ω which has C^1 frontier, and let $\mathcal{C}_{\clubsuit} \subset \mathcal{C}_u$ be the closure in Ω of the convex hull of $\Gamma \cdot \mathcal{D}'$. By Lemma 4.1, we have $\partial_{\mathrm{i}}\mathcal{C}_{\clubsuit} = \Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ and $\mathcal{C}_{\clubsuit}$ has bisaturated boundary.

Let us check that $\mathcal{C}_{\clubsuit}$ has C^1 nonideal boundary. We first observe that any supporting hyperplane Π_y of $\mathcal{C}_{\clubsuit}$ at a point $y \in \partial_{\mathrm{n}}\mathcal{C}_{\clubsuit}$ stays away from $\partial_{\mathrm{i}}\mathcal{C}_{\clubsuit}$ since $\mathcal{C}_{\clubsuit}$ has bisaturated boundary. On the other hand, since the action of Γ on $\mathcal{C}_{\clubsuit}$ is properly discontinuous, for any neighborhood $\mathcal{N}$ of $\partial_{\mathrm{i}}\mathcal{C}_{\clubsuit}$ in $\mathbb{P}(V)$ and any infinite sequence of distinct elements $\gamma_j \in \Gamma$, the translates $\gamma_j \cdot \mathcal{D}'$ are eventually all contained in $\mathcal{N}$. Therefore, in a neighborhood of y, the hypersurface $\mathrm{Fr}(\mathcal{C}_{\clubsuit})$ coincides with the convex hull of a *finite* union of translates $\bigcup_{i=1}^m \gamma_i \cdot \mathcal{D}'$, and so it is locally C^1: indeed that convex hull is dual to $\bigcap_{i=1}^m (\gamma_i \cdot \mathcal{D}')^*$, which has *strictly* convex frontier because $(\mathcal{D}')^*$ does (a convex set has C^1 frontier if and only if its dual has strictly convex frontier).

• **Construction of $\mathcal{C}_{\diamond}$:** For any $y \in \partial_{\mathrm{n}}\mathcal{C}_{\clubsuit}$, let F_y be the open stratum of $\mathrm{Fr}(\mathcal{C}_{\clubsuit})$ at y, namely the intersection of $\overline{\mathcal{C}_{\clubsuit}}$ with the unique supporting hyperplane Π_y at y. Since $\mathcal{C}_{\clubsuit}$ has bisaturated boundary, F_y is a compact convex subset of $\partial_{\mathrm{n}}\mathcal{C}_{\clubsuit}$.

We claim that F_y is disjoint from $\gamma \cdot F_y = F_{\gamma\cdot y}$ for all $\gamma \in \Gamma_0 \smallsetminus \{1\}$. Indeed, if there existed $y' \in F_y \cap F_{\gamma\cdot y}$, then by uniqueness the supporting hyperplanes would satisfy $\Pi_y = \Pi_{y'} = \Pi_{\gamma\cdot y}$, hence $F_y = F_{y'} = F_{\gamma\cdot y} = \gamma \cdot F_y$. This would imply $F_y = \gamma^m \cdot F_y$ for all $m \in \mathbb{N}$, hence $\gamma^m \cdot y \in F_y$. Using the fact that the action of Γ_0 on $\mathcal{C}_{\clubsuit}$ is properly discontinuous and taking a limit, we see that F_y would contain a point of $\partial_{\mathrm{i}}\mathcal{C}_{\clubsuit}$, which we have seen is not true. Therefore F_y is disjoint from $\gamma \cdot F_y$ for all $\gamma \in \Gamma_0 \smallsetminus \{1\}$.

For any $y \in \partial_{\mathrm{n}}\mathcal{C}_{\clubsuit}$, the subset of $\mathbb{P}(V^*)$ consisting of those projective hyperplanes near the supporting hyperplane Π_y that separate F_y from $\partial_i\mathcal{C}_{\clubsuit}$ is open and nonempty, hence $(n-1)$-dimensional where $\dim(V) = n$. Choose

$n-1$ such hyperplanes $\Pi_y^1,\dots,\Pi_y^{n-1}$ in generic position, with Π_y^i cutting off a compact region $\mathcal{Q}_y^i \supset F_y$ from $\mathcal{C}_\clubsuit$. One may imagine each Π_y^i is obtained by pushing Π_y normally into $\mathcal{C}_\clubsuit$ and then tilting slightly in one of $n-1$ independent directions. The intersection $\bigcap_{i=1}^{n-1} \Pi_y^i \subset \mathbb{P}(V)$ is reduced to a singleton. By taking each hyperplane Π_y^i very close to Π_y, we may assume that the union $\mathcal{Q}_y := \bigcup_{i=1}^{n-1} \mathcal{Q}_y^i$ is disjoint from all its γ-translates for $\gamma \in \Gamma_0 \smallsetminus \{1\}$. In addition, we ensure that F_y has a neighborhood $\mathcal{Q}'_y$ contained in $\bigcap_{i=1}^{n-1} \mathcal{Q}_y^i$.

Since the action of Γ_0 on $\partial_{\mathrm{n}}\mathcal{C}_\clubsuit$ is cocompact, there exist finitely many points $y_1,\dots,y_m \in \partial_{\mathrm{n}}\mathcal{C}_\clubsuit$ such that $\partial_{\mathrm{n}}\mathcal{C}_\clubsuit \subset \Gamma_0 \cdot (\mathcal{Q}'_{y_1} \cup \dots \cup \mathcal{Q}'_{y_m})$.

We now explain, for any $y \in \partial_{\mathrm{n}}\mathcal{C}_\clubsuit$, how to deform $\mathcal{C}_\clubsuit$ into a new, smaller properly convex Γ_0-invariant closed neighborhood of $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ in Ω with C^1 nonideal boundary, in a way that destroys all segments in $\mathcal{Q}'_y$. Repeating for $y = y_1,\dots,y_m$, this will produce a properly convex Γ_0-invariant closed neighborhood $\mathcal{C}_\diamond$ of $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ in Ω with C^1 and strictly convex nonideal boundary.

Choose an affine chart containing Ω, an auxiliary Euclidean metric g on this chart, and a smooth strictly concave function $h : \mathbb{R}^+ \to \mathbb{R}^+$ with $h(0) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} h(t) = 1$ (e.g. $h = \tanh$). We may assume that for every $1 \leq i \leq n-1$ the g-orthogonal projection π_y^i onto Π_y^i satisfies $\pi_y^i(\mathcal{Q}_y^i) \subset \Pi_y^i \cap \mathcal{C}_\clubsuit$, with $(\pi_y^i|_{\mathcal{Q}_y^i})^{-1}(\Pi_y^i \cap \partial\mathcal{C}_\clubsuit) \subset \Pi_y^i$. Define maps $\varphi_y^i : \mathcal{Q}_y^i \to \mathcal{Q}_y^i$ by the property that φ_y^i preserves each fiber $(\pi_y^i)^{-1}(y')$ (a segment), taking the point at distance t from y' to the point at distance $h(t)$. Then φ_y^i takes any segment σ of F_y to a strictly convex curve, unless σ is parallel to Π_i. The image $\varphi_y^i(\mathcal{Q}_y^i \cap \partial_{\mathrm{n}}\mathcal{C}_\clubsuit)$ is still a convex hypersurface. Extending φ_y^i by the identity on $\mathcal{Q}_y \smallsetminus \mathcal{Q}_y^i$ and repeating with varying i, we find that the composition $\varphi_y := \varphi_y^1 \circ \dots \circ \varphi_y^{n-1}$, defined on $\mathcal{Q}_y$, takes $\mathcal{Q}'_y \cap \partial_{\mathrm{n}}\mathcal{C}_\clubsuit$ to a strictly convex hypersurface. We can extend φ_y in a Γ_0-equivariant fashion to $\Gamma_0 \cdot \mathcal{Q}_y$, and extend it further by the identity on the rest of $\mathcal{C}_\clubsuit$: the set $\varphi_y(\mathcal{C}_\clubsuit)$ is still a Γ_0-invariant closed neighborhood of $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ in Ω, contained in $\mathcal{C}_u$, with C^1 nonideal boundary.

Repeating with finitely many points $y_1,\dots,y_m$ as above, we obtain a Γ_0-invariant properly convex closed neighborhood $\mathcal{C}_\diamond \subset \mathcal{C}_u$ of $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ in Ω with C^1 and strictly convex boundary.

• **Construction of $\mathcal{C}$:** Consider the finitely many Γ_0-cosets $\gamma_1\Gamma_0,\dots,\gamma_k\Gamma_0$ of Γ and the corresponding translates $\mathcal{C}_\diamond^i := \gamma_i \cdot \mathcal{C}_\diamond$; we denote by Ω^i the interior of $\mathcal{C}_\diamond^i$. Let $\mathcal{C}'$ be a Γ-invariant properly convex closed neighborhood of $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ in Ω which has C^1 (but not necessarily strictly convex) nonideal boundary and is contained in all $\mathcal{C}_\diamond^i$, $1 \leq i \leq k$. (Such a neighborhood $\mathcal{C}'$ can be constructed for instance by the same method as $\mathcal{C}_\clubsuit$ above.) Since $\mathcal{C}_\diamond^i$ has strictly convex nonideal boundary, uniform neighborhoods of $\mathcal{C}'$ in (Ω^i, d_{Ω^i}) have strictly convex nonideal boundary [Bu, (18.12)]. Therefore, by cocompactness, if $h : [0,1] \to [0,1]$ is a convex function with sufficiently fast growth (e.g. $h(t) = t^\alpha$ for large enough $\alpha > 0$), then the Γ_0-invariant function $H_i := h \circ d_{\Omega^i}(\cdot, \mathcal{C}')$ is convex on the convex region $H_i^{-1}([0,1])$, and in fact smooth and strictly convex near every point outside $\mathcal{C}'$. The function $H := \sum_{i=1}^k H_i$ is Γ-invariant and its sublevel set $\mathcal{C} := H^{-1}([0,1])$ is a Γ-invariant closed neighborhood of $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ in Ω which has C^1, strictly convex nonideal boundary. Moreover, $\mathcal{C} \subset \mathcal{C}_\diamondsuit \subset \mathcal{C}_\clubsuit \subset \mathcal{C}_u$ by construction. ☐

9.2. **Proof of the remaining implications of Theorems 1.15 and 1.20.**

Proof of (1) $\Rightarrow$ (4) *in Theorem 1.20.* Suppose Γ is convex cocompact in $\mathbb{P}(V)$, i.e. it preserves a properly convex open set $\Omega \subset \mathbb{P}(V)$ and acts cocompactly on the convex core $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$. Let $\mathcal{C}_u$ be the closed uniform 1-neighborhood of $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ in (Ω, d_Ω). The set $\mathcal{C}_u$ is properly convex [Bu, (18.12)]. The action of Γ on $\mathcal{C}_u$ is properly discontinuous, and cocompact since $\mathcal{C}_u$ is the union of the Γ-translates of the closed uniform 1-neighborhood of a compact fundamental domain of $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ in (Ω, d_Ω). By Lemma 9.2, the set $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ admits a Γ-invariant, properly convex, closed neighborhood $\mathcal{C}_{\mathsf{smooth}} \subset \mathcal{C}_u$ in Ω which has C^1, strictly convex nonideal boundary. The action of Γ on $\mathcal{C}_{\mathsf{smooth}}$ is still properly discontinuous and cocompact, and $\partial_{\mathrm{i}}\mathcal{C}_{\mathsf{smooth}} = \Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ by Lemma 4.1.(1). This proves the implication (1) $\Rightarrow$ (4) in Theorem 1.20. ☐

Proof of (iii) $\Rightarrow$ *(i) in Theorem 1.15.* Suppose that Γ acts convex cocompactly on a properly convex open set Ω and that the full orbital limit set $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ contains no nontrivial segment. As in the proof of (1) $\Rightarrow$ (4) in Theorem 1.20 just above, Γ acts properly discontinuously and cocompactly on some nonempty closed properly convex subset $\mathcal{C}_{\mathsf{smooth}}$ of Ω which has strictly convex and C^1 nonideal boundary and whose interior $\Omega_{\mathsf{smooth}} := \mathrm{Int}(\mathcal{C}_{\mathsf{smooth}})$ contains $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$. The set $\mathcal{C}_{\mathsf{smooth}}$ has bisaturated boundary (Lemma 9.1), hence the action of Γ on Ω_{smooth} is convex cocompact by Corollary 4.4.

By Lemma 4.1.(1), the ideal boundary $\partial_{\mathrm{i}}\mathcal{C}_{\mathsf{smooth}}$ is equal to $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$. Since this set contains no nontrivial segment by assumption, we deduce from Lemma 4.1.(1) that any $z \in \partial_{\mathrm{i}}\mathcal{C}_{\mathsf{smooth}}$ is an extreme point of $\partial\Omega$. Thus the full boundary of Ω_{smooth} is strictly convex.

Consider the dual convex set $\mathcal{C}^*_{\mathsf{smooth}} \subset \mathbb{P}(V^*)$ (Definition 5.1); it is properly convex. By Lemma 5.3, the set $\mathcal{C}^*_{\mathsf{smooth}}$ has bisaturated boundary and does not contain any PET (since $\mathcal{C}_{\mathsf{smooth}}$ itself does not contain any PET). By Proposition 5.4, the action of Γ on $\mathcal{C}^*_{\mathsf{smooth}}$ is properly discontinuous and cocompact. It follows from Lemma 6.2 that $\partial_{\mathrm{i}}\mathcal{C}^*_{\mathsf{smooth}}$ contains no nontrivial segment, hence each point of $\partial_{\mathrm{i}}\mathcal{C}^*_{\mathsf{smooth}}$ is an extreme point. Hence there is exactly one hyperplane supporting $\mathcal{C}_{\mathsf{smooth}}$ at any given point of $\partial_{\mathrm{i}}\mathcal{C}_{\mathsf{smooth}}$. This is also true at any given point of $\partial_{\mathrm{n}}\mathcal{C}_{\mathsf{smooth}}$ by assumption. Thus the boundary of Ω_{smooth} is C^1. ☐

10. Properties of convex cocompact groups

In this section we prove Theorem 1.17. Property (A) has already been established in Section 5.3; we now establish the other properties.

10.1. **(B): Quasi-isometric embedding.** We now establish the following very general result, using the notation $\mu_1 - \mu_n$ from (2.1). The fact that a statement of this flavor should exist was suggested to us by Yves Benoist.

Proposition 10.1. *Let Ω be a properly convex subset of $\mathbb{P}(V)$. For any $z \in \Omega$, there exists $\kappa > 0$ such that for any $g \in \mathrm{Aut}(\Omega)$,*

$$(\mu_1 - \mu_n)(g) \geq 2\, d_\Omega(z, g \cdot z) - \kappa.$$

Here is an easy consequence of Proposition 10.1.

Corollary 10.2. *Let Γ be a discrete subgroup of $G := \mathrm{PGL}(V)$.*

If Γ is naively convex cocompact in $\mathbb{P}(V)$ (Definition 1.9), then it is finitely generated and the natural inclusion $\Gamma \hookrightarrow G$ is a quasi-isometric embedding.

In particular, for any subgroup Γ' of Γ which is naively convex cocompact in $\mathbb{P}(V)$, the natural inclusion $\Gamma' \hookrightarrow \Gamma$ is a quasi-isometric embedding.

Here the finitely generated group Γ is endowed with the word metric with respect to some fixed finite generating subset. The group G is endowed with any G-invariant Riemannian metric.

Proof of Corollary 10.2. Let $K \simeq \mathrm{PO}(n)$ be a maximal compact subgroup of $G = \mathrm{PGL}(V)$ as in Section 2.2. Let $p := eK \in G/K$. There is a G-invariant metric d on G/K such that $d(p, g \cdot p) = (\mu_1 - \mu_n)(g)$ for all $g \in G$.

Let Ω be a Γ-invariant properly convex open subset of $\mathbb{P}(V)$ and $\mathcal{C}$ a nonempty Γ-invariant closed convex subset of Ω on which Γ acts cocompactly. By the Švarc–Milnor lemma, Γ is finitely generated and any orbital map $\Gamma \to (\mathcal{C}, d_\Omega)$ is a quasi-isometry. Using Proposition 10.1, we then obtain that any orbital map $\Gamma \to (G/K, d)$ is a quasi-isometric embedding. Since K is compact, the natural inclusion $\Gamma \hookrightarrow G$ is a quasi-isometric embedding. $\square$

In order to prove Proposition 10.1, we first establish the following estimate, where we consider the metric

$$d_{\mathbb{P}}([x], [x']) := |\sin \measuredangle(x, x')|$$

on $\mathbb{P}(V)$ as in Section 8.2.

Lemma 10.3. *Let Ω be a properly convex open subset of $\mathbb{P}(V)$. For any $z \in \Omega$, there exists $R_z \geq 1$ such that for any $w \in \Omega$ and $b \in \partial\Omega$ with z, w, b aligned in this order,*

$$R_z^{-1} \leq d_\Omega(z, w) \cdot d_{\mathbb{P}}(w, b) \leq R_z.$$

Proof of Lemma 10.3. Fix $z \in \Omega$. Consider an affine chart $\mathbb{R}^{n-1}$ of $\mathbb{P}(V)$, endowed with a Euclidean norm $\|\cdot\|$, in which Ω appears nested between two Euclidean balls of radii $r < R$ centered at z. There exists $B > 0$ such that for any $y_1, y_2 \in \overline{\Omega}$,

$$B^{-1}\, d_{\mathbb{P}}(y_1, y_2) \leq \|y_1 - y_2\| \leq B\, d_{\mathbb{P}}(y_1, y_2).$$

For any $w \in \Omega$ and $a, b \in \partial\Omega$ with a, z, w, b aligned in this order, we then have

$$\frac{r^2}{R \cdot B\, d_{\mathbb{P}}(w,b)} \leq \frac{\|\tilde{a} - \tilde{w}\|\,\|\tilde{z} - \tilde{b}\|}{\|\tilde{a} - \tilde{z}\|\,\|\tilde{w} - \tilde{b}\|} \leq \frac{2R^2}{r \cdot B^{-1} d_{\mathbb{P}}(w,b)}\ ,$$

and the middle term is equal to $d_\Omega(z, w)$ by definition of d_Ω. □

Proof of Proposition 10.1. Let $\|\cdot\|_V$ be a $\tilde{K}$-invariant Euclidean norm on V for which (2.2) holds. Fix $z \in \Omega$. We first make the observation that by compactness of $\partial\Omega$, there exists $0 < \varepsilon < 1/2$ with the following property: for any lift $\tilde{z} \in V \smallsetminus \{0\}$ of z and any segment $[x, x'] \subset V \smallsetminus \{0\}$ containing $\tilde{z}$ and projecting to a segment of $\overline{\Omega} \subset \mathbb{P}(V)$ with distinct endpoints in $\partial\Omega$, we have $d_{\mathbb{P}}([x], [x'])| \geq \varepsilon$, and if $\|x\|_V = \|x'\|_V$ we can write $\tilde{z} = tx + (1-t)x'$ where $\varepsilon \leq t \leq 1 - \varepsilon$.

Fix $g \in G$ and lift it to an element of $\mathrm{SL}^{\pm}(V)$, still denoted by g. Consider $\alpha, \beta \in \partial\Omega$ such that $g \cdot \alpha, z, g \cdot z, g \cdot \beta$ are aligned in this order. Lift α, β, z respectively to $\tilde{\alpha}, \tilde{\beta}, \tilde{z} \in V \smallsetminus \{0\}$ such that $\|\tilde{\alpha}\|_V = \|\tilde{\beta}\|_V$ and $\tilde{z} \in [\tilde{\alpha}, \tilde{\beta}]$. By the above observation, we have $d_{\mathbb{P}}(g \cdot \alpha, g \cdot \beta]) \geq \varepsilon$ and we may write $\tilde{z} = t\tilde{\alpha} + (1-t)\tilde{\beta}$ with $\varepsilon \leq t \leq 1 - \varepsilon$. We have

$$\begin{aligned}
\|g \cdot \tilde{z}\|_V\, \|g \cdot \tilde{\beta}\|_V\, d_{\mathbb{P}}(g \cdot z, g \cdot \beta) &= \operatorname{area}(g \cdot \tilde{z}, g \cdot \tilde{\beta}) \\
&= t \operatorname{area}(g \cdot \tilde{\alpha}, g \cdot \tilde{\beta}) \\
&= t\, \|g \cdot \tilde{\alpha}\|_V\, \|g \cdot \tilde{\beta}\|_V\, d_{\mathbb{P}}(g \cdot \alpha, g \cdot \beta),
\end{aligned}$$

by definition of $d_{\mathbb{P}}$, where $\operatorname{area}(v, w)$ denotes the area of the parallelogram of V spanned by the vectors v, w. Therefore

$$\frac{\|g \cdot \tilde{z}\|_V}{\|g \cdot \tilde{\alpha}\|_V} \geq \varepsilon^2\, d_{\mathbb{P}}(g \cdot z, g \cdot \beta)^{-1}.$$

Let $R_z > 0$ be given by Lemma 10.3. Taking $(w, b) = (g \cdot z, g \cdot \beta)$ in Lemma 10.3, we have

$$\frac{\|g \cdot \tilde{z}\|_V}{\|g \cdot \tilde{\alpha}\|_V} \geq \varepsilon^2\, R_z^{-1}\, d_\Omega(z, g \cdot z). \tag{10.1}$$

Let $\|g\|_V$ (resp. $\|g^{-1}\|_V$) be the operator norm of $g \in \mathrm{SL}^{\pm}(V)$ (resp. $g^{-1} \in \mathrm{SL}^{\pm}(V)$). Then

$$\|g \cdot \tilde{z}\|_V \leq \|g\|_V\, \|z\|_V \leq \|g\|_V\, \|\alpha\|_V$$

and

$$\|g \cdot \tilde{\alpha}\|_V \geq \frac{\|\alpha\|_V}{\|g^{-1}\|_V}.$$

Therefore, taking the logarithm in (10.1) and using (2.2), we obtain

$$(\mu_1 - \mu_n)(g) = \log\left(\|g\|_V\, \|g^{-1}\|_V\right) \geq 2\, d_\Omega(z, g \cdot z) - |\log(\varepsilon^2\, R_z^{-1})|. \quad □$$

10.2. **(C): No unipotent elements.** Property (C) of Theorem 1.17 is contained in the following more general statement.

Proposition 10.4. *Let Γ be a discrete subgroup of* $\mathrm{PGL}(V)$ *which is naively convex cocompact in* $\mathbb{P}(V)$ *(Definition 1.9). Then Γ does not contain any unipotent element.*

Proof. Let Ω be a Γ-invariant properly convex open subset of $\mathbb{P}(V)$ and $\mathcal{C}$ a nonempty Γ-invariant closed convex subset of Ω on which Γ acts cocompactly. By [CLT1, Prop. 2.13], we only need to check that any element $\gamma \in \Gamma$ of infinite order achieves its translation length

$$R := \inf_{y\in\mathcal{C}} d_\Omega(y, \gamma \cdot y) \geq 0.$$

Consider $(y_m) \in \mathcal{C}^{\mathbb{N}}$ such that $d_\Omega(y_m, \gamma \cdot y_m) \to R$. For any m, there exists $\gamma_m \in \Gamma$ such that $\gamma_m \cdot y_m$ belongs to some fixed compact fundamental domain for the action of Γ on $\mathcal{C}$. Up to passing to a subsequence, $\gamma_m \cdot y_m$ converges to some $y \in \mathcal{C}$. As $m \to +\infty$, we have $d_\Omega(y_m, \gamma \cdot y_m) = R + o(1)$, which means that $\gamma_m\gamma\gamma_m^{-1} \in \Gamma$ sends $\gamma_m \cdot y_m \in \mathcal{C}$ at distance $\leq R + o(1)$ from itself. Since the action of Γ on $\mathcal{C}$ is properly discontinuous, we deduce that the discrete sequence $(\gamma_m\gamma\gamma_m^{-1})_{m\in\mathbb{N}}$ is bounded; up to passing to a subsequence, we may therefore assume that it is constant, equal to $\gamma_\infty\gamma\gamma_\infty^{-1}$ for some $\gamma_\infty \in \Gamma$. We then have $d_\Omega(z, \gamma \cdot z) = R$ where $z := \gamma_\infty^{-1} \cdot y$. ☐

10.3. **(D): Stability.** Property (D) of Theorem 1.17 follows from the equivalence (2) ⇔ (4) of Theorem 1.20 and from the work of Cooper–Long–Tillmann (namely [CLT2, Th. 0.1] with empty collection $\mathcal{V}$ of generalized cusps). Indeed, the equivalence (2) ⇔ (4) of Theorem 1.20 shows that Γ is convex cocompact in $\mathbb{P}(V)$ if and only if it acts properly discontinuously and cocompactly on a properly convex subset $\mathcal{C}$ of $\mathbb{P}(V)$ with strictly convex nonideal boundary, and this condition is stable under small perturbations by [CLT2].

10.4. **(E): Semisimplification.** Let Γ be a discrete subgroup of $\mathrm{PGL}(V)$. The Zariski closure H of Γ in $\mathrm{PGL}(V)$ admits a Levi decomposition $H = L \ltimes R_u(H)$ where L is reductive (called a *Levi factor*) and $R_u(H)$ is the unipotent radical of H. The projection of Γ to L is discrete (see [R, Th. 8.24]) and does not depend, up to conjugation in $\mathrm{PGL}(V)$, on the choice of the Levi factor L. We shall use the following terminology.

Definition 10.5. The *semisimplification of* Γ is the projection of Γ to a Levi factor of the Zariski closure of Γ. It is a discrete subgroup of $\mathrm{PGL}(V)$, well defined up to conjugation.

The semisimplification of Γ, like the Levi factor containing it, acts projectively on V in a semisimple way: namely, any invariant linear subspace of V admits an invariant complementary subspace.

Suppose that the semisimplification of Γ is convex cocompact in $\mathbb{P}(V)$. This semisimplification is a limit of $\mathrm{PGL}(V)$-conjugates of Γ. Since convex cocompactness is open (property (D) above) and invariant under conjugation, we deduce that Γ is convex cocompact in $\mathbb{P}(V)$. This proves property (E) of Theorem 1.17.

10.5. **(F): Inclusion into a larger space.** Let $V' = \mathbb{R}^{n'}$ and let

$$\begin{cases} j: & V & \hookrightarrow & V \oplus V' \\ j^*: & V^* & \hookrightarrow & (V \oplus V')^* \simeq V^* \oplus (V')^* \end{cases}$$

be the natural inclusions. We use the same letters for the induced inclusions of projective spaces. Let $i : \mathrm{SL}^{\pm}(V) \hookrightarrow \mathrm{SL}^{\pm}(V \oplus V')$ be the natural inclusion, whose image acts trivially on the second factor.

Suppose $\Gamma \subset \mathrm{PGL}(V)$ acts convex cocompactly on some nonempty properly convex open subset Ω of $\mathbb{P}(V)$. By Proposition 5.6, we may choose Ω so that the dual action of Γ on the dual convex set $\Omega^* \subset \mathbb{P}(V^*)$ is also convex cocompact. Let $\hat{\Gamma}$ be the lift of Γ to $\mathrm{SL}^{\pm}(V)$ that preserves a properly convex cone of V lifting Ω (see Remark 3.1).

Lemma 10.6. *(1) Let $\mathcal{K}$ be a compact subset of $\mathbb{P}(V \oplus V')$ that does not meet any projective hyperplane $z^* \in j^*(\overline{\Omega^*})$. Then any accumulation point in $\mathbb{P}(V \oplus V')$ of the $i(\hat{\Gamma})$-orbit of $\mathcal{K}$ is contained in $j(\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma))$.*

(2) Let $\mathcal{K}^*$ be a compact subset of $\mathbb{P}((V \oplus V')^*)$ whose elements correspond to projective hyperplanes disjoint from $j(\overline{\Omega})$ in $\mathbb{P}(V \oplus V')$. Then any accumulation point in $\mathbb{P}((V \oplus V')^*)$ of the $i(\hat{\Gamma})$-orbit of $\mathcal{K}^*$ is contained in $j^*(\Lambda^{\mathsf{orb}}_{\Omega^*}(\Gamma))$.

Proof. The two statements are dual to each other, so we only need to prove (1). Let $\pi : \mathbb{P}(V \oplus V') \smallsetminus \mathbb{P}(V') \to \mathbb{P}(V)$ be the map induced by the projection onto the first factor of $V \oplus V'$, so that $\pi \circ j$ is the identity of $\mathbb{P}(V)$. Note that V' is contained in every projective hyperplane of $\mathbb{P}(V \oplus V')$ corresponding to an element of $j^*(\overline{\Omega}^*)$, hence π is defined on $\mathcal{K}$. Consider a sequence $(z_m)_{m \in \mathbb{N}}$ of points of $\mathcal{K}$ and a sequence $(\gamma_m)_{m \in \mathbb{N}}$ of pairwise disjoint elements of $\hat{\Gamma}$ such that $(i(\gamma_m) \cdot z_m)_{m \in \mathbb{N}}$ converges in $\mathbb{P}(V \oplus V')$. By construction, for any m, the point z_m does not belong to any hyperplane $z^* \in j^*(\overline{\Omega^*})$, hence $\pi(z_m)$ does not belong to any hyperplane in $\overline{\Omega^*}$, i.e. $\pi(z_m) \in \Omega$. Since $\hat{\Gamma}$ acts properly discontinuously on Ω, the point $y_\infty := \lim_m \gamma_m \cdot \pi(z_m)$ is contained in $\partial\Omega$. Up to passing to a subsequence, we may assume that $(\pi(z_m))_{m \in \mathbb{N}} \subset \pi(\mathcal{K})$ converges to some $y \in \Omega$. Then $d_\Omega(\gamma_m \cdot \pi(z_m), \gamma_m \cdot y) = d_\Omega(\pi(z_m), y) \to 0$, and so $\gamma_m \cdot y \to y_\infty$ by Corollary 3.5. In particular, $y_\infty \in \Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$. Now lift z_m to a vector $x_m + x'_m \in V \oplus V'$, with $(x_m)_{m \in \mathbb{N}} \subset V \oplus \{0\}$ and $(x'_m)_{m \in \mathbb{N}} \subset \{0\} \oplus V'$ bounded. The image of x_m in $\mathbb{P}(V)$ is $\pi(z_m)$. By Lemma 3.2, the sequence $(i(\gamma_m) \cdot x_m)_{m \in \mathbb{N}}$ tends to infinity in V. On the other hand, $i(\gamma_m) \cdot x'_m = x'_m$ remains bounded in V'. Therefore,

$$\lim_m i(\gamma_m) \cdot z_m = \lim_m i(\gamma_m) \cdot j \circ \pi(z_m) = \lim_m j(\gamma_m \cdot \pi(z_m)) \in j(\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)). \qquad \square$$

The set $j(\Omega)$ is contained in the Γ-invariant open subset

$$\mathcal{O}_{\max} := \mathbb{P}(V \oplus V') \smallsetminus \bigcup_{z^* \in j^*(\overline{\Omega^*})} z^*$$

of $\mathbb{P}(V \oplus V')$, which is convex but not properly convex. This set $\mathcal{O}_{\max}$ is the union of all projective lines of $\mathbb{P}(V \oplus V')$ intersecting both $j(\Omega)$ and $\mathbb{P}(\{0\} \times V')$.

Lemma 10.7. *The set $j(\Omega)$ is contained in a Γ-invariant* properly convex *open set $\mathcal{O} \subset \mathcal{O}_{\max}$.*

Proof. We argue similarly to the proof of Lemma 8.5. Choose projective hyperplanes $H_0, \dots, H_N$ of $\mathbb{P}(V \oplus V')$ bounding an open simplex Δ containing $j(\overline{\Omega})$. Let $\mathcal{U} := \Delta \cap \mathcal{O}_{\max}$. Define

$$\mathcal{O} := \bigcap_{\gamma \in \hat{\Gamma}} i(\gamma) \cdot \mathcal{U}.$$

Then $j(\Omega) \subset \mathcal{O} \subset \mathcal{O}_{\max}$ and $\mathcal{O}$ is properly convex. We claim that $\mathcal{O}$ is open. Indeed, suppose by contradiction that there exists a point $z \in \partial_{\mathrm{n}}\mathcal{O}$. Then there is a sequence (γ_m) in $\hat{\Gamma}$ and $k \in \{0, \dots, N\}$ such that $\gamma_m \cdot H_k$ converges to a hyperplane H_∞ containing z. This is impossible since, by Lemma 10.6.(2), the hyperplane $H_\infty \in j^*(\Lambda^{\mathsf{orb}}_\Omega(\Gamma))$ supports the open set $\mathcal{O}_{\max}$ which contains z. ☐

Finally, observe that $\Lambda^{\mathsf{orb}}_{\mathcal{O}}(i(\hat{\Gamma})) = j(\Lambda^{\mathsf{orb}}_\Omega(\Gamma))$ by Lemma 10.6.(1). Hence the action of $i(\hat{\Gamma})$ on $\mathcal{C}^{\mathsf{cor}}_{\mathcal{O}}(i(\hat{\Gamma})) = j(\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma))$ is cocompact and we conclude that $i(\hat{\Gamma})$ is convex cocompact in $\mathbb{P}(V \oplus V')$.

10.6. A consequence of properties (A), (D), and (F) of Theorem 1.17.

The following result uses reasoning similar to Section 10.4.

Proposition 10.8. *Let Γ be a discrete subgroup of $\mathrm{SL}^{\pm}(V)$ acting trivially on some linear subspace V_0 of V. Then the induced action of Γ on $\mathbb{P}(V/V_0)$ is convex cocompact if and only if the action of Γ on $\mathbb{P}(V)$ is convex cocompact.*

Proof. Throughout the proof we fix a complement V_1 to V_0 in V so that $V = V_0 \oplus V_1$. We first use properties (D) and (F) to prove the forward implication. We can write all elements $\gamma \in \Gamma$ as upper triangular block matrices with respect to the decomposition $V = V_0 \oplus V_1$:

$$[\gamma]_{V_0 \oplus V_1} = \begin{bmatrix} I & B \\ 0 & A \end{bmatrix}.$$

Let Γ^{diag} be the discrete subgroup of $\mathrm{PGL}(V)$ obtained by considering only the two diagonal blocks. The action of Γ^{diag} on V_1 identifies with the action of Γ on V/V_0. Therefore the action of Γ^{diag} on $\mathbb{P}(V_1)$ is convex cocompact. By property (F) above, the action of Γ^{diag} on $\mathbb{P}(V)$ is convex cocompact. We now note that Γ^{diag} is the limit of the conjugates $g_m \Gamma g_m^{-1}$ of Γ, where g_m is a diagonal matrix acting on V_0 by $1/m$ and on V_1 by m. Since convex cocompactness is open (property (D)), we deduce that the action of Γ on $\mathbb{P}(V)$ is convex cocompact.

We now check the converse implication. Suppose Γ is convex cocompact in $\mathbb{P}(V)$. By property (A) of Theorem 1.17, the group Γ is also convex cocompact in the dual $\mathbb{P}(V^*)$, and it will be easiest to work there. There is a dual splitting $V^* = V_0^* \oplus V_1^*$, where V_0^* and V_1^* are the linear functionals which vanish on V_1 and V_0 respectively. Note that V_1^* is independent of the choice of the complement V_1 to V_0 in V, hence V_1^* is invariant under Γ. The matrix of $\gamma \in \Gamma$ for the action on V^* with respect to the concatenation of a basis for V_0^* and a basis for V_1^* has the form

$$[\gamma]_{V_0^* \oplus V_1^*} = \begin{bmatrix} I & 0 \\ {}^t B_\gamma & {}^t A_\gamma \end{bmatrix}.$$

By Lemma 3.2, the accumulation points of the orbit of any compact set of points in $\mathbb{P}(V^*) \smallsetminus \mathbb{P}(V_0^*)$ lie in $\mathbb{P}(V_1^*)$. Let $\mathcal{C}^*$ be a properly convex subset of $\mathbb{P}(V^*)$ with bisaturated boundary on which the action of Γ is properly discontinuous and cocompact. Then the ideal boundary $\partial_{\mathrm{i}}\mathcal{C}^*$ is contained in $\mathbb{P}(V_1^*)$. The intersection $\mathcal{C}^* \cap \mathbb{P}(V_1^*)$ is a properly convex subset of $\mathbb{P}(V_1^*)$ with bisaturated boundary and the action of Γ is properly discontinuous and cocompact. Hence the restriction of Γ is convex cocompact in $\mathbb{P}(V_1^*)$, and so the restriction of Γ (for the dual action, with the matrix of γ being A_γ) is convex cocompact in $\mathbb{P}(V_1)$. We now conclude in the same way as for the forward implication using property (F) of Theorem 1.17. $\square$

10.7. **A consequence of properties (E) and (F) of Theorem 1.17.**

Proposition 10.9. *Let $V'' = V \oplus V'$ be a direct sum of finite-dimensional real vector spaces, and Γ a discrete subgroup of $\mathrm{SL}^{\pm}(V) \subset \mathrm{SL}^{\pm}(V'')$. Any map $u : \Gamma \to \mathrm{Hom}(V', V)$ satisfying the cocycle relation*

$$u(\gamma_1\gamma_2) = u(\gamma_1) + \gamma_1 \cdot u(\gamma_2)$$

for all $\gamma_1, \gamma_2 \in \Gamma$ defines a discrete subgroup Γ^u of $\mathrm{SL}^{\pm}(V'')$, hence of $\mathrm{PGL}(V'')$. If Γ is convex cocompact in $\mathbb{P}(V)$, then Γ^u is convex cocompact in $\mathbb{P}(V'')$.

Recall that $u : \Gamma \to \mathrm{Hom}(V', V)$ is said to be a *coboundary* if there exists $\varphi \in \mathrm{Hom}(V', V)$ such that $u(\gamma) = \varphi - \gamma \cdot \varphi$ for all $\gamma \in \Gamma$. This is equivalent to Γ^u being conjugate to a subgroup of $\mathrm{SL}^{\pm}(V)$.

Remark 10.10. If the cocycle u is not a coboundary, then Γ^u is a discrete subgroup of $\mathrm{PGL}(V'')$ which is convex cocompact in $\mathbb{P}(V'')$ but not completely reducible. Such a cocycle u always exists e.g. if Γ is a free group. See also Example 10.11.

Example 10.11. We briefly examine the special case where $V = \mathbb{R}^3$, where $V' = \mathbb{R}^1$, and where Γ is a discrete subgroup of $\mathrm{PO}(2,1) \subset \mathrm{PGL}(\mathbb{R}^3)$ isomorphic to the fundamental group of a closed orientable surface S_g of genus $g \geq 2$. In this case, Γ represents a point of the Teichmüller space $\mathcal{T}(S_g)$ and the space of cohomology classes of cocycles $u : \Gamma \to \mathrm{Hom}(V', V) \simeq V$

identifies with the tangent space to $\mathcal{T}(S_g)$ at Γ (see e.g. [DGK1, § 2.3]) and has dimension $6g-6$.

(1) By Proposition 10.9, the group Γ^u preserves and acts convex cocompactly on a nonempty properly convex open subset Ω of $\mathbb{P}(\mathbb{R}^3 \oplus \mathbb{R}^1)$. The Γ^u-invariant convex open set $\Omega_{\max} \supset \Omega$ given by Proposition 4.5.(3) turns out to be properly convex as soon as u is not a coboundary. It can be described as follows. The group Γ^u preserves an affine chart $U \simeq \mathbb{R}^3$ of $\mathbb{P}(\mathbb{R}^3 \oplus \mathbb{R}^1)$ and a flat Lorentzian metric on U. The action on U is not properly discontinuous. However, Mess [Me] described two maximal globally hyperbolic domains of discontinuity, called *domains of dependence*, one oriented to the future and one oriented to the past. The domain $\Omega_{\max}$ is the union of the two domains of dependence plus a copy of the hyperbolic plane $\mathbb{H}^2$ in $\mathbb{P}(\mathbb{R}^3 \oplus 0)$. We have $\Lambda^{\mathsf{orb}}_{\Omega_{\max}}(\Gamma^u) \subset \partial\mathbb{H}^2 \subset \mathbb{P}(\mathbb{R}^3 \oplus 0)$ and $\mathcal{C}^{\mathsf{cor}}_{\Omega_{\max}}(\Gamma^u) \subset \mathbb{H}^2$.

(2) By Theorem 1.17.(A), the group Γ^u also acts convex cocompactly on a nonempty properly convex open subset Ω^* of the dual projective space $\mathbb{P}((\mathbb{R}^3 \oplus \mathbb{R})^*)$. In this case the Γ^u-invariant convex open set $(\Omega^*)_{\max} \supset \Omega^*$, given by Proposition 4.5.(3) applied to Ω^*, is *not* properly convex: it is the suspension HP^3 of the dual copy $(\mathbb{H}^2)^* \subset \mathbb{P}((\mathbb{R}^3)^*)$ of the hyperbolic plane with the point $\mathbb{P}((\mathbb{R}^1)^*)$. The convex set $\mathrm{HP}^3 \subset \mathbb{P}((\mathbb{R}^3 \oplus \mathbb{R})^*)$ is the projective model for *half-pipe geometry*, a transitional geometry lying between hyperbolic geometry and anti-de Sitter geometry (see [Dan]). The full orbital limit set $\Lambda^{\mathsf{orb}}_{\Omega^*}(\Gamma) = \overline{\Omega^*} \cap \partial\mathrm{HP}^3$ is *not* contained in a hyperplane if u is not a coboundary. Indeed, $\mathcal{C}^{\mathsf{cor}}_{\Omega^*}(\Gamma^u)$ may be thought of as a rescaled limit of the collapsing convex cores for quasi-Fuchsian subgroups of $\mathrm{PO}(3,1)$ (or of $\mathrm{PO}(2,2)$) which converge to the Fuchsian group Γ. This situation was described in some detail by Kerckhoff in a 2010 lecture at the workshop on Geometry, topology, and dynamics of character varieties at the National University of Singapore, and other lectures in 2011 and 2012 about the work [DK] (still in preparation).

Proof of Proposition 10.9. Suppose Γ is convex cocompact in $\mathbb{P}(V)$. Then its image $i(\Gamma)$ in $\mathrm{PGL}(V'')$ is convex cocompact in $\mathbb{P}(V'')$ by Theorem 1.17.(F). But $i(\Gamma)$ is the semisimplification of Γ^u. Therefore Γ^u is convex cocompact in $\mathbb{P}(V'')$ by Theorem 1.17.(E). $\square$

11. Convex cocompactness in $\mathbb{H}^{p,q-1}$

Fix $p, q \in \mathbb{N}^*$. Recall from Section 1.8 that the projective space $\mathbb{P}(\mathbb{R}^{p+q})$ is the disjoint union of

$$\mathbb{H}^{p,q-1} = \big\{[x] \in \mathbb{P}(\mathbb{R}^{p,q}) \mid \langle x, x\rangle_{p,q} < 0\big\},$$

of $\mathbb{S}^{p-1,q} = \{[x] \in \mathbb{P}(\mathbb{R}^{p,q}) \mid \langle x, x\rangle_{p,q} > 0\}$, and of

$$\partial\mathbb{H}^{p,q-1} = \partial\mathbb{S}^{p-1,q} = \big\{[x] \in \mathbb{P}(\mathbb{R}^{p,q}) \mid \langle x, x\rangle_{p,q} = 0\big\}.$$

For instance, Figure 5 shows

$$\mathbb{P}(\mathbb{R}^4) = \mathbb{H}^{3,0} \sqcup \big(\partial\mathbb{H}^{3,0} = \partial\mathbb{S}^{2,1}\big) \sqcup \mathbb{S}^{2,1}$$

and

$$\mathbb{P}(\mathbb{R}^4) = \mathbb{H}^{2,1} \sqcup \left(\partial\mathbb{H}^{2,1} = \partial\mathbb{S}^{1,2}\right) \sqcup \mathbb{S}^{1,2}.$$

As explained in [DGK3, § 2.1], the space $\mathbb{H}^{p,q-1}$ has a natural $\mathrm{PO}(p,q)$-

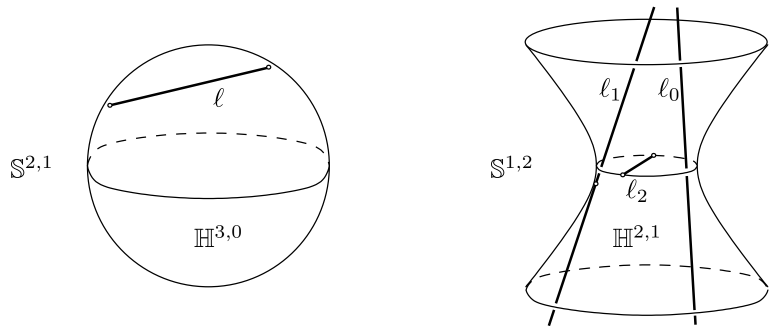


FIGURE 5. Left: $\mathbb{H}^{3,0}$ with a geodesic line ℓ (necessarily spacelike), and $\mathbb{S}^{2,1}$. Right: $\mathbb{H}^{2,1}$ with three geodesic lines ℓ_2 (spacelike), ℓ_1 (lightlike), and ℓ_0 (timelike), and $\mathbb{S}^{1,2}$.

invariant pseudo-Riemannian structure of constant negative curvature, for which the geodesic lines are the intersections of $\mathbb{H}^{p,q-1}$ with projective lines in $\mathbb{P}(\mathbb{R}^{p,q})$. Such a line is called *spacelike* (resp. *lightlike*, resp. *timelike*) if it meets $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$ in two (resp. one, resp. zero) points: see Figure 5.

Remark 11.1. An element $g \in \mathrm{PO}(p,q)$ is proximal in $\mathbb{P}(\mathbb{R}^{p+q})$ (Definition 2.1) if and only if it admits a unique attracting fixed point ξ_g^+ in $\partial\mathbb{H}^{p,q-1}$, in which case we shall say that g is proximal in $\partial\mathbb{H}^{p,q-1}$. In particular, for a discrete subgroup Γ of $\mathrm{PO}(p,q) \subset \mathrm{PGL}(\mathbb{R}^{p+q})$, the proximal limit set Λ_Γ of Γ in $\mathbb{P}(\mathbb{R}^{p,q})$ (Definition 2.2) is contained in $\partial\mathbb{H}^{p,q-1}$, and called the proximal limit set of Γ in $\partial\mathbb{H}^{p,q-1}$.

In this section we prove Theorems 1.25 and 1.29.

For Theorem 1.25, the equivalences (7) $\Leftrightarrow$ (8) $\Leftrightarrow$ (9) $\Leftrightarrow$ (10) follow from the equivalences (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) by replacing the symmetric bilinear form $\langle\cdot,\cdot\rangle_{p,q}$ by $-\langle\cdot,\cdot\rangle_{p,q} \simeq \langle\cdot,\cdot\rangle_{q,p}$: see [DGK3, Rem. 4.1]. The implication (3) $\Rightarrow$ (6) is contained in the forward implication of Theorem 1.4, which has been established in Section 7. Here we shall prove the implications (6) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3) and (1) $\Rightarrow$ (2). We note that (2) $\Rightarrow$ (1) is contained in (4) $\Rightarrow$ (3) and (8) $\Rightarrow$ (7).

11.1. **Proof of the implication** (6) $\Rightarrow$ (5) **in Theorem 1.25.** Suppose that Γ is word hyperbolic, that the natural inclusion $\Gamma \hookrightarrow \mathrm{PO}(p,q)$ is $P_1^{p,q}$-Anosov, and that the proximal limit set $\Lambda_\Gamma \subset \partial\mathbb{H}^{p,q-1}$ is negative.

By definition of negativity, the set Λ_Γ lifts to a cone $\widetilde{\Lambda}_\Gamma$ of $\mathbb{R}^{p,q} \smallsetminus \{0\}$ on which all inner products $\langle\cdot,\cdot\rangle_{p,q}$ of noncollinear points are negative. The set

$$\Omega_{\max} := \mathbb{P}\big(\big\{x \in \mathbb{R}^{p,q} \mid \langle x, x'\rangle_{p,q} < 0 \quad \forall x' \in \widetilde{\Lambda}_\Gamma\big\}\big)$$

is a connected component of $\mathbb{P}(\mathbb{R}^{p,q}) \smallsetminus \bigcup_{z\in\Lambda_\Gamma} z^\perp$ which is open and convex. It is nonempty because it contains $x_1 + x_2$ for any noncollinear $x_1, x_2 \in \widetilde{\Lambda}_\Gamma$. Note that if $\widehat{\Lambda}_\Gamma$ is another cone of $\mathbb{R}^{p,q} \smallsetminus \{0\}$ lifting Λ_Γ on which all inner products $\langle\cdot,\cdot\rangle_{p,q}$ of noncollinear points are negative, then either $\widehat{\Lambda}_\Gamma = \widetilde{\Lambda}_\Gamma$ or $\widehat{\Lambda}_\Gamma = -\widetilde{\Lambda}_\Gamma$ (using negativity). Thus $\Omega_{\max}$ is well defined independently of the lift $\widetilde{\Lambda}_\Gamma$, and so $\Omega_{\max}$ is Γ-invariant because Λ_Γ is.

By Proposition 8.1, the group Γ acts convex cocompactly on some nonempty, properly convex open subset $\Omega \subset \Omega_{\max}$, and $\Lambda^{\mathsf{orb}}_\Omega(\Gamma) = \Lambda_\Gamma$. In particular, the convex hull $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ of Λ_Γ in Ω has compact quotient by Γ.

As in [DGK3, Lem. 3.6.(1)], this convex hull $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ is contained in $\mathbb{H}^{p,q-1}$: indeed, any point of $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ lifts to a vector of the form $x = \sum_{i=1}^k t_i x_i$ with $k \geq 2$, where $x_1, \dots, x_k \in \widetilde{\Lambda}_\Gamma$ are distinct and $t_1, \dots, t_k > 0$; for all $i \neq j$ we have $\langle x_i, x_i\rangle_{p,q} = 0$ and $\langle x_i, x_j\rangle_{p,q} < 0$ by negativity of Λ_Γ, hence $\langle x, x\rangle_{p,q} < 0$.

Since $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ has compact quotient by Γ, it is easy to check that for small enough $r > 0$ the open uniform r-neighborhood $\mathcal{U}$ of $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ in (Ω, d_Ω) is contained in $\mathbb{H}^{p,q-1}$ and properly convex (see [DGK3, Lem. 6.3]). The full orbital limit set $\Lambda^{\mathsf{orb}}_{\mathcal{U}}(\Gamma)$ is a nonempty closed Γ-invariant subset of $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$, hence its convex hull $\mathcal{C}^{\mathsf{cor}}_{\mathcal{U}}(\Gamma)$ in $\mathcal{U}$ is a nonempty closed Γ-invariant subset of $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$. (In fact $\mathcal{C}^{\mathsf{cor}}_{\mathcal{U}}(\Gamma) = \mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ by Lemma 4.1.(1).) The nonempty set $\mathcal{C}^{\mathsf{cor}}_{\mathcal{U}}(\Gamma)$ has compact quotient by Γ. Thus Γ acts convex cocompactly on $\mathcal{U} \subset \mathbb{H}^{p,q-1}$, and Λ_Γ is transverse since it is negative.

11.2. **Proof of the implication** (5) $\Rightarrow$ (4) **in Theorem 1.25.** Suppose that Γ acts convex cocompactly on some nonempty properly convex open subset Ω of $\mathbb{H}^{p,q-1}$ and that the proximal limit set Λ_Γ is transverse. The set Λ_Γ is contained in $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$. By Lemma 4.1.(1)–(2), the convex hull of Λ_Γ in Ω is the convex hull $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ of $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ in Ω, and the ideal boundary $\partial_{\mathrm{i}}\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ is equal to $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$. By Corollary 3.3, this set is contained in $\partial\mathbb{H}^{p,q-1}$. It follows that $\Lambda_\Gamma = \Lambda^{\mathsf{orb}}_\Omega(\Gamma)$: indeed, otherwise $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ would contain a nontrivial segment between two points of Λ_Γ, this segment would be contained in $\partial\mathbb{H}^{p,q-1}$, and this would contradict the assumption that Λ_Γ is transverse. Thus $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ is a closed properly convex subset of $\mathbb{H}^{p,q-1}$ on which Γ acts properly discontinuously and cocompactly, and whose ideal boundary does not contain any nontrivial projective line segment.

It could be the case that $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ has empty interior. However, for any $r > 0$ the closed uniform r-neighborhood $\mathcal{C}_r$ of $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ in (Ω, d_Ω) has nonempty interior, and is still properly convex with compact quotient by Γ. By Lemma 4.1.(1), we have $\partial_{\mathrm{i}}\mathcal{C}_r = \Lambda^{\mathsf{orb}}_\Omega(\Gamma)$, hence $\partial_{\mathrm{i}}\mathcal{C}_r$ does not contain any nontrivial projective line segment. This shows that Γ is $\mathbb{H}^{p,q-1}$-convex cocompact.

11.3. **Proof of the implication** (1) $\Rightarrow$ (2) **in Theorem 1.25.** The proof relies on the following proposition, which is stated in [DGK3, Prop. 3.7] for irreducible Γ. The proof for general Γ is literally the same; we recall it for the reader's convenience.

Proposition 11.2. *For $p,q \in \mathbb{N}^*$, let Γ be a discrete subgroup of $\mathrm{PO}(p,q)$ preserving a nonempty properly convex open subset Ω of $\mathbb{P}(\mathbb{R}^{p,q})$. Let $\Lambda_\Gamma \subset \partial\mathbb{H}^{p,q-1}$ be the proximal limit set of Γ (Definition 2.2 and Remark 11.1). If Λ_Γ contains at least two points and is transverse, and if the action of Γ on Λ_Γ is minimal (i.e. every orbit is dense), then Λ_Γ is negative or positive.*

Recall from Remark 2.3 that if Γ is irreducible with $\Lambda_\Gamma \neq \emptyset$, then the action of Γ on Λ_Γ is always minimal.

Proof. Suppose that Λ_Γ contains at least two points and is transverse, and that the action of Γ on Λ_Γ is minimal. By Proposition 4.5.(2), the sets Ω and Λ_Γ lift to cones $\widetilde{\Omega}$ and $\widetilde{\Lambda}_\Gamma$ of $V \smallsetminus \{0\}$ with $\widetilde{\Omega}$ properly convex containing $\widetilde{\Lambda}_\Gamma$ in its boundary, and Ω^* and Λ_Γ^* lift to cones $\widetilde{\Omega}^*$ and $\widetilde{\Lambda}_\Gamma^*$ of $V^* \smallsetminus \{0\}$ with $\widetilde{\Omega}^*$ properly convex containing $\widetilde{\Lambda}_\Gamma^*$ in its boundary, such that $\ell(x) \geq 0$ for all $x \in \widetilde{\Lambda}_\Gamma$ and $\ell \in \widetilde{\Lambda}_\Gamma^*$. By Remark 3.1, the group Γ lifts to a discrete subgroup $\hat{\Gamma}$ of $\mathrm{O}(p,q)$ preserving $\widetilde{\Omega}$ (hence also $\widetilde{\Lambda}_\Gamma$, $\widetilde{\Omega}^*$, and $\widetilde{\Lambda}_\Gamma^*$). Note that the map $\psi : x \mapsto \langle x, \cdot \rangle_{p,q}$ from $\mathbb{R}^{p,q}$ to $(\mathbb{R}^{p,q})^*$ induces a homeomorphism $\Lambda_\Gamma \simeq \Lambda_\Gamma^*$. For any $x \in \widetilde{\Lambda}_\Gamma$ we have $\psi(x) \in \widetilde{\Lambda}_\Gamma^* \cup -\widetilde{\Lambda}_\Gamma^*$. Let F^+ (resp. F^-) be the subcone of $\widetilde{\Lambda}_\Gamma$ consisting of those vectors x such that $\psi(x) \in \widetilde{\Lambda}_\Gamma^*$ (resp. $\psi(x) \in -\widetilde{\Lambda}_\Gamma^*$). By construction, we have $x \in F^+$ if and only if $\langle x, x' \rangle_{p,q} \geq 0$ for all $x' \in \widetilde{\Lambda}_\Gamma$; in particular, F^+ is closed in $\widetilde{\Lambda}_\Gamma$ and $\hat{\Gamma}$-invariant. Similarly, F^- is closed and $\hat{\Gamma}$-invariant. The sets F^+ and F^- are disjoint since Λ_Γ contains at least two points and is transverse. Thus F^+ and F^- are disjoint, $\hat{\Gamma}$-invariant, closed subcones of $\widetilde{\Lambda}_\Gamma$, whose projections to $\mathbb{P}(\mathbb{R}^{p,q})$ are disjoint, Γ-invariant, closed subsets of Λ_Γ. Since the action of Γ on Λ_Γ is minimal, Λ_Γ is the smallest nonempty Γ-invariant closed subset of $\mathbb{P}(\mathbb{R}^{p,q})$, and so $\{F^+, F^-\} = \{\widetilde{\Lambda}_\Gamma, \emptyset\}$. If $\widetilde{\Lambda}_\Gamma = F^+$ then Λ_Γ is nonnegative, hence positive by transversality. Similarly, if $\widetilde{\Lambda}_\Gamma = F^-$ then Λ_Γ is negative. ☐

We can now prove the implication (1) ⇒ (2) in Theorem 1.25.

Suppose Γ is strongly convex cocompact in $\mathbb{P}(\mathbb{R}^{p+q})$. By the implication (i) ⇒ (vi) of Theorem 1.15, the group Γ is word hyperbolic and the natural inclusion $\Gamma \hookrightarrow \mathrm{PO}(p,q)$ is $P_1^{p,q}$-Anosov. If $\#\Lambda_\Gamma > 2$, then the action of Γ on $\partial_\infty\Gamma$, hence on Λ_Γ, is minimal, and so Proposition 11.2 implies that the set Λ_Γ is negative or positive. This last conclusion also holds, vacuously, if $\#\Lambda_\Gamma \leq 2$. Thus the implications (6) ⇒ (4) and (10) ⇒ (8) of Theorem 1.25 (proved in Sections 11.1 and 11.2 just above) show that Γ is $\mathbb{H}^{p,q-1}$-convex cocompact or $\mathbb{H}^{q,p-1}$-convex cocompact. This completes the proof of (1) ⇒ (2) in Theorem 1.25.

11.4. **Proof of the implication (4) ⇒ (3) in Theorem 1.25.** Suppose $\Gamma \subset \mathrm{PO}(p,q)$ is $\mathbb{H}^{p,q-1}$-convex cocompact, i.e. it acts properly discontinuously and cocompactly on a closed convex subset $\mathcal{C}$ of $\mathbb{H}^{p,q-1}$ such that $\mathcal{C}$ has nonempty interior and $\partial_{\mathrm{i}}\mathcal{C}$ does not contain any nontrivial projective line segment. We shall first show that Γ satisfies condition (v) of Theorem 1.15: namely, Γ preserves a nonempty properly convex open subset Ω of $\mathbb{P}(\mathbb{R}^{p,q})$ and acts cocompactly on a closed convex subset $\mathcal{C}'$ of Ω with nonempty interior, such that $\partial_{\mathrm{i}}\mathcal{C}'$ does not contain any nontrivial segment.

One naive idea would be to take $\Omega = \mathrm{Int}(\mathcal{C})$ and $\mathcal{C}'$ to be the convex hull $\mathcal{C}_0$ of $\partial_{\mathrm{i}}\mathcal{C}$ in $\mathcal{C}$ (or some small thickening), but $\mathcal{C}_0$ might not be contained in $\mathrm{Int}(\mathcal{C})$ (e.g. if $\mathcal{C} = \mathcal{C}_0$). So we must find a larger open set Ω. Lemma 11.3 below implies that the Γ-invariant convex open subset $\Omega_{\max} \supset \mathrm{Int}(\mathcal{C})$ of Proposition 4.5 contains $\mathcal{C}$. If $\Omega_{\max}$ is properly convex (e.g. if Γ is irreducible), then we may take $\Omega = \Omega_{\max}$ and $\mathcal{C}' = \mathcal{C}$. However, $\Omega_{\max}$ might not be properly convex; we shall show (Lemma 11.4) that the Γ-invariant properly convex open set $\Omega = \mathrm{Int}(\mathcal{C})^*$ (realized in the same projective space via the quadratic form) contains $\mathcal{C}_0$ (though possibly not $\mathcal{C}$) and we shall take $\mathcal{C}'$ to be the intersection of $\mathcal{C}$ with a neighborhood of $\mathcal{C}_0$ in Ω.

The following key observation is similar to [DGK3, Lem. 4.2].

Lemma 11.3. *For $p, q \in \mathbb{N}^*$, let Γ be a discrete subgroup of* $\mathrm{PO}(p,q)$ *acting properly discontinuously and cocompactly on a nonempty properly convex closed subset $\mathcal{C}$ of $\mathbb{H}^{p,q-1}$. Then $\mathcal{C}$ does not meet any hyperplane $z^\perp$ with $z \in \partial_{\mathrm{i}}\mathcal{C}$. In other words, any point of $\mathcal{C}$ sees any point of $\partial_{\mathrm{i}}\mathcal{C}$ in a spacelike direction.*

Proof. Suppose by contradiction that $\mathcal{C}$ meets $z^\perp$ for some $z \in \partial_{\mathrm{i}}\mathcal{C}$. Then $z^\perp$ contains a ray $[y, z) \subset \partial_{\mathrm{n}}\mathcal{C}$. Let $(a_m)_{m\in\mathbb{N}}$ be a sequence of points of $[y, z)$ converging to z (see Figure 6). Since Γ acts cocompactly on $\mathcal{C}$, for any m there exists $\gamma_m \in \Gamma$ such that $\gamma_m \cdot a_m$ belongs to a fixed compact subset of $\partial_{\mathrm{n}}\mathcal{C}$. Up to taking a subsequence, the sequences $(\gamma_m \cdot a_m)_m$ and $(\gamma_m \cdot y)_m$ and $(\gamma_m \cdot z)_m$ converge respectively to some points $a_\infty, y_\infty, z_\infty$ in $\mathbb{P}(\mathbb{R}^{p,q})$. We have $a_\infty \in \partial_{\mathrm{n}}\mathcal{C}$ and $y_\infty \in \partial_{\mathrm{i}}\mathcal{C}$ (because the action of Γ on $\mathcal{C}$ is properly discontinuous) and $z_\infty \in \partial_{\mathrm{i}}\mathcal{C}$ (because $\partial_{\mathrm{i}}\mathcal{C} = \mathrm{Fr}(\mathcal{C}) \cap \partial\mathbb{H}^{p,q-1}$ is closed in $\mathbb{P}(\mathbb{R}^{p,q})$). The segment $[y_\infty, z_\infty]$ is contained in $z_\infty^\perp$, hence its intersection with $\mathbb{H}^{p,q-1}$ is contained in a lightlike geodesic and can meet $\partial\mathbb{H}^{p,q-1}$ only at z_∞. Therefore $y_\infty = z_\infty$ and the closure of $\mathcal{C}$ in $\mathbb{P}(\mathbb{R}^{p,q})$ contains a full projective line, contradicting the proper convexity of $\mathcal{C}$. ☐

Lemma 11.4. *In the setting of Lemma 11.3, suppose $\mathcal{C}$ has nonempty interior. Then the convex hull $\mathcal{C}_0$ of $\partial_{\mathrm{i}}\mathcal{C}$ in $\mathcal{C}$ is contained in the Γ-invariant properly convex open set*

$$\Omega := \big\{y \in \mathbb{P}(\mathbb{R}^{p,q}) \mid y^\perp \cap \overline{\mathcal{C}} = \emptyset\big\}.$$

Note that Ω is the dual of $\mathrm{Int}(\mathcal{C})$ realized in $\mathbb{R}^{p,q}$ (rather than $(\mathbb{R}^{p,q})^*$) via the symmetric bilinear form $\langle\cdot,\cdot\rangle_{p,q}$.

Proof. The properly convex set $\mathcal{C} \subset \mathbb{H}^{p,q-1}$ lifts to a properly convex cone $\widetilde{\mathcal{C}}$ of $\mathbb{R}^{p,q} \smallsetminus \{0\}$ such that $\langle x, x\rangle_{p,q} < 0$ for all $x \in \widetilde{\mathcal{C}}$. We denote by $\widetilde{\mathcal{C}}_0 \subset \widetilde{\mathcal{C}}$

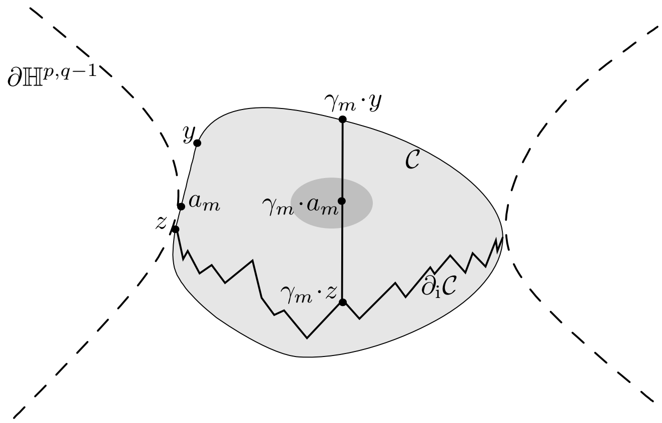


FIGURE 6. Illustration for the proof of Lemma 11.3

the preimage of $\mathcal{C}_0$. The ideal boundary $\partial_{\mathrm{i}}\mathcal{C}$ lifts to the intersection $\widetilde{\partial_{\mathrm{i}}\mathcal{C}} \subset \mathbb{R}^{p,q} \smallsetminus \{0\}$ of the closure $\overline{\widetilde{\mathcal{C}}}$ of $\widetilde{\mathcal{C}}$ with the null cone of $\langle\cdot,\cdot\rangle_{p,q}$ minus $\{0\}$. For any $x \in \overline{\widetilde{\mathcal{C}}}$ and $x' \in \widetilde{\partial_{\mathrm{i}}\mathcal{C}}$, we have $\langle x, x'\rangle_{p,q} \leq 0$: indeed, this is easily seen by considering $tx+x' \in \mathbb{R}^{p,q}$, which for small $t > 0$ must belong to $\overline{\widetilde{\mathcal{C}}}$ hence have nonpositive norm; see also [DGK3, Lem. 3.6.(1)]. By Lemma 11.3 we have in fact $\langle x, x'\rangle_{p,q} < 0$ for all $x \in \widetilde{\mathcal{C}}$ and $x' \in \widetilde{\partial_{\mathrm{i}}\mathcal{C}}$. In particular, this holds for $x \in \widetilde{\mathcal{C}_0}$. Now consider $x \in \widetilde{\mathcal{C}_0}$ and $x' \in \widetilde{\mathcal{C}}$. Since $\mathcal{C}_0$ is the convex hull of $\partial_{\mathrm{i}}\mathcal{C}$ in $\mathcal{C}$, we may write $x = \sum_{i=1}^{k} t_i x_i$ where $x_1, \dots, x_k \in \widetilde{\partial_{\mathrm{i}}\mathcal{C}}$ and $t_1, \dots, t_k > 0$. By Lemma 11.3 we have $\langle x_i, x'\rangle_{p,q} < 0$ for all i, hence $\langle x, x'\rangle_{p,q} < 0$. This proves that $\mathcal{C}_0$ is contained in Ω. The open set Ω is properly convex because $\mathcal{C}$ has nonempty interior. ☐

Corollary 11.5. *In the setting of Lemma 11.4, the group Γ acts cocompactly on some closed properly convex subset $\mathcal{C}'$ of Ω with nonempty interior which is contained in $\mathcal{C} \subset \mathbb{H}^{p,q-1}$, with $\partial_{\mathrm{i}}\mathcal{C}' = \partial_{\mathrm{i}}\mathcal{C}$.*

Proof. Since the action of Γ on $\mathcal{C}_0$ is cocompact, it is easy to check that for any small enough $r > 0$ the closed r-neighborhood $\mathcal{C}_r$ of $\mathcal{C}_0$ in (Ω, d_Ω) is contained in $\mathbb{H}^{p,q-1}$ (see [DGK3, Lem. 6.3]). The set $\mathcal{C}' := \mathcal{C}_r \cap \mathcal{C}$ is then a closed properly convex subset of Ω with nonempty interior, and Γ acts properly discontinuously and cocompactly on $\mathcal{C}'$. Since $\mathcal{C}'$ is also a closed subset of $\mathbb{H}^{p,q-1}$, we have $\partial_{\mathrm{i}}\mathcal{C}' = \overline{\mathcal{C}'} \cap \partial\mathbb{H}^{p,q-1} = \partial_{\mathrm{i}}\mathcal{C}_r \cap \partial_{\mathrm{i}}\mathcal{C} = \partial_{\mathrm{i}}\mathcal{C}$. ☐

If $\partial_{\mathrm{i}}\mathcal{C}$ does not contain any nontrivial segment, then neither does $\partial_{\mathrm{i}}\mathcal{C}'$ for $\mathcal{C}'$ as in Lemma 11.5. This proves that if Γ is $\mathbb{H}^{p,q-1}$-convex cocompact (i.e. satisfies (4) of Theorem 1.25, i.e. (2) up to switching p and q), then it satisfies condition (v) of Theorem 1.15, as announced.

Now, Theorem 1.15 states that (v) is equivalent to strong convex cocompactness: this yields the implication (2) ⇒ (1) of Theorem 1.25. Condition (vi) of Theorem 1.15, equivalent to (v), also says that the group Γ is word hyperbolic and that the natural inclusion $\Gamma \hookrightarrow \mathrm{PO}(p,q)$ is $P_1^{p,q}$-Anosov. Since in addition the proximal limit set Λ_Γ is the ideal boundary $\partial_{\mathrm{i}}\mathcal{C}'$, we find that Λ_Γ is negative.

This completes the proof of the implication (4) ⇒ (3) in Theorem 1.25.

11.5. **End of the proof of Theorem 1.25.** Suppose Γ is irreducible and acts convex cocompactly on some nonempty properly convex open subset Ω of $\mathbb{P}(\mathbb{R}^{p+q})$. Let us prove that Γ is strongly convex cocompact in $\mathbb{P}(\mathbb{R}^{p+q})$.

Let Λ_Γ be the proximal limit set of Γ in $\partial\mathbb{H}^{p,q-1}$ (Definition 2.2 and Remark 11.1); it is a nonempty closed Γ-invariant subset of $\partial\mathbb{H}^{p,q-1}$ which is contained in the full orbital limit set $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ (see Remark 2.3). By minimality (Lemma 4.1.(2)), the set $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ is the convex hull of Λ_Γ in Ω. By [DGK3, Prop. 3.7], the set $\Lambda_\Gamma \subset \partial\mathbb{H}^{p,q-1}$ is nonpositive or nonnegative, i.e. it lifts to a cone $\widetilde{\Lambda}_\Gamma$ of $\mathbb{R}^{p,q} \smallsetminus \{0\}$ on which $\langle\cdot,\cdot\rangle_{p,q}$ is everywhere nonpositive or everywhere nonnegative; in the first (resp. second) case $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ is contained in $\mathbb{H}^{p,q-1} \cup \partial\mathbb{H}^{p,q-1}$ (resp. $\mathbb{S}^{p-1,q} \cup \partial\mathbb{H}^{p,q-1}$) (see [DGK3, Lem. 3.6.(1)]). By Proposition 4.5 (see also [DGK3, Prop. 3.7]), there is a unique largest Γ-invariant properly convex open domain $\Omega_{\max}$ of $\mathbb{P}(V)$ containing Ω, namely the projectivization of the interior of the set of $x' \in \mathbb{R}^{p,q}$ such that $\langle x, x'\rangle_{p,q} \leq 0$ for all $x \in \widetilde{\Lambda}_\Gamma$ (resp. $\langle x, x'\rangle_{p,q} \geq 0$ for all $x \in \widetilde{\Lambda}_\Gamma$). Suppose by contradiction that $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ contains a PET. By Lemma 4.1.(1), we have $\partial_{\mathrm{i}}\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma) = \Lambda^{\mathsf{orb}}_\Omega(\Gamma) \subset \partial\mathbb{H}^{p,q-1}$, hence the edges of the PET lie in $\partial\mathbb{H}^{p,q-1}$. In particular, $\langle a,b\rangle_{p,q} = \langle a,c\rangle_{p,q} = 0$, and so the PET is entirely contained in $a^\perp \cap \overline{\Omega_{\max}} \subset \partial\Omega_{\max}$, contradicting the fact that $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma) \subset \Omega \subset \Omega_{\max}$. This shows that $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ does not contain any PET, and so Γ is strongly convex cocompact in $\mathbb{P}(\mathbb{R}^{p+q})$ by Theorem 1.15.

11.6. **Proof of Theorem 1.29.** The implications (4) ⇒ (3) ⇒ (2) of Theorem 1.29 hold trivially. We now prove (1) ⇔ (2) ⇒ (4). We start with the following observation.

Lemma 11.6. *For $p,q \in \mathbb{N}^*$, let Γ be an infinite discrete subgroup of $\mathrm{PO}(p,q)$ acting properly discontinuously and cocompactly on a closed convex subset $\mathcal{C}$ of $\mathbb{H}^{p,q-1}$ with nonempty interior. Then $\mathcal{C}$ has bisaturated boundary if and only if $\partial_{\mathrm{n}}\mathcal{C}$ does not contain any infinite geodesic line of $\mathbb{H}^{p,q-1}$.*

Proof. Suppose $\partial_{\mathrm{n}}\mathcal{C}$ contains an infinite geodesic line of $\mathbb{H}^{p,q-1}$. Since $\mathcal{C}$ is properly convex and closed in $\mathbb{H}^{p,q-1}$, this line must meet $\partial\mathbb{H}^{p,q-1}$ in a point of $\partial_{\mathrm{i}}\mathcal{C}$, and so $\mathcal{C}$ does not have bisaturated boundary.

Conversely, suppose $\mathcal{C}$ does not have bisaturated boundary. Since $\partial_{\mathrm{i}}\mathcal{C} = \mathrm{Fr}(\mathcal{C}) \cap \partial\mathbb{H}^{p,q-1}$ is closed in $\mathbb{P}(\mathbb{R}^{p,q})$, there exists a ray $[y,z) \subset \partial_{\mathrm{n}}\mathcal{C}$ terminating at a point $z \in \partial_{\mathrm{i}}\mathcal{C}$. Let $(a_m)_{m\in\mathbb{N}}$ be a sequence of points of $[y,z)$ converging to z (see Figure 6). Since Γ acts cocompactly on $\mathcal{C}$, for any m there exists $\gamma_m \in \Gamma$ such that $\gamma_m \cdot a_m$ belongs to a fixed compact subset of $\partial_{\mathrm{n}}\mathcal{C}$. Up to taking a subsequence, the sequences $(\gamma_m \cdot a_m)_m$ and $(\gamma_m \cdot y)_m$ and $(\gamma_m \cdot z)_m$ converge respectively to some points $a_\infty, y_\infty, z_\infty$ in $\mathbb{P}(V)$. We have $a_\infty \in \partial_{\mathrm{n}}\mathcal{C}$

and $y_\infty \in \partial_{\mathrm{i}}\mathcal{C}$ (because the action of Γ on $\mathcal{C}$ is properly discontinuous) and $z_\infty \in \partial_{\mathrm{i}}\mathcal{C}$ (because $\partial_{\mathrm{i}}\mathcal{C}$ is closed). Moreover, $a_\infty \in (y_\infty, z_\infty)$. Thus (y_∞, z_∞) is an infinite geodesic line of $\mathbb{H}^{p,q-1}$ contained in $\partial_{\mathrm{n}}\mathcal{C}$. □

Suppose condition (1) of Theorem 1.29 holds, i.e. $\Gamma < \mathrm{PO}(p,q)$ is $\mathbb{H}^{p,q-1}$-convex cocompact. By Corollary 11.5, the group Γ preserves a properly convex open subset Ω of $\mathbb{P}(\mathbb{R}^{p,q})$ and acts cocompactly on a closed convex subset $\mathcal{C}'$ of Ω with nonempty interior which is contained in $\mathbb{H}^{p,q-1}$, and whose ideal boundary $\partial_{\mathrm{i}}\mathcal{C}'$ does not contain any nontrivial segment. As in the proof of Corollary 11.5, for small enough $r > 0$ the closed r-neighborhood $\mathcal{C}'_r$ of $\mathcal{C}'$ in (Ω, d_Ω) is contained in $\mathbb{H}^{p,q-1}$. By Lemma 4.1.(1)–(2), we have $\partial_{\mathrm{i}}\mathcal{C}'_r = \partial_{\mathrm{i}}\mathcal{C}' = \Lambda^{\mathsf{orb}}_\Omega(\Gamma)$, hence $\mathcal{C}'_r$ has bisaturated boundary by Lemma 4.1.(3). In particular, $\partial_{\mathrm{n}}\mathcal{C}'_r$ does not contain any infinite geodesic line of $\mathbb{H}^{p,q-1}$ by Lemma 11.6. Thus condition (2) of Theorem 1.29 holds, with $\mathcal{C}_{\mathsf{bisat}} = \mathcal{C}'_r$.

Conversely, suppose condition (2) of Theorem 1.29 holds, i.e. Γ acts properly discontinuously and cocompactly on a closed convex subset $\mathcal{C}_{\mathsf{bisat}}$ of $\mathbb{H}^{p,q-1}$ such that $\partial_{\mathrm{n}}\mathcal{C}_{\mathsf{bisat}}$ does not contain any infinite geodesic line of $\mathbb{H}^{p,q-1}$. By Lemma 11.6, the set $\mathcal{C}_{\mathsf{bisat}}$ has bisaturated boundary. By Corollary 4.4, the group Γ acts convex cocompactly on $\Omega := \mathrm{Int}(\mathcal{C}_{\mathsf{bisat}})$ and $\Lambda^{\mathsf{orb}}_\Omega(\Gamma) = \partial_{\mathrm{i}}\mathcal{C}_{\mathsf{bisat}}$. By Lemma 9.2, the group Γ acts cocompactly on a closed convex subset $\mathcal{C}_{\mathsf{smooth}} \supset \mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ of Ω whose nonideal boundary is strictly convex and C^1, and $\partial_{\mathrm{i}}\mathcal{C}_{\mathsf{smooth}} = \Lambda^{\mathsf{orb}}_\Omega(\Gamma) \subset \partial\mathbb{H}^{p,q-1}$ (see Lemma 4.1.(1)). In particular, $\mathcal{C}_{\mathsf{smooth}}$ is closed in $\mathbb{H}^{p,q-1}$ and condition (4) of Theorem 1.29 holds. Since $\mathcal{C}_{\mathsf{bisat}}$ has bisaturated boundary, any inextendable segment in $\partial_{\mathrm{i}}\mathcal{C}_{\mathsf{bisat}}$ is inextendable in $\partial\Omega = \partial_{\mathrm{i}}\mathcal{C}_{\mathsf{bisat}} \cup \partial_{\mathrm{n}}\mathcal{C}_{\mathsf{bisat}}$. By Lemma 6.2, if $\partial_{\mathrm{i}}\mathcal{C}_{\mathsf{smooth}} = \partial_{\mathrm{i}}\mathcal{C}_{\mathsf{bisat}}$ contained a nontrivial segment, then $\mathcal{C}_{\mathsf{smooth}}$ would contain a PET, but that is not possible since closed subsets of $\mathbb{H}^{p,q-1}$ do not contain PETs; see Remark 11.7 below. Therefore $\partial_{\mathrm{i}}\mathcal{C}_{\mathsf{smooth}} = \partial_{\mathrm{i}}\mathcal{C}_{\mathsf{bisat}}$ does not contain any nontrivial segment, and so Γ is $\mathbb{H}^{p,q-1}$-convex cocompact, i.e. condition (1) of Theorem 1.29 holds. This concludes the proof of Theorem 1.29.

In the proof we have used the following elementary observation.

Remark 11.7. For $p, q \in \mathbb{N}^*$ with $p+q \geq 3$, a closed subset of $\mathbb{H}^{p,q-1}$ cannot contain a PET. Indeed, any triangle of $\mathbb{P}(\mathbb{R}^{p,q})$ whose edges lie in $\partial\mathbb{H}^{p,q-1}$ must have interior in $\partial\mathbb{H}^{p,q-1}$, because in this case the symmetric bilinear form is zero on the projective span.

11.7. **$\mathbb{H}^{p,q-1}$-convex cocompact groups whose boundary is a $(p-1)$-sphere.** In what follows we use the term *standard $(p-1)$-sphere in $\partial\mathbb{H}^{p,q-1}$* to mean the intersection with $\partial\mathbb{H}^{p,q-1}$ of the projectivization of a $(p+1)$-dimensional subspace of $\mathbb{R}^{p+q}$ of signature $(p,1)$.

Lemma 11.8. *For $p, q \geq 1$, let Γ be an infinite discrete subgroup of $\mathrm{PO}(p,q)$ which is $\mathbb{H}^{p,q-1}$-convex cocompact, acting properly discontinuously and cocompactly on some closed convex subset $\mathcal{C}$ of $\mathbb{H}^{p,q-1}$ with nonempty interior* *whose ideal boundary $\partial_{\mathrm{i}}\mathcal{C} \subset \partial\mathbb{H}^{p,q-1}$ does not contain any nontrivial projective line segment. Let $\partial_\infty\Gamma$ be the Gromov boundary of Γ, and let $\mathcal{S}$ be any standard $(p-1)$-sphere in $\partial\mathbb{H}^{p,q-1}$. Then*

(1) *the Γ-equivariant boundary map $\xi : \partial_\infty\Gamma \to \partial\mathbb{H}^{p,q-1}$, with image $\partial_{\mathrm{i}}\mathcal{C}$, is homotopic to an embedding $\partial_\infty\Gamma \to \partial\mathbb{H}^{p,q-1}$ whose image is contained in $\mathcal{S}$; in particular, $\partial_\infty\Gamma$ has Lebesgue covering dimension $\leq p-1$.*

Suppose $\partial_\infty\Gamma$ is homeomorphic to a $(p-1)$-dimensional sphere and $p \geq q$. Then

(2) *the unique maximal Γ-invariant convex open subset $\Omega_{\max}$ of $\mathbb{P}(\mathbb{R}^{p,q})$ containing $\mathcal{C}$ (see Proposition 4.5) is contained in $\mathbb{H}^{p,q-1}$;*
(3) *any supporting hyperplane of $\mathcal{C}$ at a point of $\partial_{\mathrm{n}}\mathcal{C}$ is the projectivization of a linear hyperplane of $\mathbb{R}^{p,q}$ of signature $(p, q-1)$.*

In particular, if $q = 2$ and $\partial_\infty\Gamma$ is homeomorphic to a $(p-1)$-dimensional sphere with $p \geq 2$, then any hyperplane tangent to $\partial_{\mathrm{n}}\mathcal{C}$ is spacelike.

Lemma 11.8.(1) has the following consequences.

Corollary 11.9. *Let $p, q \geq 1$.*

(1) *Any infinite discrete subgroup of $\mathrm{PO}(p,q)$ which is $\mathbb{H}^{p,q-1}$-convex cocompact has virtual cohomological dimension $\leq p$.*
(2) *Let Γ be a word hyperbolic group with connected Gromov boundary $\partial_\infty\Gamma$ and virtual cohomological dimension $> q$. If there is a $P_1^{p,q}$-Anosov representation $\rho : \Gamma \to \mathrm{PO}(p,q)$, then $p > q$ and the group $\rho(\Gamma)$ is $\mathbb{H}^{p,q-1}$-convex cocompact.*

Proof of Corollary 11.9. (1) For a word hyperbolic group Γ, the Lebesgue covering dimension of $\partial_\infty\Gamma$ is equal to the virtual cohomological dimension of Γ minus one [BeM]. We conclude using Lemma 11.8.(1).

(2) By Corollary 1.27, since $\partial_\infty\Gamma$ is connected, the group $\rho(\Gamma)$ is $\mathbb{H}^{p,q-1}$-convex cocompact or $\mathbb{H}^{q,p-1}$-convex cocompact. We conclude using (1). □

Proof of Lemma 11.8. We work in an affine chart $\mathbb{A}$ containing $\overline{\mathcal{C}}$, and choose coordinates $(x_1, \dots, x_{p+q})$ on $\mathbb{R}^{p,q}$ so that the quadratic form $\langle\cdot,\cdot\rangle_{p,q}$ takes the usual form $x_1^2 + \cdots x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$, the affine chart $\mathbb{A}$ is defined by $x_{p+q} \neq 0$, and our chosen standard $(p-1)$-sphere in $\partial\mathbb{H}^{p,q-1}$ is

$$\mathcal{S} = \big\{[x_1 : \ldots : x_p : 0 : \ldots : 0 : x_{p+q}] \in \mathbb{P}(\mathbb{R}^{p,q}) \mid x_1^2 + \cdots + x_p^2 = x_{p+q}^2\big\}.$$

For any $0 \leq t \leq 1$, consider the map $f_t : \mathbb{A} \to \mathbb{A}$ sending $[x_1 : \ldots : x_{p+q}]$ to

$$\Big[x_1 : \ldots : x_p : \sqrt{1-t}\, x_{p+1} : \cdots : \sqrt{1-t}\, x_{p+q-1} : \sqrt{1 + t\,\alpha_{(x_{p+1},\dots,x_{p+q})}}\, x_{p+q}\Big],$$

where $\alpha_{(x_{p+1},\dots,x_{p+q})} = (x_{p+1}^2 + \cdots + x_{p+q-1}^2)/x_{p+q}^2$. Then $(f_t)_{0\leq t\leq 1}$ restricts to a homotopy of maps $\mathbb{A} \cap \partial\mathbb{H}^{p,q-1} \to \mathbb{A} \cap \partial\mathbb{H}^{p,q-1}$ from the identity map to the map $f_1|_{\mathbb{A}\cap\partial\mathbb{H}^{p,q-1}}$, which has image $\mathcal{S} \subset \mathbb{P}(\mathbb{R}^{p,1})$. Thus $(f_t \circ \xi)_{0\leq t\leq 1}$ defines a homotopy between $\xi : \partial_\infty\Gamma \to \partial\mathbb{H}^{p,q-1}$ and the continuous map $f_1 \circ \xi : \partial_\infty\Gamma \to \partial\mathbb{H}^{p,q-1}$ whose image lies in $\mathcal{S}$. By the Cauchy–Schwarz inequality for Euclidean inner products, two points of $\mathbb{A} \cap \partial\mathbb{H}^{p,q-1}$ which have the same image under f_1 are connected by a line segment in $\mathbb{A}$ whose interior lies outside of $\overline{\mathbb{H}^{p,q-1}}$. Since $\mathcal{C}$ is contained in $\mathbb{H}^{p,q-1}$, we deduce that the restriction of f_1 to $\partial_{\mathrm{i}}\mathcal{C}$ is injective. Since $\partial_\infty\Gamma$ is compact, we obtain that $f_1 \circ \xi : \partial_\infty\Gamma \to \mathcal{S}$ is an embedding. This proves (1).

Henceforth we assume that $\partial_\infty\Gamma$ is homeomorphic to a $(p-1)$-dimensional sphere and that $p \geq q$. Let $\Lambda_\Gamma = \xi(\partial_\infty\Gamma) \subset \partial\mathbb{H}^{p,q-1}$ be the proximal limit set of Γ in $\mathbb{P}(\mathbb{R}^{p+q})$. The map $f_1 : \Lambda_\Gamma \to \mathcal{S}$ is an embedding, hence a homeomorphism since Λ_Γ is compact.

Let us prove (2). By definition of $\Omega_{\max}$ (see Proposition 4.5), it is sufficient to prove that any point of $\partial\mathbb{H}^{p,q-1}$ is contained in $z^\perp$ for some $z \in \Lambda_\Gamma$. Therefore it is sufficient to prove that Λ_Γ intersects $\mathbb{P}(W)$ for every maximal totally isotropic subspace W of $\mathbb{R}^{p,q}$. Let W be such a subspace. We work in the coordinates from above. Up to changing the first p coordinates by applying an element of $\mathrm{O}(p)$, we may assume that in the splitting

$$\mathbb{R}^{p,q} = \mathbb{R}^{p-q,0} \oplus \mathbb{R}^{q,0} \oplus \mathbb{R}^{0,q}$$

defined by these coordinates, the space W is $\{(0, x', -x') \mid x' \in \mathbb{R}^q\}$. Consider the map $\varphi : \mathbb{S}^{p-1} \to \mathbb{S}^{q-1}$ sending a unit vector $(x, x') \in \mathbb{R}^{p,0} = \mathbb{R}^{p-q,0} \oplus \mathbb{R}^{q,0}$ to

$$\varphi(x, x') = \pi_q \circ (f_1|_{\Lambda_\Gamma})^{-1}(x, x', a),$$

where $a = (0, \dots, 0, 1) \in \mathbb{R}^{0,q}$ and $\pi_q : \mathbb{R}^{p,q} \to \mathbb{R}^{0,q}$ is the projection onto the last q coordinates. Then φ is homotopic to a constant map (namely the map sending all points to a) via the homotopy $\varphi_t = \pi_q \circ f_t \circ (f_1|_{\Lambda_\Gamma})^{-1}$. Now consider the restriction $\psi : \mathbb{S}^{q-1} \to \mathbb{S}^{q-1}$ of φ to the unit sphere in $\mathbb{R}^{q,0}$, which is also homotopic to a constant map via the restriction of the homotopy φ_t. If we had $\Lambda_\Gamma \cap \mathbb{P}(W) = \emptyset$, then we would have $\psi(x') = \varphi(0, x') \neq -x'$ for all $x' \in \mathbb{S}^{q-1} \subset \mathbb{R}^q$, and

$$\psi_t(x') = \frac{(1-t)\,\psi(x') + tx'}{\|(1-t)\,\psi(x') + tx'\|}$$

would define a homotopy from ψ to the identity map on $\mathbb{S}^{q-1}$, showing that a constant map is homotopic to the identity map: contradiction since $\mathbb{S}^{q-1}$ is not contractible. This completes the proof of (2): namely, $\Omega_{\max} \subset \mathbb{H}^{p,q-1}$. We note that $\partial\Omega_{\max} \cap \partial\mathbb{H}^{p,q-1} = \Lambda_\Gamma$, since we saw that f_1 maps $\partial\Omega_{\max} \cap \partial\mathbb{H}^{p,q-1}$ injectively to $\mathcal{S}$.

Finally, we prove (3). The dual convex $\mathrm{Int}(\mathcal{C})^*$ to $\mathrm{Int}(\mathcal{C})$ naturally identifies, via $\langle\cdot,\cdot\rangle_{p,q}$, with a properly convex subset of $\mathbb{P}(\mathbb{R}^{p+q})$ which must be contained in $\Omega_{\max} \subset \mathbb{H}^{p,q-1}$. By Lemma 11.3, we have $\mathcal{C} \cap z^\perp = \emptyset$ for any $z \in \partial_{\mathrm{i}}\mathcal{C} = \Lambda_\Gamma$, hence $\mathcal{C} \subset \Omega_{\max}$. Hence, a projective hyperplane $y^\perp$ supporting $\mathcal{C}$ at some point of $\partial_{\mathrm{n}}\mathcal{C}$ is dual to a point $y \in \overline{\Omega_{\max}} \smallsetminus \Lambda_\Gamma$, which is contained in $\mathbb{H}^{p,q-1}$ by the previous paragraph. Hence $\langle y, y\rangle_{p,q} < 0$ and so $y^\perp$ is the projectivization of a linear hyperplane of signature $(p, q-1)$. $\square$

11.8. $\mathbb{H}^{p,1}$**-convex cocompactness and global hyperbolicity.** Recall that a Lorentzian manifold M is said to be *globally hyperbolic* if it is causal (i.e.

contains no timelike loop) and for any two points $x, x' \in M$, the intersection $J^+(x) \cap J^-(x')$ is compact (possibly empty). Here we denote by $J^+(x)$ (resp. $J^-(x)$) the set of points of M which are seen from x by a future-pointing (resp. past-pointing) timelike or lightlike geodesic. Equivalently [CGe], the Lorentzian manifold M admits a *Cauchy hypersurface*, i.e. an achronal subset which intersects every inextendible timelike curve in exactly one point.

We make the following observation; it extends [BM, § 4.2], which focused on the case that $\Gamma \subset \mathrm{PO}(p, 2)$ is isomorphic to a uniform lattice of $\mathrm{PO}(p, 1)$. The case considered here is a bit more general, see [LM]. We argue in an elementary way, without using the notion of CT-regularity.

Proposition 11.10. *For $p \geq 2$, let Γ be a torsion-free infinite discrete subgroup of $\mathrm{PO}(p, 2)$ which is $\mathbb{H}^{p,1}$-convex cocompact and whose Gromov boundary $\partial_\infty \Gamma$ is homeomorphic to a $(p-1)$-dimensional sphere. For any nonempty properly convex open subset Ω of $\mathbb{H}^{p,1}$ on which Γ acts convex cocompactly, the quotient $M = \Gamma \backslash \Omega$ is a globally hyperbolic Lorentzian manifold.*

Proof. Consider two points $x, x' \in M = \Gamma\backslash\Omega$. There exists $R > 0$ such that any lifts $y, y' \in \Omega$ of x, x' belong to the uniform R-neighborhood $\mathcal{C}$ of $\mathcal{C}^{\mathsf{cor}}_\Omega(\Gamma)$ in Ω for the Hilbert metric d_Ω. Let $J^+(y)$ (resp. $J^-(y)$) be the set of points of Ω which are seen from y by a future-pointing (resp. past-pointing) timelike or lightlike geodesic. By Lemma 11.8.(3), all supporting hyperplanes of $\mathcal{C}$ at points of $\partial_{\mathrm{n}}\mathcal{C}$ are spacelike. We may decompose $\partial_{\mathrm{n}}\mathcal{C}$ into two disjoint open subsets, namely the subset $\partial_{\mathrm{n}}^+\mathcal{C}$ of points for which the outward pointing normal to a supporting plane is future pointing, and the subset $\partial_{\mathrm{n}}^-\mathcal{C}$ of points for which it is past pointing. Indeed, $\partial_{\mathrm{n}}^+\mathcal{C}$ and $\partial_{\mathrm{n}}^-\mathcal{C}$ are the two path connected components of the complement $\partial_{\mathrm{n}}\mathcal{C}$ of the embedded $(p-1)$-sphere Λ_Γ in the p-sphere $\mathrm{Fr}(\mathcal{C})$. The set $\Omega \smallsetminus \mathcal{C}$ similarly has two components, a component to the future of $\partial_{\mathrm{n}}^+\mathcal{C}$ and a component to the past of $\partial_{\mathrm{n}}^-\mathcal{C}$. Any point of $J^+(y) \cap (\Omega \smallsetminus \mathcal{C})$ lies in the future component of $\Omega \smallsetminus \mathcal{C}$ and similarly, any point of $J^-(y') \cap (\Omega \smallsetminus \mathcal{C})$ lies in the past component of $\Omega \smallsetminus \mathcal{C}$. By proper discontinuity of the Γ-action on $\mathcal{C}$, in order to check that $J^+(x) \cap J^-(x')$ is compact in M, it is therefore enough to check that $J^+(y) \cap \mathcal{C}$ and $J^-(y') \cap \mathcal{C}$ are compact in Ω. This follows from the fact that the ideal boundary of $\mathcal{C}$ is $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ (Lemma 4.1.(1)) and that any point of Ω sees any point of $\Lambda^{\mathsf{orb}}_\Omega(\Gamma)$ in a spacelike direction (Lemma 11.3). ☐

11.9. Examples of $\mathbb{H}^{p,q-1}$-convex cocompact groups.

11.9.1. *Quasi-Fuchsian $\mathbb{H}^{p,q-1}$-convex cocompact groups.* Let H be a real semisimple Lie group of real rank 1 and $\tau : H \to \mathrm{PO}(p, q)$ a representation whose image contains an element which is proximal in $\partial\mathbb{H}^{p,q-1}$ (see Remark 11.1). By [DGK3, Prop. 7.1], for any word hyperbolic group Γ and any (classical) convex cocompact representation $\sigma_0 : \Gamma \to H$,

(1) the composition $\rho_0 := \tau \circ \sigma_0 : \Gamma \to \mathrm{PO}(p, q)$ is $P_1^{p,q}$-Anosov and the proximal limit set $\Lambda_{\rho_0(\Gamma)} \subset \partial\mathbb{H}^{p,q-1}$ is negative or positive;

(2) the connected component $\mathcal{T}_{\rho_0}$ of ρ_0 in the space of $P_1^{p,q}$-Anosov representations from Γ to $\mathrm{PO}(p,q)$ is a neighborhood of ρ_0 in $\mathrm{Hom}(\Gamma, \mathrm{PO}(p,q))$ consisting entirely of representations ρ with negative proximal limit set $\Lambda_{\rho(\Gamma)} \subset \partial\mathbb{H}^{p,q-1}$ or entirely of representations ρ with positive proximal limit set $\Lambda_{\rho(\Gamma)} \subset \partial\mathbb{H}^{p,q-1}$.

Here is an immediate consequence of this result and of Theorem 1.25.

Corollary 11.11. *In this setting, either $\rho(\Gamma)$ is $\mathbb{H}^{p,q-1}$-convex cocompact for all $\rho \in \mathcal{T}_{\rho_0}$, or $\rho(\Gamma)$ is $\mathbb{H}^{q,p-1}$-convex cocompact (after identifying $\mathrm{PO}(p,q)$ with $\mathrm{PO}(q,p)$) for all $\rho \in \mathcal{T}_{\rho_0}$.*

This improves [DGK3, Cor. 7.2], which assumed ρ to be irreducible.

Here are two examples. We refer to [DGK3, § 7] for more details.

Example 11.12. Let Γ be the fundamental group of a convex cocompact (e.g. closed) hyperbolic manifold M of dimension $m \geq 2$, with holonomy $\sigma_0 : \Gamma \to \mathrm{PO}(m,1) = \mathrm{Isom}(\mathbb{H}^m)$. The representation σ_0 lifts to a representation $\widetilde{\sigma}_0 : \Gamma \to H := \mathrm{O}(m,1)$. For $p, q \in \mathbb{N}^*$ with $p \geq m, q \geq 1$, let $\tau : \mathrm{O}(m,1) \to \mathrm{PO}(p,q)$ be induced by the natural embedding $\mathbb{R}^{m,1} \hookrightarrow \mathbb{R}^{p,q}$. Then $\rho_0 := \tau \circ \widetilde{\sigma}_0 : \Gamma \to \mathrm{PO}(p,q)$ is $P_1^{p,q}$-Anosov, and one checks that the set $\Lambda_{\rho_0(\Gamma)} \subset \partial\mathbb{H}^{p,q-1}$ is negative. Let $\mathcal{T}_{\rho_0}$ be the connected component of ρ_0 in the space of $P_1^{p,q}$-Anosov representations from Γ to $\mathrm{PO}(p,q)$. By Corollary 11.11, for any $\rho \in \mathcal{T}_{\rho_0}$, the group $\rho(\Gamma)$ is $\mathbb{H}^{p,q-1}$-convex cocompact.

By [Me, Ba], when $p = m$ and $q = 2$ and when the hyperbolic m-manifold M is closed, the space $\mathcal{T}_{\rho_0}$ of Example 11.12 is a full connected component of $\mathrm{Hom}(\Gamma, \mathrm{PO}(p,2))$, consisting of so-called *AdS quasi-Fuchsian* representations; that $\rho(\Gamma)$ is $\mathbb{H}^{p,1}$-convex cocompact in that case follows from [Me, BM].

Example 11.13. For $n \geq 2$, let

$$\tau_n : \mathrm{SL}(\mathbb{R}^2) \longrightarrow \mathrm{SL}(\mathbb{R}^n) \tag{11.1}$$

be the irreducible n-dimensional linear representation of $\mathrm{SL}(\mathbb{R}^2)$ obtained from the action of $\mathrm{SL}(\mathbb{R}^2)$ on the $(n-1)^{\mathrm{st}}$ symmetric power $\mathrm{Sym}^{n-1}(\mathbb{R}^2) \simeq \mathbb{R}^n$. The image of τ_n preserves the nondegenerate bilinear form $B_n := -\omega^{\otimes(n-1)}$ induced from the area form ω of $\mathbb{R}^2$. This form is symmetric if n is odd, and antisymmetric (i.e. symplectic) if n is even. Suppose $n = 2m+1$ is odd. The symmetric bilinear form B_n has signature

$$(k_n, \ell_n) := \begin{cases} (m+1, m) & \text{if } m \text{ is odd,} \\ (m, m+1) & \text{if } m \text{ is even.} \end{cases} \tag{11.2}$$

If we identify the orthogonal group $\mathrm{O}(B_n)$ (containing the image of τ_n) with $\mathrm{O}(k_n, \ell_n)$, then there is a unique τ_n-equivariant embedding $\iota_n : \partial_\infty\mathbb{H}^2 \hookrightarrow \partial_{\mathbb{P}}\mathbb{H}^{k_n,\ell_n-1}$, and an easy computation shows that its image $\Lambda_n := \iota(\partial_\infty\mathbb{H}^2)$ is negative. For $p \geq k_n$ and $q \geq \ell_n$, the representation $\tau_n : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{O}(B_n) \simeq \mathrm{O}(k_n, \ell_n)$ and the natural embedding $\mathbb{R}^{k_n,\ell_n} \hookrightarrow \mathbb{R}^{p,q}$ induce a representation $\tau : H = \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PO}(p,q)$ whose image contains an element which is proximal in $\partial\mathbb{H}^{p,q-1}$, and a τ-equivariant embedding $\iota : \partial_\infty\mathbb{H}^2 \hookrightarrow \partial_{\mathbb{P}}\mathbb{H}^{k_n,\ell_n-1} \hookrightarrow$

$\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$. The set $\Lambda := \iota(\partial_\infty \mathbb{H}^2) \subset \partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$ is negative by construction. Let Γ be the fundamental group of a convex cocompact orientable hyperbolic surface, with holonomy $\sigma_0 : \Gamma \to \mathrm{PSL}_2(\mathbb{R})$. The representation σ_0 lifts to a representation $\widetilde{\sigma}_0 : \Gamma \to H := \mathrm{SL}_2(\mathbb{R})$. Let $\rho_0 := \tau \circ \widetilde{\sigma}_0 : \Gamma \to G := \mathrm{PO}(p,q)$. The proximal limit set $\Lambda_{\rho_0(\Gamma)} = \iota(\Lambda_{\sigma_0(\Gamma)}) \subset \Lambda$ is negative. Thus Corollary 11.11 implies that ρ_0 is $\mathbb{H}^{p,q-1}$-convex cocompact.

It follows from [L, FG] (see e.g. [BIW2, § 6.1]) that when $(p,q) = (k_n, \ell_n)$ or $(m+1, m+1)$ and when Γ is a closed surface group, the space $\mathcal{T}_{\rho_0}$ of Example 11.13 is a full connected component of $\mathrm{Hom}(\Gamma, \mathrm{PO}(p,q))$, consisting of so-called *Hitchin representations*. Example 11.13 thus states the following.

Corollary 11.14. *Let Γ be the fundamental group of a closed orientable hyperbolic surface and let $m \geq 1$.*

For any Hitchin representation $\rho : \Gamma \to \mathrm{PO}(m+1, m)$, the group $\rho(\Gamma)$ is $\mathbb{H}^{m+1,m-1}$-convex cocompact if m is odd, and $\mathbb{H}^{m,m}$-convex cocompact if m is even.

For any Hitchin representation $\rho : \Gamma \to \mathrm{PO}(m+1, m+1)$, the group $\rho(\Gamma)$ is $\mathbb{H}^{m+1,m}$-convex cocompact.

By [BIW1, BIW3], when $p = m+1 = 2$ and Γ is a closed surface group, the space $\mathcal{T}_{\rho_0}$ of Example 11.13 is a full connected component of $\mathrm{Hom}(\Gamma, \mathrm{PO}(2,q))$, consisting of so-called *maximal representations*. Example 11.13 thus states the following.

Corollary 11.15. *Let Γ be the fundamental group of a closed orientable hyperbolic surface and let $q \geq 1$. Any connected component of* $\mathrm{Hom}(\Gamma, \mathrm{PO}(2,q))$ *consisting of maximal representations and containing a Fuchsian representation $\rho_0 : \Gamma \to \mathrm{PO}(2,1)_0 \hookrightarrow \mathrm{PO}(2,q)$ consists entirely of $\mathbb{H}^{2,q-1}$-convex cocompact representations.*

11.9.2. *Groups with connected boundary.* We now briefly discuss a class of examples that does not necessarily come from the deformation of "Fuchsian" representations as above.

Suppose the word hyperbolic group Γ has connected boundary $\partial_\infty \Gamma$ (for instance Γ is the fundamental group of a closed negatively-curved Riemannian manifold). By [DGK3, Prop. 1.10 & Prop. 3.5], any connected component in the space of $P_1^{p,q}$-Anosov representations from Γ to $\mathrm{PO}(p,q)$ consists entirely of representations ρ with negative proximal limit set $\Lambda_{\rho(\Gamma)} \subset \partial\mathbb{H}^{p,q-1}$ or entirely of representations with ρ with positive proximal limit set $\Lambda_{\rho(\Gamma)} \subset \partial\mathbb{H}^{p,q-1}$. Theorem 1.25 then implies that for any connected component $\mathcal{T}$ in the space of P_1-Anosov representations of Γ with values in $\mathrm{PO}(p,q) \subset \mathrm{PGL}(\mathbb{R}^{p+q})$, either $\rho(\Gamma)$ is $\mathbb{H}^{p,q-1}$-convex cocompact for all $\rho \in \mathcal{T}$, or $\rho(\Gamma)$ is $\mathbb{H}^{q,p-1}$-convex cocompact for all $\rho \in \mathcal{T}$, as in Corollary 1.27.

This applies for instance to the case that Γ is the fundamental group of a closed hyperbolic surface and $\mathcal{T}$ is a connected component of $\mathrm{Hom}(\Gamma, \mathrm{PO}(2,q))$ consisting of maximal representations [BIW1, BIW3]. By [GW2], for $q = 3$

there exist such connected components $\mathcal{T}$ that consist entirely of Zariski-dense representations, hence that do not come from the deformation of "Fuchsian" representations as in Corollary 11.15.

12. Examples of groups which are convex cocompact in $\mathbb{P}(V)$

In Section 11.9 we constructed examples of discrete subgroups of $\mathrm{PO}(p,q) \subset \mathrm{PGL}(\mathbb{R}^{p,q})$ which are $\mathbb{H}^{p,q-1}$-convex cocompact; these groups are strongly convex cocompact in $\mathbb{P}(\mathbb{R}^{p,q})$ by Theorem 1.25. We now discuss several constructions of discrete subgroups of $\mathrm{PGL}(V)$ which are convex cocompact in $\mathbb{P}(V)$ but which do not necessarily preserve a nonzero quadratic form on V.

Based on Theorem 1.15.(D)–(F), another fruitful source of groups that are convex cocompact in $\mathbb{P}(V)$ is the continuous deformation of "Fuchsian" groups, coming from an algebraic embedding. These "Fuchsian" groups can be for instance:

(1) convex cocompact subgroups (in the classical sense) of appropriate rank-one Lie subgroups H of $\mathrm{PGL}(V)$, as in Section 11.9.1;
(2) discrete subgroups of $\mathrm{PGL}(V)$ dividing a properly convex open subset of some projective subspace $\mathbb{P}(V_0)$ of $\mathbb{P}(V)$ and acting trivially on a complementary subspace.

The groups of (1) are always word hyperbolic; in Section 12.1, we explain how to choose H so that they are strongly convex cocompact in $\mathbb{P}(V)$, and we prove Proposition 1.7. The groups of (2) are not necessarily word hyperbolic; they are convex cocompact in $\mathbb{P}(V)$ by Theorem 1.15.(F), but not necessarily strongly convex cocompact; we discuss them in Section 12.2.2.

We also mention other constructions of convex cocompact groups in Sections 12.2.3 and 12.3, which do not involve any deformation.

12.1. "Quasi-Fuchsian" strongly convex cocompact groups.

We start by discussing a similar construction to Section 11.9.1, but for discrete subgroups of $\mathrm{PGL}(V)$ which do not necessarily preserve a nonzero quadratic form on V.

Let Γ be a word hyperbolic group, H a real semisimple Lie group of real rank one, and $\tau : H \to \mathrm{PGL}(V)$ a representation whose image contains a proximal element. Any (classical) convex cocompact representation $\sigma_0 : \Gamma \to H$ is Anosov, hence the composition $\tau \circ \sigma_0 : \Gamma \to \mathrm{PGL}(V)$ is P_1-Anosov (see [L, Prop. 3.1] and [GW3, Prop. 4.7]). In particular, by Theorem 1.4, the group $\tau \circ \sigma_0(\Gamma)$ is strongly convex cocompact in $\mathbb{P}(V)$ as soon as it preserves a nonempty properly convex open subset of $\mathbb{P}(V)$.

Suppose this is the case. By Theorem 1.17 (see Remark 1.18), the group $\tau \circ \sigma(\Gamma)$ remains strongly convex cocompact in $\mathbb{P}(V)$ for any $\sigma \in \mathrm{Hom}(\Gamma, H)$ close enough to σ_0. Sometimes $\tau \circ \sigma(\Gamma)$ also remains strongly convex cocompact for some $\sigma \in \mathrm{Hom}(\Gamma, H)$ which are continuous deformations of σ_0 quite far away from σ_0; we now discuss this in view of proving Proposition 1.7.

12.1.1. *Connected open sets of strongly convex cocompact representations.* We prove the following.

Proposition 12.1. *Let Γ be a word hyperbolic group and $\mathcal{A}$ a connected open subset of* $\mathrm{Hom}(\Gamma, \mathrm{PGL}(V))$ *consisting entirely of P_1-Anosov representations.*

1. *Suppose every representation in $\mathcal{A}$ is irreducible. If $\rho(\Gamma)$ is convex cocompact in $\mathbb{P}(V)$ for some $\rho \in \mathcal{A}$, then $\rho(\Gamma)$ is convex cocompact in $\mathbb{P}(V)$ for all $\rho \in \mathcal{A}$.*
2. *Suppose $\mathcal{A}$ contains representations ρ which are not irreducible, and that $\rho(\Gamma)$ is convex cocompact in $\mathbb{P}(V)$ for all these ρ. Then $\rho(\Gamma)$ is convex cocompact in $\mathbb{P}(V)$ for all $\rho \in \mathcal{A}$.*

The proof of Proposition 12.1 reduces to the following lemma.

Lemma 12.2. *Let Γ be a discrete group and $(\rho_m) \in \mathrm{Hom}(\Gamma, \mathrm{PGL}(V))^{\mathbb{N}}$ a sequence of discrete and faithful representations converging to a discrete and faithful representation $\rho_\infty \in \mathrm{Hom}(\Gamma, \mathrm{PGL}(V))$ which is irreducible. If each $\rho_m(\Gamma)$ preserves a nonempty properly convex domain $\Omega_m \subset \mathbb{P}(V)$, then so does $\rho_\infty(\Gamma)$.*

Proof of Lemma 12.2. Up to passing to a subsequence, the closed convex subsets $\overline{\Omega_m}$ converge to a closed subset $\mathcal{C}$ in the Hausdorff topology on compact subsets. The set $\mathcal{C}$ is invariant under $\rho_\infty(\Gamma)$. Suppose by contradiction that $\rho_\infty(\Gamma)$ fails to preserve any nonempty properly convex open domain. Then either $\mathcal{C}$ has empty interior or $\mathcal{C}$ is not contained in any affine chart. If $\mathcal{C}$ has empty interior, then it is contained in a smallest plane of positive codimension and that plane is preserved by $\rho_\infty(\Gamma)$, contradicting the fact that ρ_∞ is irreducible. If $\mathcal{C}$ is not contained in any affine chart, then the limit $\mathcal{C}'^*$ of the closures of the dual convex domains $\overline{\Omega_m^*}$ has empty interior. Hence the dual action of Γ on $\mathbb{P}(V^*)$ preserves a plane of positive codimension in $\mathbb{P}(V^*)$, and so Γ preserves some nontrivial subspace in $\mathbb{P}(V)$, contradicting again the fact that ρ_∞ is irreducible. $\square$

Proof of Proposition 12.1. First, observe that the finite normal subgroups $\mathrm{Ker}(\rho) \subset \Gamma$ are constant over $\rho \in \mathcal{A}$, since representations of a finite group are rigid up to conjugation. Hence by passing to the quotient group $\Gamma/\mathrm{Ker}(\rho)$, we may assume each representation $\rho \in \mathcal{A}$ is faithful. Suppose $\rho(\Gamma)$ is convex cocompact in $\mathbb{P}(V)$ for all $\rho \in \mathcal{A}$ which are not irreducible. By Theorem 1.17.(D), the property of being convex cocompact in $\mathbb{P}(V)$ is open in $\mathrm{Hom}(\Gamma, \mathrm{PGL}(V))$. By Lemma 12.2, the property of preserving a nonempty properly convex open subset of $\mathbb{P}(V)$ is closed in $\mathcal{A}$. Since $\mathcal{A}$ consists entirely of P_1-Anosov representations, Theorem 1.15 implies that the property of being convex cocompact in $\mathbb{P}(V)$ is closed in $\mathcal{A}$. $\square$

12.1.2. *Hitchin representations.* We now prove Proposition 1.7 by applying Proposition 12.1.(1) in the following specific context.

Let Γ be the fundamental group of a closed orientable hyperbolic surface S. For $n \geq 2$, let $\tau_n : \mathrm{SL}(\mathbb{R}^2) \to \mathrm{SL}(\mathbb{R}^n)$ be the irreducible n-dimensional linear representation from (11.1). We still denote by τ_n the representation $\mathrm{PSL}(\mathbb{R}^2) \to \mathrm{PSL}(\mathbb{R}^n)$ obtained by modding out by $\{\pm I\}$. A representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}(\mathbb{R}^n))$ is said to be *Fuchsian* if it is of the form $\rho = \tau_n \circ \rho_0$ where $\rho_0 : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ is discrete and faithful. By definition, a *Hitchin representation* is a continuous deformation of a Fuchsian representation; the *Hitchin component* $\mathrm{Hit}_n(S)$ is the space of Hitchin representations $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}(\mathbb{R}^n))$ modulo conjugation by $\mathrm{PGL}(\mathbb{R}^n)$. Hitchin [H] used Higgs bundles techniques to parametrize $\mathrm{Hit}_n(S)$, showing in particular that it is homeomorphic to a ball of dimension $(n^2-1)(2g-2)$. Labourie [L] proved that any Hitchin representation is discrete and faithful, and moreover that it is an Anosov representation with respect to a minimal parabolic subgroup of $\mathrm{PSL}(\mathbb{R}^n)$. In particular, any Hitchin representation is P_1-Anosov.

In order to prove Proposition 1.7, we first consider the case of Fuchsian representations.

Lemma 12.3. *Let $\rho : \Gamma \to \mathrm{PSL}(\mathbb{R}^2) \to \mathrm{PSL}(\mathbb{R}^n)$ be Fuchsian.*

(1) If n is odd, then $\rho(\Gamma)$ is strongly convex cocompact in $\mathbb{P}(\mathbb{R}^n)$.

(2) If n is even, then the boundary map of the P_1-Anosov representation ρ defines a nontrivial loop in $\mathbb{P}(\mathbb{R}^n)$ and $\rho(\Gamma)$ does not preserve any nonempty properly convex open subset of $\mathbb{P}(\mathbb{R}^n)$.

For even n, the fact that a Fuchsian representation ρ cannot preserve a properly convex open subset of $\mathbb{P}(\mathbb{R}^n)$ also follows from [B2, Th. 1.5], since in this case ρ takes values in the projective symplectic group $\mathrm{PSp}(n/2,\mathbb{R})$.

Proof. (1) Suppose $n = 2k+1$ is odd. Then ρ takes values in the projective orthogonal group $\mathrm{PSO}(k, k+1) \simeq \mathrm{SO}(k, k+1)$, and so $\rho(\Gamma)$ is strongly convex cocompact in $\mathbb{P}(\mathbb{R}^n)$ by [DGK3, Prop. 1.17 & 1.19].

(2) Suppose $n = 2k$ is even. Then ρ takes values in the symplectic group $\mathrm{Sp}(n,\mathbb{R})$. It is well known that, in natural coordinates identifying $\partial_\infty\Gamma \simeq \mathbb{S}^1$ with $\mathbb{P}(\mathbb{R}^2)$, the boundary map $\xi : \partial_\infty\Gamma \to \mathbb{P}(\mathbb{R}^n)$ of the P_1-Anosov representation ρ is the *Veronese curve*

$$\mathbb{P}(\mathbb{R}^2) \ni [x,y] \longmapsto [x^{n-1}, x^{n-2}y, \dots, xy^{n-2}, y^{n-1}] \in \mathbb{P}(\mathbb{R}^n).$$

This map is homotopic to the map $[x,y] \mapsto [x^{n-1}, 0, \dots, 0, y^{n-1}]$ which, since $n-1$ is odd, is a homeomorphism from $\mathbb{P}(\mathbb{R}^2)$ to the projective plane spanned by the first and last coordinate vectors, hence is nontrivial in $\pi_1(\mathbb{P}(\mathbb{R}^n))$. Hence the image of the boundary map ξ crosses every hyperplane in $\mathbb{P}(\mathbb{R}^n)$. It follows that $\rho(\Gamma)$ does not preserve any properly convex open subset of $\mathbb{P}(\mathbb{R}^n)$, because the image of ξ must lie in the boundary of any $\rho(\Gamma)$-invariant such set. ☐

Proof of Proposition 1.7. (1) Suppose n is odd. By [L, FG], all Hitchin representations $\rho : \Gamma \to \mathrm{PSL}(\mathbb{R}^n)$ are P_1-Anosov. Moreover, these representations ρ are irreducible (see [L, Lem. 10.1]), and $\rho(\Gamma)$ is convex cocompact in $\mathbb{P}(V)$ as soon as ρ is Fuchsian, by Lemma 12.3. Applying Proposition 12.1.(1), we obtain that for any Hitchin representation $\rho : \Gamma \to \mathrm{PSL}(\mathbb{R}^n)$ the group $\rho(\Gamma)$

is convex cocompact in $\mathbb{P}(V)$, hence strongly convex cocompact since Γ is word hyperbolic (Theorem 1.15).

(2) Suppose n is even. By Lemma 12.3, for any Fuchsian $\rho : \Gamma \to \mathrm{PSL}(\mathbb{R}^n)$, the image of the boundary map of the P_1-Anosov representation ρ defines a nontrivial loop in $\mathbb{P}(\mathbb{R}^n)$. Since the assignment of a boundary map to an Anosov representation is continuous [GW3, Th. 5.13], we obtain that for any Hitchin $\rho : \Gamma \to \mathrm{PSL}(\mathbb{R}^n)$, the image of the boundary map of the P_1-Anosov representation ρ defines a nontrivial loop in $\mathbb{P}(\mathbb{R}^n)$. In particular, $\rho(\Gamma)$ does not preserve any nonempty properly convex open subset of $\mathbb{P}(\mathbb{R}^n)$, by the same reasoning as in the proof of Lemma 12.3. More generally, ρ does not preserve any nonempty properly convex subset of $\mathbb{P}(\mathbb{R}^n)$, since any such set would have nonempty interior by irreducibility of ρ. ☐

12.2. **Convex cocompact groups coming from divisible convex sets.** Recall that any discrete subgroup Γ of $\mathrm{PGL}(V)$ dividing a properly convex open subset $\Omega \subset \mathbb{P}(V)$ is convex cocompact in $\mathbb{P}(V)$ (Example 1.13). The group Γ is strongly convex cocompact if and only if it is word hyperbolic (Theorem 1.15), and this is equivalent to the fact that Ω is strictly convex (by [B3] or Theorem 1.15 again).

After briefly reviewing some examples of nonhyperbolic groups dividing properly convex open sets (Section 12.2.1), we explain how they can be used to construct more examples of convex cocompact groups which are not strongly convex cocompact, either by deformation (Section 12.2.2) or by taking subgroups (Section 12.2.3).

12.2.1. *Examples of nonhyperbolic groups dividing properly convex open sets.* Examples of virtually *abelian*, nonhyperbolic groups dividing a properly convex open subset $\Omega \subset \mathbb{P}(V)$ are easily constructed: see Examples 3.7 and 3.8. In these examples the properly convex (but not strictly convex) set Ω is *decomposable*: there is a direct sum decomposition $V = V_1 \oplus V_2$ and properly convex open cones $\widetilde{\Omega}_1$ of V_1 and $\widetilde{\Omega}_2$ of V_2 such that $\Omega \subset \mathbb{P}(V)$ is the projection of $\widetilde{\Omega}_1 + \widetilde{\Omega}_2 := \{v_1 + v_2 \mid v_i \in \widetilde{\Omega}_i\}$.

Examples of nonhyperbolic groups dividing an *indecomposable* properly convex open set Ω can classically be built by letting a uniform lattice of $\mathrm{SL}(\mathbb{R}^m)$ act on the Riemannian symmetric space of $\mathrm{SL}(\mathbb{R}^m)$, realized as a properly convex subset of $\mathbb{P}(\mathbb{R}^{m(m-1)/2})$ (see [B7, § 2.4]). A similar construction works over the complex numbers, the quaternions, or the octonions. Such examples are called *symmetric*.

Benoist [B6, Prop. 4.2] gave further examples by considering nonhyperbolic Coxeter groups

$$W = \big\langle s_1, \ldots, s_5 \mid (s_i s_j)^{m_{i,j}} = 1 \ \forall\ 1 \leq i, j \leq 5 \big\rangle$$

for which the matrix $(m_{i,j})_{1\leq i,j\leq 5}$ is given by

$$\begin{pmatrix} 1 & 3 & 3 & 2 & 2 \\ 3 & 1 & 3 & 2 & 2 \\ 3 & 3 & 1 & m & 2 \\ 2 & 2 & m & 1 & \infty \\ 2 & 2 & 2 & \infty & 1 \end{pmatrix}$$

with $m \in \{3,4,5\}$. He constructed explicit injective and discrete representations $\rho : W \to \mathrm{PGL}(\mathbb{R}^4)$ for which the group $\rho(W)$ divides a properly convex open set $\Omega \subset \mathbb{P}(\mathbb{R}^4)$. Similar examples can be constructed in $\mathbb{P}(\mathbb{R}^n)$ for $n = 5,6,7$ [B6, Rem. 4.3]. These groups admit abelian subgroups isomorphic to $\mathbb{Z}^{n-2}$.

Ballas–Danciger–Lee [BDL] gave more examples of nonhyperbolic groups dividing properly convex open sets $\Omega \subset \mathbb{P}(\mathbb{R}^4)$. In particular, if a finite-volume hyperbolic 3-manifold M with torus cusps is *infinitesimally projectively rigid relative to the cusps* (see [BDL, Def. 3.1]), then the cusp group becomes diagonalizable in small deformations of the holonomy of M and the double DM of the manifold along its torus boundary components admits a properly convex projective structure. The universal cover $\widetilde{DM}$ identifies with a properly convex open subset of $\mathbb{P}(\mathbb{R}^4)$ which is divided by the fundamental group $\pi_1(DM)$.

Recently, for $n = 5,6,7$, Choi–Lee–Marquis [CLM] have constructed other examples of injective and discrete representations ρ of certain nonhyperbolic Coxeter groups W into $\mathrm{PGL}(\mathbb{R}^n)$ for which the group $\rho(W)$ divides a properly convex open set $\Omega \subset \mathbb{P}(\mathbb{R}^n)$. These groups admit abelian subgroups isomorphic to $\mathbb{Z}^{n-3}$ but not to $\mathbb{Z}^{n-2}$.

12.2.2. *Convex cocompact deformations of groups dividing a convex set.* Let Γ be a discrete subgroup of $\mathrm{PGL}(V)$ dividing a nonempty properly convex open subset Ω of $\mathbb{P}(V)$. By Theorem 1.17.(D)–(F), any small deformation of the inclusion of Γ into a larger $\mathrm{PGL}(V \oplus V')$ is convex cocompact in $\mathbb{P}(V \oplus V')$.

For nonhyperbolic Γ, this works well in particular for $V = \mathbb{R}^4$. Indeed, Benoist [B6] described a beautiful structure theory for indecomposable divisible convex sets in $\mathbb{P}(\mathbb{R}^4)$: the failure of strict convexity is due to the presence of PETs; the stabilizer in Γ of each PET of Ω is virtually the fundamental group $\mathbb{Z}^2$ of a torus, and the group Γ splits as an amalgamated product or HNN extension over the stabilizer. Thus the inclusion of (the lift to $\mathrm{SL}^{\pm}(\mathbb{R}^4)$ of) Γ into a larger $\mathrm{PGL}(\mathbb{R}^4 \oplus \mathbb{R}^{n'})$ may be deformed nontrivially by a Johnson–Millson bending construction [JM] along PETs. Such small deformations give examples of nonhyperbolic subgroups which are convex cocompact in $\mathbb{P}(\mathbb{R}^4 \oplus \mathbb{R}^{n'})$, but do not divide a properly convex set. The deformations may be chosen irreducible (indeed this is generically the case).

12.2.3. *Convex cocompact groups coming from divisible convex sets by taking subgroups.* We now explain that nonhyperbolic groups dividing a properly convex open set $\Omega \subset \mathbb{P}(\mathbb{R}^4)$ admit nonhyperbolic subgroups that are still convex cocompact in $\mathbb{P}(\mathbb{R}^4)$, but that do not divide any properly convex open subset of $\mathbb{P}(\mathbb{R}^4)$.

Let Γ be a torsion-free, nonhyperbolic, discrete subgroup of $\mathrm{PGL}(\mathbb{R}^4)$ dividing a nonempty properly convex open subset Ω of $\mathbb{P}(\mathbb{R}^4)$, as above. By [B6], the PETs in Ω descend to a finite collection of pairwise disjoint planar tori and Klein bottles in the closed manifold $N := \Gamma\backslash\Omega$. Cutting N along this collection yields a collection of compact manifolds M with torus and Klein bottle boundary components. Each such M is a compact convex projective manifold with boundary, but from topological considerations, Benoist [B6] proved that $M \smallsetminus \partial M$ also admits a finite-volume hyperbolic structure (in which the boundary tori and Klein bottles are cusps). The universal cover of M identifies with a convex subset $\mathcal{C}_M$ of Ω whose interior is a connected component of the complement in Ω of the union of all PETs, and whose nonideal boundary $\partial_{\mathrm{n}}\mathcal{C}_M$ is a disjoint union of PETs. The fundamental group of M identifies with the subgroup Γ_M of Γ that preserves $\mathcal{C}_M$, and it acts properly discontinuously and cocompactly on $\mathcal{C}_M$.

Lemma 12.4. *The action of Γ_M on Ω is convex cocompact.*

Proof. By Lemma 4.1.(1), it is enough to check that $\mathcal{C}^{\mathsf{cor}}_{\Omega}(\Gamma_M) \subset \mathcal{C}_M$. Since each component of $\partial_{\mathrm{n}}\mathcal{C}_M$ is planar, $\mathcal{C}_M$ is the convex hull of its ideal boundary $\partial_{\mathrm{i}}\mathcal{C}_M$, and so it is enough to show that $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma_M) \subset \partial_{\mathrm{i}}\mathcal{C}_M$. Suppose by contradiction that this is not the case: namely, there exists $y \in \Omega \smallsetminus \mathcal{C}_M$ and a sequence (γ_m) in Γ_M such that $(\gamma_m \cdot y)$ converges to some $y_\infty \in \Lambda^{\mathsf{orb}}_{\Omega}(\Gamma_M) \smallsetminus \partial_{\mathrm{i}}\mathcal{C}_M$. Let z be the point of $\partial_{\mathrm{n}}\mathcal{C}_M$ which is closest to y for the Hilbert metric d_Ω; it is contained in a PET of $\partial_{\mathrm{n}}\mathcal{C}_M$. Let $a, b \in \partial\Omega$ be such that a, y, z, b are aligned in that order. Up to taking a subsequence, we may assume that $(\gamma_m \cdot z)$, $(\gamma_m \cdot a)$, and $(\gamma_m \cdot b)$ converge respectively to some $z_\infty, a_\infty, b_\infty \in \partial\Omega$, with $z_\infty \in \partial_{\mathrm{i}}\mathcal{C}_M$ and $a_\infty, y_\infty, z, b_\infty$ aligned in that order. The cross-ratio $[a_\infty, y_\infty, z_\infty, b_\infty]$ is equal to $[a, y, z, b] \in (1, +\infty)$, hence the points $a_\infty, y_\infty, z, b_\infty$ are pairwise distinct and contained in a segment of $\partial\Omega$. However, any segment on $\partial\Omega$ lies on the boundary of some PET [B6]. Thus we have found a PET whose closure intersects the closure of $\mathcal{C}_M$ but is not contained in it; its closure must cross the closure of a second PET on $\partial_{\mathrm{n}}\mathcal{C}_M$, contradicting the fact [B6] that PETs have disjoint closures. □

The deformation construction of Ballas–Danciger–Lee [BDL] discussed in section §12.2.1 yields many more examples, similar to Lemma 12.4, of convex cocompact subgroups in $\mathrm{PGL}(\mathbb{R}^4)$ for which the nonideal boundary of the convex core consists of a union of PETs. As in §12.2.1, let M be a compact, connected, orientable 3-manifold with a union of tori as boundary whose interior admits a finite-volume complete hyperbolic structure which is infinitesimally projectively rigid relative to the cusps in the sense of [BDL, Def. 3.1]. Then [BDL, Th. 1.3] gives properly convex projective deformations of the hyperbolic structure on M where each boundary component is a totally geodesic (i.e. planar) torus. It is an easy corollary of techniques in [BDL] that the holonomy representations in $\mathrm{PGL}(\mathbb{R}^4)$ for such convex projective structures have image that is convex cocompact in $\mathbb{P}(\mathbb{R}^4)$. One way to see this is that each planar torus boundary component admits an explicit strictly convex thickening.

12.3. **Convex cocompact groups as free products.** Being convex cocompact in $\mathbb{P}(V)$ is a much more flexible property than dividing a properly convex open subset of $\mathbb{P}(V)$, and there is a rich world of examples, which we shall explore in forthcoming work [DGK5]. In particular, we shall prove the following.

Proposition 12.5 ([DGK5]). *Let Γ_1 and Γ_2 be infinite discrete subgroups of $\mathrm{PGL}(V)$ which are convex cocompact in $\mathbb{P}(V)$ but do not divide any nonempty properly convex open subset of $\mathbb{P}(V)$. Then there exists $g \in \mathrm{PGL}(V)$ such that the group generated by Γ_1 and $g\Gamma_2 g^{-1}$ is isomorphic to the free product $\Gamma_1 * \Gamma_2$ and is convex cocompact in $\mathbb{P}(V)$.*

This yields many examples of nonhyperbolic convex cocompact groups, for instance Coxeter groups by taking e.g. $\Gamma_1 = \Gamma_2$ to be the subgroup of $\mathrm{PGL}(\mathbb{R}^n)$ of Example 3.8, embedded into $\mathrm{PGL}(V)$ for $V = \mathbb{R}^{n+1}$.

Appendix A. Some open questions

Here we list some open questions about discrete subgroups Γ of $\mathrm{PGL}(V)$ that are convex cocompact in $\mathbb{P}(V)$. The case that Γ is word hyperbolic boils down to Anosov representations by Theorem 1.4 and is reasonably understood; on the other hand, the case that Γ is *not* word hyperbolic corresponds to a new class of discrete groups whose study is still in its infancy. Most of the following questions are interesting even in the case that Γ divides a (nonstrictly convex) properly convex open subset of $\mathbb{P}(V)$.

We fix a discrete subgroup Γ of $\mathrm{PGL}(V)$ acting convex cocompactly on a nonempty properly convex open subset Ω of $\mathbb{P}(V)$.

Question A.1. Which finitely generated subgroups Γ' of Γ are convex cocompact in $\mathbb{P}(V)$?

Such subgroups are necessarily quasi-isometrically embedded in Γ, by Corollary 10.2. Conversely, when Γ is word hyperbolic, any quasi-isometrically embedded (or equivalently quasi-convex) subgroup Γ' of Γ is convex cocompact in $\mathbb{P}(V)$: indeed, the boundary maps for the P_1-Anosov representation $\Gamma \hookrightarrow \mathrm{PGL}(V)$ induce boundary maps for $\Gamma' \hookrightarrow \mathrm{PGL}(V)$ that make it P_1-Anosov, hence Γ' is convex cocompact in $\mathbb{P}(V)$ by Theorem 1.4.

When Γ is not word hyperbolic, Question A.1 becomes more subtle. For instance, let $\Gamma \simeq \mathbb{Z}^2$ be the subgroup of $\mathrm{PGL}(\mathbb{R}^3)$ consisting of all diagonal matrices whose entries are powers of some fixed $t > 0$; it is convex cocompact in $\mathbb{P}(\mathbb{R}^3)$ (see Example 3.7). Any cyclic subgroup $\Gamma' = \langle \gamma' \rangle$ of Γ is quasi-isometrically embedded in Γ. However, Γ' is convex cocompact in $\mathbb{P}(V)$ if and only if γ' has distinct eigenvalues (see Examples 3.9.(1)–(2)).

Question A.2. Assume Ω is indecomposable. Under what conditions is Γ relatively hyperbolic, relative to a family of virtually abelian subgroups?

This is always the case if $\dim(V) \leq 3$. If $\dim(V) = 4$ and Γ divides Ω, then this is also seen to be true from work of Benoist [B6]: in this case there are finitely many conjugacy classes of PET stabilizers in Γ, which are virtually $\mathbb{Z}^2$, and Γ is relatively hyperbolic with respect to the PET stabilizers (using [Dah, Th. 0.1]). However, when $\dim(V) = m(m-1)/2$ for some $m \geq 3$, we can take for $\Omega \subset \mathbb{P}(V)$ the projective realization of the Riemannian symmetric space of $\mathrm{SL}_m(\mathbb{R})$ (see [B7, § 2.4]) and for Γ a uniform lattice of $\mathrm{SL}_m(\mathbb{R})$: this Γ is *not* relatively hyperbolic with respect to any subgroups, see [BDM].

Question A.3. If Γ is not word hyperbolic, must there be a properly embedded maximal k-simplex invariant under some subgroup isomorphic to $\mathbb{Z}^k$ for some $k \geq 2$?

This question is a specialization, to the class of convex-cocompact subgroups of $\mathrm{PGL}(V)$, of the following more general question (see [Be, Q 1.1]): if $\mathcal{G}$ is a finitely generated group admitting a finite $K(\mathcal{G}, 1)$, must it be word hyperbolic as soon as it does not contain a Baumslag–Solitar group $BS(m, n)$? Here we view $\mathbb{Z}^2$ as the Baumslag–Solitar group $BS(1, 1)$. Note that our convex cocompact group $\Gamma \subset \mathrm{PGL}(V)$ cannot contain $BS(m, n)$ for $m, n \geq 2$ since it does not contain any unipotent element by Theorem 1.17.(C).

In light of the equivalence (ii) $\Leftrightarrow$ (vi) of Theorem 1.15, we ask the following:

Question A.4. When Γ is not word hyperbolic, is there a dynamical description, similar to P_1-Anosov, that characterizes the action of Γ on $\mathbb{P}(V)$, for example in terms of $\Lambda^{\mathsf{orb}}_{\Omega}(\Gamma)$ and divergence of Cartan projections?

While this question is vague, a good answer could lead to the definition of new classes of nonhyperbolic discrete subgroups in other higher-rank reductive Lie groups for which there is not necessarily a good notion of convexity.

A variant of the following question was asked by Olivier Guichard.

Question A.5. In $\mathrm{Hom}(\Gamma, \mathrm{PGL}(V))$, does the interior of the set of naively convex cocompact representations consist of convex cocompact representations?

Here we say that a representation is naively convex cocompact (resp. convex cocompact) if it is faithful and if its image is naively convex cocompact in $\mathbb{P}(V)$ (resp. convex cocompact in $\mathbb{P}(V)$) in the sense of Definition 1.9 (resp. Definition 1.11).

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