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On an Elementary Proof of Rivin's Characterization of Convex Ideal Hyperbolic Polyhedra by their Dihedral Angles★

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Abstract. In 1832, Jakob Steiner (*Systematische Entwicklung der Abhängigkeit geometrischer Gestalten von einander*, Reimer (Berlin)) asked for a characterization of those planar graphs which are combinatorially equivalent to polyhedra inscribed in the sphere. This question was answered in the 1990s by Igor Rivin (*Ann. of Math.* **143** (1996), 51–70), as a byproduct of his classification of ideal polyhedra in hyperbolic 3-space. Rivin also proposed a more direct approach to these results in Rivin (*Ann. of Math.* **139** (1994), 553–580). In this paper, we prove a combinatorial result (Theorem 6) which enables one to complete the program of Rivin (*Ann. Math.* **139** (1994), 553–580).★★

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1. Introduction

The boundary of a convex polyhedron in $\mathbb{R}^3$ is homeomorphic to the 2-sphere, and is naturally tessellated by its decomposition into faces, edges and vertices. Two polyhedra are *combinatorially equivalent* when the corresponding tessellations of the 2-sphere are isotopic.

Let $\mathbb{P}$ be such a spherical tessellation, obtained from a convex polyhedron Π in $\mathbb{R}^3$, and considered up to isotopy. Then $\mathbb{P}$ admits a *dual tessellation* $\mathbb{P}^*$ of the 2-sphere, whose faces, edges and vertices respectively correspond to the vertices, edges and faces of $\mathbb{P}$. In particular, there is a natural bijection between the edge sets $\mathbb{E}$ and $\mathbb{E}^*$ of $\mathbb{P}$ and $\mathbb{P}^*$. For every function $\alpha : \mathbb{E} \to X$ valued in a set X, we will denote by $\alpha^*: \mathbb{E}^* \to X$ the composition of α and of the identification $\mathbb{E} \cong \mathbb{E}^*$.

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★★The gap was actually filled by Rivin himself as early as 1996; a full account of his more general argument, using network flow analysis and duality between linear programs, can be found in his recently published article [6]. He and the referee kindly made me aware of the existence of this paper, which was unknown to me at the time of submission.

We consider the problem of deciding when the polyhedron Π is combinatorially equivalent to an inscribed polyhedron, namely to a convex polyhedron Π^{in} all whose vertices are on the standard sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ (this is Jakob Steiner's original question [7]). In the Beltrami–Klein projective model for the hyperbolic space $\mathbb{H}^3$, such an inscribed polyhedron Π^{in} can also be interpreted as an *ideal* hyperbolic polyhedron, namely one with all of its vertices on the sphere at infinity.

Call a subset Λ of the dual edge set $\mathbb{E}^*$ a *simple closed curve* when the union of its elements is a topological circle. When this topological circle is the boundary of a face, we will say abusively that Λ itself is the boundary of a face. Rivin's result is the following.

THEOREM 1 (Rivin). *The convex Euclidean polyhedron Π is combinatorially equivalent to an inscribed polyhedron Π^{in} if and only if there is a function $\alpha: \mathbb{E} \to \mathbb{R}$ such that*

(1) $0 < \alpha(e) < \pi$ *for every e in $\mathbb{E}$;*
(2) $\sum_{\varepsilon\in\Lambda} \alpha^*(\varepsilon) = 2\pi$ *if $\Lambda \subset \mathbb{E}^*$ is the boundary of a face of $\mathbb{P}^*$;*
(3) $\sum_{\varepsilon\in\Lambda} \alpha^*(\varepsilon) > 2\pi$ *for every simple closed curve $\Lambda \subset \mathbb{E}^*$ which is not the boundary of a face of $\mathbb{P}^*$.*

Moreover, for a function $\alpha: \mathbb{E} \to \mathbb{R}$ satisfying these conditions, the ideal polyhedron Π^{in} can be chosen so that for every e in $\mathbb{E}$, the exterior dihedral angle of Π^{in} along the edge corresponding to e is equal to $\alpha(e)$, and Π^{in} is then unique up to hyperbolic isometry.

Theorem 1 has a converse: namely, given a realization of $\mathbb{P}$ as an ideal hyperbolic polyhedron Π^{in}, the exterior dihedral angle function $\alpha: \mathbb{E} \to \mathbb{R}$ associated to Π^{in} will always satisfy Conditions 1–3 of Theorem 1 (only Condition 3 is not straightforward: see [4] or [1]). So Theorem 1 expresses an affine parametrization of the space of isometric classes of ideal hyperbolic realizations of $\mathbb{P}$.

Theorem 1 reduces Steiner's question to solving a system of affine inequalities, i.e. to a linear programming problem. It can be shown [6] that this problem is time-polynomial in the number of edges of $\mathbb{P}$.

As an illustration of Theorem 1, consider a convex Euclidean polyhedron Π_0 having v vertices and f faces, with $v \leqslant f$. Add a small pyramid on each face of Π_0, and let Π be the resulting convex polyhedron. The edges of Π fall into two categories: those which are also edges of Π_0, and those which go from a vertex of Π_0 to the apex of a pyramid. The criterion of Theorem 1 shows that Π is not combinatorially equivalent to an inscribed polyhedron. Indeed, if there was a function $\alpha: \mathbb{E} \to \mathbb{R}$ satisfying the conditions of the theorem, the weights $\alpha(e)$ of the edges adjacent to the apices of the pyramids would add up to $2\pi f$ (by Condition 2), but this same sum would be redistributed over only v vertices of Π_0; so at least one edge e of Π_0 would have a nonpositive weight $\alpha(e) \leqslant 0$, contradicting Condition 1.

The original proof [5] of Theorem 1 by Rivin is based on the characterization [2] of convex hyperbolic polyhedra by their dual metrics, in the compact and ideal cases. The proof uses Alexandrov's nonexplicit correspondence between polyhedra and singular metrics on the sphere, thus losing track of the combinatorics of the polyhedron halfway in the argument. In [3], Rivin established Theorem 4 below, thus indicating a new approach, where the sought hyperbolic polyhedron appears as a point maximizing a certain functional over an affine domain. To obtain a new proof of Theorem 1, it remained to establish that this domain can never be empty, a combinatorial result which is the aim of this paper (Theorem 6 below).

Section 1 gives an outline of the results of [3] (including Theorem 4) which are useful to the proof of Theorem 1. The further sections are devoted to proving Theorem 6. A more detailed exposition of the proof of Theorem 1, gathering all the elements of [3] and of the current paper, can be found in [1].

2. Facts About Hyperbolic Polyhedra

2.1. HYPERBOLIC POLYHEDRA AND PLANE TESSELLATIONS

We start with the data of a map $\alpha\colon\ \mathbb{E} \to \mathbb{R}$ satisfying Conditions 1–3 of Theorem 1, and aim at realizing it as the dihedral angle data of an ideal polyhedron which is combinatorially equivalent to Π. If we find such an ideal polyhedron and make sure it is unique, Theorem 1 will be proved.

Choose a vertex v_∞ of the spherical tessellation $\mathbb{P}$ associated to Π and split all adjacent faces into triangles with apex v_∞. Then, triangulate all remaining faces of $\mathbb{P}$ by adding diagonal edges in any arbitrary way. Denote by P (not boldface) the resulting spherical tessellation, and by $E \supset \mathbb{E}$ its edge set. Extending α to E by making it 0 on $E - \mathbb{E}$, this enables us to work with P which has only triangular faces (but certain vanishing dihedral angles $\alpha(e)$).

DEFINITION 2. Let D denote the triangulated topological disk obtained from P by removing v_∞ together with all adjacent edges and faces. Each face of D is a triangle, whose three corners are called *combinatorial angles*. The set of all combinatorial angles of D is denoted by A.

Now suppose that the problem is solved, namely that we are able to realize the tessellation P and the edge weights $\alpha\colon\ E \to \mathbb{R}$ by an ideal polyhedron, also noted Π, in $\mathbb{H}^3$. Switch to Poincaré's upper half-space model for $\mathbb{H}^3$, in such a way that the vertex v_∞ corresponds to the point ∞. Project Π vertically down to $\mathbb{C}$ in this upper half-space model. Because Π is star-shaped with respect to v_∞, this provides an embedding $\sigma\colon\ D \to \mathbb{C}$ sending each face of D to a nondegenerate Euclidean triangle.

A face f of D determines a Euclidean triangle $\sigma(f) \subset \mathbb{C}$ and an ideal tetrahedron $T_f \subset \Pi$ which is the preimage of $\sigma(f)$ under the projection $\Pi \to \mathbb{C}$. Namely, T_f is the ideal tetrahedron whose vertices are ∞ and the three vertices of $\sigma(f)$. In an ideal tetrahedron of $\mathbb{H}^3$, disjoint edges have the same dihedral angles. It follows that, if e is an edge of f, the dihedral angle of T_f along the edge corresponding to e is equal to the angle of the Euclidean triangle $\sigma(f)$ at the vertex opposite e.

Each combinatorial angle a in A corresponds to the angular sector at a vertex of the Euclidean triangle $\sigma(f)$, for some face f of D. Let $\theta(a) \in]0, \pi[$ denote the corresponding Euclidean angle of $\sigma(f)$.

Next we explain how the function $\theta : A \to]0, \pi[$ encodes the complete data of $\alpha : E \to [0, \pi[$. If e is an edge of the boundary of D, it faces exactly one combinatorial angle a in A and the above analysis implies that the interior dihedral angle $\pi - \alpha(e)$ of Π along the edge corresponding to e is equal to $\theta(a)$. Similarly, if e is an edge of D which is not on the boundary, it faces exactly two combinatorial angles a, a' in A, and $\pi - \alpha(e) = \theta(a) + \theta(a')$. Finally, if v is a vertex of the boundary of D, it determines an edge vv_∞ of P, and $\pi - \alpha(vv_\infty)$ is equal to the interior angle of $\sigma(D)$ at $\sigma(v)$, namely to the sum of the angles $\theta(a)$ as a ranges over all combinatorial angles that are adjacent to v.

We can summarize these properties by introducing three functions $\mathcal{F} : A \longrightarrow \{\text{faces of } D\}$, $\mathcal{E} : A \longrightarrow \{\text{edges of } D\}$ and $\mathcal{V} : A \longrightarrow \{\text{vertices of } D\}$ with the property that the combinatorial angle a in A lies in the triangular face $\mathcal{F}(a)$, at the vertex $\mathcal{V}(a)$ and opposite the edge $\mathcal{E}(a)$ in $\mathcal{F}(a)$. The function $\theta : A \to]0, \pi[$ then satisfies the following conditions:

$$\sum_{a \in \mathcal{E}^{-1}(e)} \theta(a) = \pi - \alpha(e) \quad \text{for all edges } e \text{ of } D, \tag{1}$$

$$\sum_{a \in \mathcal{F}^{-1}(f)} \theta(a) = \pi \quad \text{for all faces } f \text{ of } D, \tag{2}$$

$$\sum_{a \in \mathcal{V}^{-1}(v)} \theta(a) = \pi - \alpha(v_\infty v) \quad \text{for all boundary vertices } v \text{ of } D, \tag{3}$$

$$\sum_{a \in \mathcal{V}^{-1}(v)} \theta(a) = 2\pi \quad \text{for all interior vertices } v \text{ of } D. \tag{4}$$

Conditions (2) and (4) hold for any Euclidean tessellation, but will play an important rôle later on. In particular, we now have a constraint associated to each vertex, edge and face of D.

Conversely, it is not hard to show (see, for example, [1]):

PROPOSITION 3. *Any Euclidean embedding* $\sigma : D \to \mathbb{C}$ *whose associated angle function* $\theta : A \to]0, \pi[$ *satisfies Conditions* (1)–(4) *is the vertical projection of an ideal hyperbolic polyhedron* Π *which realizes* P *with the dihedral angle data* α *and the vertex* v_∞ *at infinity.*

2.2. RIVIN'S CHARACTERIZATION OF θ

Not every positive solution θ to Equations (1)–(4) can be realized by a Euclidean embedding of D. For instance, start with an actual Euclidean embedding of D, with associated angle function $\theta : A \to]0, \pi[$. Pick an interior vertex v of D, and let $a_1, \dots, a_n \in A$ be the combinatorial angles adjacent to v. For each i, let b_i and c_i be the two other combinatorial angles of the triangular face containing a_i, in such a way that a_i, b_i, c_i counterclockwise occur in this order. For a small $\delta > 0$, let $\theta' : A \to]0, \pi[$ be defined by the property that, for every i, $\theta'(b_i) = \theta(b_i) + \delta$ and $\theta'(c_i) = \theta(c_i) - \delta$; away from the b_i and c_i, just put $\theta' = \theta$. Then, θ' is also a solution to Equations (1)–(4), but one easily sees that it cannot be realized by a Euclidean embedding of D. Indeed, edge lengths cannot match; see Figure 1.

However, Rivin showed in [3] that this is essentially the only obstruction. In addition, using the fact that *each* vertex, edge and face of D is associated to an affine constraint, he proved the following result.

THEOREM 4 (Rivin). *Let Θ^+ be the space of solutions $\theta : A \to]0, \pi[$ to Equations* (1)–(4). *The θ in Θ^+ which can be realized as the angle data of a Euclidean embedding of D are exactly the critical points of the Lobachevski volume function $\mathcal{L} : \Theta^+ \longrightarrow \mathbb{R}$ defined by*

$$\mathcal{L}(\theta) = -\sum_{a \in A} \int_0^{\theta(a)} \log 2 \sin(u)\, \mathrm{d}u.$$

Note that Θ^+ is a convex polytope in the affine space of solutions $\theta : A \to \mathbb{R}$ to (1)–(4). Rivin also proved:

THEOREM 5 (Rivin). *If the convex polytope Θ^+ is nonempty, then the Lobachevski volume function $\mathcal{L} : \Theta^+ \longrightarrow \mathbb{R}$ admits a unique critical point.*

This simply follows from the fact that $\mathcal{L}$ is concave and from a careful analysis of $\mathcal{L}$ near the boundary of Θ^+, where partial derivatives diverge (see [3] or [1]).

The main result of this paper is the following theorem:

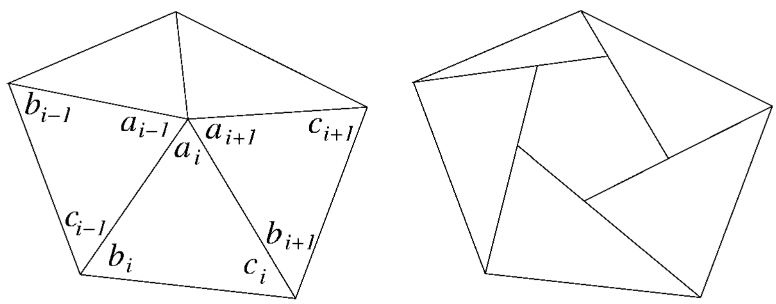

Figure 1.

THEOREM 6. *If the dihedral angle data* α *satisfies the conditions of Theorem 1, then the space* Θ^+ *is nonempty. Namely, Equations* (1)–(4) *admit at least one positive solution.*

Note that, in Theorem 4, the Euclidean embedding of D is completely determined by θ up to similitude of $\mathbb{C}$, namely up to the restriction of an isometry of the hyperbolic space $\mathbb{H}^3$ fixing ∞; hence uniqueness. Therefore, Theorem 1 follows from Proposition 3 and Theorems 4–6.

3. Existence of Linear Solutions

To prove Theorem 6, we first consider the space Θ of solutions $\theta: \ A \to]-\infty, \infty[$ to Equations (1)–(4), postponing to the next section the problem of finding *positive* solutions to these equations.

PROPOSITION 7. *The space* Θ *is a nonempty affine subspace of* $\mathbb{R}^A$, *whose dimension is equal to the number of interior vertices of* D.

Proof. Equations (1)–(4) are linear, so Θ is clearly an affine subspace of $\mathbb{R}^A$, possibly empty.

We first show that we can omit Equations (3) and (4).

LEMMA 8. *Equations* (3) *and* (4) *are redundant with Equations* (1) *and* (2). *In other words, every solution to Equations* (1) *and* (2) *is also a solution of Equations* (3) *and* (4).

Proof. Suppose that θ satisfies Equations (1) and (2).

For a boundary vertex v of D, let $a_1, a_2, \ldots, a_n \in A$ be the combinatorial angles of D at v, occurring counterclockwise in this order at v. Let b_i and c_i be the other two angles of the face of D containing a_i, in such a way that a_i, b_i and c_i occur in this order counterclockwise in this face. Then,

$$\begin{aligned}\sum_{i=1}^{n} \theta(a_i) &= \sum_{i=1}^{n} \pi - \theta(b_i) - \theta(c_i) \\ &= -\pi + (\pi - \theta(c_1)) + \left(\sum_{i=1}^{n-1} \pi - \theta(b_i) - \theta(c_{i+1}) \right) + (\pi - \theta(b_n)) \\ &= -\pi + \sum_{vy \in E, y \neq v_\infty} \alpha(vy) = -\pi + (2\pi - \alpha(v_\infty v)) = \pi - \alpha(v_\infty v),\end{aligned}$$

where vy denotes the edge joining the vertices v and y. Here we used the fact that α satisfies Condition 2 of Theorem 1. This proves that θ also satisfies Equation (3).

Similarly, if v is an interior vertex of D, label the combinatorial angles of D at v as $a_1, a_2, \ldots, a_n, a_{n+1} = a_1$, counterclockwise. Again, let b_i and c_i be the other two angles of the face of D containing a_i, with the same counterclockwise convention as above. Then,

$$\sum_{i=1}^{n} \theta(a_i) = \sum_{i=1}^{n} \pi - \theta(b_i) - \theta(c_i) = \sum_{i=1}^{n} \pi - \theta(b_i) - \theta(c_{i+1})$$
$$= \sum_{vy \in E} \alpha(vy) = 2\pi$$

which proves that θ also satisfies Equation (4). □

By Lemma 8, we can therefore focus attention on Equations (1) and (2).

These equations are not independent either. Rewrite them as

$$u_e(\theta) = \pi - \alpha(e) \quad \text{for all edges } e \text{ of } D,$$
$$u_f(\theta) = \pi \quad \text{for all faces } f \text{ of } D,$$

where the linear forms $u_e, u_f: \mathbb{R}^A \to \mathbb{R}$ are defined by $u_e(\theta) = \sum_{a \in \mathcal{E}^{-1}(e)} \theta(a)$ and $u_f(\theta) = \sum_{a \in \mathcal{F}^{-1}(f)} \theta(a)$. Define V_D, E_D, F_D as the sets of vertices, edges, and faces of D, respectively. A *relation* between the u_e and u_f is an equation of the form $\sum_{e \in E_D} \lambda(e) \cdot u_e + \sum_{f \in F_D} \mu(f) \cdot u_f = 0$ for two functions $\lambda: E_D \to \mathbb{R}$ and $\mu: F_D \to \mathbb{R}$.

LEMMA 9. *The relations between the u_e, $e \in E_D$ and u_f, $f \in F_D$ form a vector space of dimension one.*

Proof. For such a relation, set $u = \sum_{e \in E_D} \lambda(e) \cdot u_e = -\sum_{f \in F_D} \mu(f) \cdot u_f$. For all $a \in A$, denote by $\kappa_a: \mathbb{R}^A \to \mathbb{R}$ the ath-coordinate function. Then we can rewrite $u = \sum_{a \in A} \lambda(\mathcal{E}(a)) \cdot \kappa_a = \sum_{a \in A} -\mu(\mathcal{F}(a)) \cdot \kappa_a$ (by definition of u_e, u_f). Consequently, there exists a function $w: A \to \mathbb{R}$, such that $w = \lambda \circ \mathcal{E} = -\mu \circ \mathcal{F}$. In particular, w is constant on each triangle, and the constants for adjacent triangles are equal. By connectedness of D, it follows that w is constant. Therefore, the functions λ and μ are constant, with opposite values.

Conversely, two constant functions $\lambda: E \to \mathbb{R}$ and $\mu: F \to \mathbb{R}$ with opposite values define a relation between the u_e and u_f, since $\sum_{e \in E_D} u_e = \sum_{f \in F_D} u_f = \sum_{a \in A} \kappa_a$.

This concludes the proof of Lemma 9. □

LEMMA 10. *Equations* (1) *and* (2) *are compatible.*

Proof. For this we have to check that, for every relation

$$\sum_{e \in E_D} \lambda(e) \cdot u_e + \sum_{f \in F_D} \mu(f) \cdot u_f = 0$$

between the left-hand terms of Equations (1) and (2), we have a similar relation

$$\sum_{e \in E_D} \lambda(e)(\pi - \alpha(e)) + \sum_{f \in F_D} \mu(f)\pi = 0$$

between the right-hand terms.

By the proof of Lemma 9, λ and μ are constant and opposite, so it suffices to check that

$$\sum_{e\in E_D}(\pi-\alpha(e))-\sum_{f\in F_D}\pi=0$$

And indeed,

$$\begin{aligned}
&\sum_{e\in E_D}(\pi-\alpha(e))-\sum_{f\in F_D}\pi\\
&\quad=|E_D|\pi-\sum_{e\in E_D}\alpha(e)-|F_D|\pi\\
&\quad=|E_D|\pi-\frac{1}{2}\sum_{v\in V_D}\sum_{vy\in E}\alpha(vy)+\frac{1}{2}\sum_{v_\infty y\in E}\alpha(v_\infty y)-|F_D|\pi\\
&\quad=|E_D|\pi-|V_D|\pi+\pi-|F_D|\pi=0
\end{aligned}$$

using the facts that α satisfies Condition 2 of Theorem 1 and that the disk D has Euler characteristic one. □

Combining Lemmas 9 and 10 and noting that A has $3|F_D|$ elements, we conclude that the space Θ of solutions to Equations (1) and (2) is a (nonempty) affine subspace of $\mathbb{R}^A$ of dimension $3|F_D|-|E_D|-|F_D|+1=3|F_D|-2|E_D|+|V_D|$.

Let $|V_{\partial D}|$ denote the number of vertices of the boundary of D, which is also the number of edges of this boundary. Then $3|F_D|=2|E_D|-|V_{\partial D}|$, so that the dimension $3|F_D|-2|E_D|+|V_D|$ of Θ is equal to the number $|V_D|-|V_{\partial D}|$ of interior vertices of D.

This completes the proof of Proposition 7. □

It can be shown that the transformations around interior vertices considered at the beginning of Section 1.2 form a basis of the tangent space for the affine space Θ.

4. Finding a Positive Solution

In this section we provide a positive solution $\theta:\ A\rightarrow\,]0,+\infty[$ to Equations (1) and (2) (and therefore to Equations (1)–(4) by Lemma 8). Note that such a solution will have its image in $]0,\pi[$, by Equation (2).

Before going any further, let us look more closely into the triangulation process which yielded the spherical triangulation P from the original spherical tessellation $\mathbb{P}$, and into its consequences on the extension $\alpha:\ E\rightarrow\mathbb{R}$ of $\alpha:\ \mathbb{E}\rightarrow\mathbb{R}$.

The dual tessellation P^* is obtained from $\mathbb{P}^*$ by replacing the neighborhood of each vertex (a star) by a trivalent tree. The edges so introduced are mapped to 0 by the dual map α^*: $E^*\rightarrow\mathbb{R}$. In particular, a simple closed edge path Λ in $\mathbb{P}^*$ determines in a unique way a simple closed edge path Λ' in P^* which satisfies $\int_{\Lambda'}\alpha^*=\int_\Lambda\alpha^*$, using the notation $\int_\Lambda\alpha^*=\sum_{\varepsilon\in\Lambda}\alpha^*(\varepsilon)$.

To check that Conditions 2 and 3 of Theorem 1 still hold for a given path Λ' in the new edge set E^*, we must do the opposite: pull Λ' back to an ‘old’ closed path Λ in $\mathbb{E}^*$ by collapsing every trivalent tree to a point. Then $\int_{\Lambda'}\alpha^*=\int_\Lambda\alpha^*$. However, the path Λ may not be simple (it may contain the same vertex twice); but at any rate Λ is a *union* of simple closed paths, so $\int_{\Lambda'} \alpha^* \geqslant 2\pi$ in all cases. To check that equality holds if and only if Λ' is the boundary of a face, notice that the rotation index of a given path around a point x is constant when x ranges over a face: since the graphs of P^* and $\mathbb{P}^*$ differ only in neighborhoods of the vertices, the rotation indices are the same for Λ and Λ' with respect to any face, so the property of surrounding one (and only one) face is preserved. Thus the extended α^* still satisfies Conditions 2 and 3 of Theorem 1.

Also, the triangulation was done in a very special way near the vertex v_∞: every edge added across a face adjacent to v_∞ had v_∞ as an end point (see Figure 2).

To translate this property into the language of dual objects, let the *special face* denote the face of P^* that is dual to the vertex v_∞. A *peripheral edge* of P^* is one which touches the special face but is not contained in its boundary; equivalently, a peripheral edge is dual to a boundary edge of the disk D. In particular, from the above observation on the triangulation process, every peripheral edge corresponds to an edge of the original $\mathbb{P}^*$ and consequently has nonzero image under α^*.

We can summarize these properties as follows:

$$0 \leqslant \alpha(e) < \pi \quad \text{for every edge } e \in E, \tag{5}$$

$$\int_\Lambda \alpha^* = 2\pi \quad \text{for each boundary } \Lambda \text{ of a face of } P^*, \tag{6}$$

$$\int_\Lambda \alpha^* > 2\pi \quad \text{for any other simple closed edge path } \Lambda \text{ in } P^*, \tag{7}$$

$$(\alpha^*)^{-1}(0) \quad \text{is a union of disjoint trees in } E^*, \tag{8}$$

$$(\alpha^*)^{-1}(0) \quad \text{contains no peripheral edge of } E^*. \tag{9}$$

Let $\vec{E}^*$ be the set of *oriented* edges of P^* (each edge in E^* appears twice in $\vec{E}^*$, with opposite orientations). We represent an element of $\vec{E}^*$ by a little arrow across an edge e in E, determining a transverse orientation for e (and consequently an orientation for the dual edge in E^*). Let $\widetilde{A} \supset A$ be the set of combinatorial angles of faces

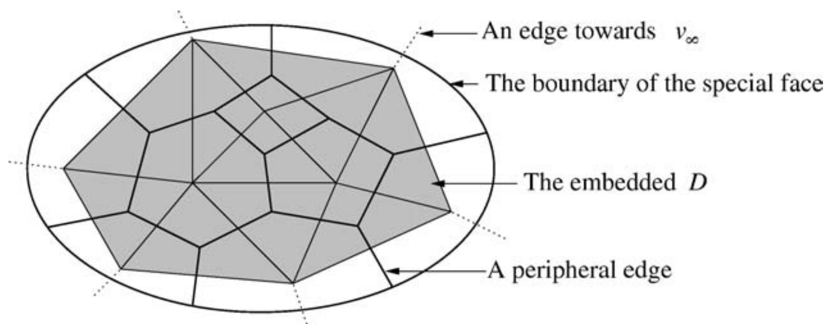

Figure 2. The disk D with P^*.

of P (not just of D). There is a natural bijection $\vec{E}^* \longrightarrow \widetilde{A}$, which to an arrow $\uparrow$ across an edge $e \in E$ associates the combinatorial angle $\widehat{\uparrow}$ in $\widetilde{A}$ opposite e in the face of P towards which the arrow $\uparrow$ is pointing.

Consider some $\theta: A \rightarrow \mathbb{R}$ which satisfies (1)–(4). We use θ to decorate the edges of D in the following way: for each arrow $\uparrow$ in $\vec{E}^*$ across an edge e of D, consider the opposite $a = \widehat{\uparrow} \in \widetilde{A}$, and draw across e a colored representation of $\uparrow$, where the color is chosen as follows:

$$\theta(a) < 0 \quad \Longrightarrow \quad \uparrow \text{ is red}$$

$$\theta(a) = 0 \quad \Longrightarrow \quad \uparrow \text{ is yellow}$$

$$\theta(a) > 0 \quad \Longrightarrow \quad \uparrow \text{ is green}$$

At this point, this process does not apply for the outward-pointing arrows across the boundary edges of D because, for such an arrow $\uparrow$, the angle $a = \widehat{\uparrow}$ is in $\widetilde{A} - A$ so that $\theta(a)$ is not defined; in this case, we decide that $\theta(a) = 0$ and color the arrow yellow. Note that this convention is consistent with the edge condition (1).

Now every edge e of D is overcrossed by two arrows of opposite orientations, and at least one of these two arrows is green; indeed, by (1), the two angles facing e must add up to $\pi - \alpha(e) \in]0, \pi]$.

Our goal is to find a θ for which all the arrows are green, except the outward-pointing arrows on the boundary of D. The main technical step is the following.

LEMMA 11. *Let J be a simple closed curve made up of edges of D. Suppose that all the edges in J are overcrossed by nongreen arrows pointing towards the same side of J. Then J is in fact the boundary of D, endowed with the outward-pointing yellow arrows.*

Proof. The proof will take a while, and will use several steps and sublemmas.

The curve J separates the surface of P into two topological disks, B_0 from which the nongreen arrows point away, and B_1 towards which they point. We are going to look carefully at the maximal subgraph Γ of the 1-skeleton of P^* that is contained in B_1: in particular, the vertices and edges of Γ are dual to the faces and edges of P that are contained in B_1, and Γ is connected since so is B_1. Lemma 11 amounts to proving that Γ is reduced to the boundary of the special face of P^* (the face dual to the vertex v_∞).

DEFINITION 12. *If Γ' is a subgraph of the 1-skeleton of P^*, define the inward-pointing frontier of Γ' as the subset $\Delta(\Gamma')$ of $\overrightarrow{E}^*$ that consists of those oriented edges of P^* which end on a vertex of Γ' but are not contained in Γ'.*

Recall that we also interpreted elements of $\vec{E}^*$ as arrows across edges of P.

SUBLEMMA 13. *The inward-pointing frontier $\Delta\Gamma$ of Γ is the set of nongreen arrows across edges of J.*

Proof. The inclusion $\supset$ is clear by definition of Γ. The converse is also easy, since any oriented edge from a point in B_0 to a point in B_1 must cross J. $\square$

Now we distinguish two cases, according to the component of the complement of J that contains the vertex v_∞.

Case 1: the vertex v_∞ is in B_0. This is the simpler case. The key property here is that Γ is contained in D, since so is B_1.

SUBLEMMA 14. *For any subgraph Γ' of Γ,*

$$\sum_{\uparrow\in\Delta\Gamma'} \theta(\widehat{\uparrow}) + \pi(E_{\Gamma'} - V_{\Gamma'}) = \int_{\Gamma'} \alpha^*,$$

where $E_{\Gamma'}$ and $V_{\Gamma'}$ are the number of edges and vertices of Γ', and where we use the integral notation $\int_{\Gamma'} \alpha^* = \sum_{\varepsilon\in\Gamma'} \alpha^*(\varepsilon)$.

Proof. This is seen by induction on the graph Γ'. By the Face Condition (2) applied to dual objects, the $\theta(\widehat{\uparrow})$ must add up to π when $\uparrow$ runs over the set of three arrows pointing to the same vertex. This proves the identity for any graph Γ' consisting just of vertices. For general Γ', the identity already holds for the 0-skeleton of Γ', so we only need to check that it survives when an edge ε of E^* is added between existing vertices. When such an ε is added, the two arrows $\leftarrow$ and $\rightarrow$ running along ε are suppressed from $\Delta\Gamma'$: both sides of the identity are thus increased by $\alpha^*(\varepsilon)$ since the Edge Condition (1) precisely states that $\theta(\widehat{\leftarrow}) + \theta(\widehat{\rightarrow}) = \pi - \alpha^*(\varepsilon)$. $\square$

Let R be the set of (open) faces of P^* which are entirely contained in B_1. The boundary of a face of R is always included in Γ, and each edge of Γ belongs to at most two faces of R. By (6), it follows that

$$\int_\Gamma \alpha^* \geqslant \sum_{r\in R} \frac{1}{2} \int_{\partial r} \alpha^* = \pi|R|$$

The disk B_1 is the union of the faces r in R, of Γ and of an annulus component of $B_1 - \Gamma$. It follows that the union of Γ and of the r in R has Euler characteristic one. Combining this with Sublemma 14 (applied to $\Gamma' = \Gamma$), we conclude that

$$\sum_{\uparrow\in\Delta\Gamma} \theta(\widehat{\uparrow}) \geqslant \pi(|R| - E_\Gamma + V_\Gamma) = \pi,$$

which contradicts the nonpositiveness of the $\theta(\widehat{\uparrow})$ for $\uparrow$ in $\Delta\Gamma$ (Sublemma 13). Therefore, this Case 1 cannot occur.

Case 2: The vertex v_∞ is in B_1. Then Γ contains the boundary circle $\mathcal{C}$ of the special face dual to the vertex v_∞.

Note that every $\uparrow$ in $\Delta\mathcal{C}$ corresponds to an outward-pointing arrow across an edge of ∂D; in particular, we have already decided by convention that $\theta(\widehat{\uparrow}) = 0$ and that the arrow is colored yellow in this case.

Sublemma 14 is now replaced by the following relation.

SUBLEMMA 15. *For any subgraph Γ' of Γ containing $\mathcal{C}$,*

$$\sum_{\uparrow \in \Delta\Gamma'} \theta(\widehat{\uparrow}) + \pi(E_{\Gamma'} - V_{\Gamma'}) = \int_{\Gamma'} \alpha^* - 2\pi$$

Proof. The proof is analogous to that of Sublemma 14, by induction on Γ'. The relation holds if $\Gamma' = \mathcal{C}$ (both sides vanish in this case), and remains true when one adds isolated vertices or edges between existing vertices. □

Again, consider the set R of faces of P^* that are entirely contained in B_1. The set R contains the special face (dual to v_∞). The complement $\partial P - \Gamma$ consists of the r in R and of one disk component $B_2 \supset B_0$ containing the boundary $\partial B_0 = \partial B_1 = J$. (Note that B_2 cannot be just a face of P^*, since it contains the edges of P^* that are dual to the edges of P contained in J.) In particular, the union of Γ and of the r in R again has Euler characteristic one. Note that some edges of Γ may have both sides in the closure of B_2.

Applying (6) and (7) to a simple closed curve contained in the boundary of each component of $\partial P - \Gamma$, one gets

$$\int_\Gamma \alpha^* \geqslant \sum_{r \in R} \frac{1}{2} \int_{\partial \bar{r}} \alpha^* + \frac{1}{2} \int_{\partial B_2} \alpha^* \tag{10}$$

$$\geqslant (|R| + 1)\pi \tag{11}$$

from which we conclude that

$$\sum_{\uparrow \in \Delta\Gamma} \theta(\widehat{\uparrow}) \geqslant (|R| - E_\Gamma + V_\Gamma - 1)\pi = 0 \tag{12}$$

By Sublemma 13, the $\theta(\widehat{\uparrow})$ with $\uparrow$ in $\Delta\Gamma$ are nonpositive; it follows that (12) is actually an equality, and therefore that both (10) and (11) are also equalities.

Recall that the topological sphere ∂P is the union of Γ, B_2 and the faces of R. By (7), equality in (11) can only occur if ∂B_2 is the union of the boundary of a face f of P^* and of a few edges ε with $\alpha^*(\varepsilon) = 0$. Because B_2 is not reduced to a face of P^*, it follows that R consists of only this face f, which therefore must be the special face dual to the vertex v_∞: as a consequence, $\mathcal{C} \subset \partial B_2$. In addition, since *peripheral* edges have positive weight by (9), ∂B_2 (which is connected) cannot contain any additional edges ε with $\alpha^*(\varepsilon) = 0$. This proves that ∂B_2 is equal to the boundary $\mathcal{C}$ of the special face.

Finally, Γ is reduced to the special circle $\mathcal{C}$, which implies that $J = \partial D$.

This completes the proof of Lemma 11. □

We now return to the affine set Θ of solutions $\theta : A \to \mathbb{R}$ to (1)–(4), which is nonempty by Proposition 7. Consider the function $\varphi : \Theta \longrightarrow [0, +\infty[$ defined by $\varphi(\theta) = \sum_{a \in A} \theta(a)^-$, where $x^- = \max\{-x, 0\}$ for any real number x.

Whenever $\theta(a) > \pi + 2M$ in a face with combinatorial angles a, b, c, it follows from (2) that one of $\theta(b), \theta(c)$ must be smaller than $-M$, which implies $\varphi(\theta) > M$. Therefore, $\varphi^{-1}[0, r]$ is compact for any r (i.e. φ is a *proper* map), and φ has a well-defined minimum.

Let us focus attention to a θ that realizes the minimum of φ and has the largest possible number of green arrows among such realizations of the minimum. We will show that θ is a positive solution to (1)–(4).

Consider a vertex v of D.

LEMMA 16. *If an edge containing v carries a nongreen arrow, then some edge containing v carries a nongreen arrow which turns in the opposite direction, as seen from v. Moreover, these two edges are distinct.*

Proof. The second statement is a necessity, since each dual edge carries at least one green arrow.

The first statement is clear if v is a boundary vertex, because of the outward-pointing yellow arrows.

If v is an interior vertex, we prove the first statement using the modification considered at the beginning of Section 1.2. Label the combinatorial angles of D at v as $a_1, a_2, \ldots, a_n, a_{n+1} = a_1$, counterclockwise. Let b_i and c_i be the other two angles of the face of D containing a_i, so that a_i, b_i, c_i counterclockwise occur in this order around the boundary of this face. By symmetry, we only need to assume $\theta(b_i) \leqslant 0$ for some i, and $\theta(c_j) > 0$ for all j, and to reach a contradiction.

Pick a small $\delta > 0$, add δ to each $\theta(b_i)$ and subtract δ from each $\theta(c_j)$. This provides a new solution θ_δ to (1) and (2). If some $\theta(b_i)$ is strictly negative, then $\varphi(\theta_\delta) < \varphi(\theta)$ for $\delta > 0$ small enough, contradicting the hypothesis that θ minimizes φ. Otherwise, the smallest $\theta(b_i)$ was 0; then $\varphi(\theta_\delta) = \varphi(\theta)$ for $\delta > 0$ small enough, but θ_δ has more green arrows than θ, again contradicting our choice of θ. This proves Lemma 16. □

We are now ready to prove that the outward-pointing arrows across D are the only nongreen arrows. Suppose that an interior edge e_1 of D carries a nongreen arrow. We construct a Jordan curve J contradicting Lemma 11 as follows.

Arbitrarily orient the edge e_1. Lemma 16 provides another edge $e_2 \neq e_1$ adjacent to the end-point of e_1, with a nongreen arrow turning in the opposite direction, meaning that the two nongreen arrows of e_1 and e_2 point to the same side of $e_1 \cup e_2$. Applying the same process at the end point of e_2 and iterating the construction, one obtains an arc $\gamma_n = e_1 \cup e_2 \cup \cdots \cup e_n$ of edges of D, with nongreen arrows crossing the e_i towards the same side of γ_n.

If the path returns to one of its own vertices before hitting the boundary ∂D, it provides a simple closed curve J contradicting Lemma 11.

If γ_n hits the boundary ∂D, start again from e_1 but this time in the opposite direction. Then either one returns again to an earlier vertex, thereby contradicting Lemma 11, or this new arc will eventually reach the boundary as well. In this second case, we now have a simple arc γ made up of interior edges of D, with both ends in ∂D, and such that all edges of γ are crossed by non-green arrows pointing towards the same side of γ. Adding to γ the appropriate arc $\gamma' \subset \partial D$ with its outward pointing yellow arrows again provides a Jordan curve J contradicting Lemma 11.

This proves that all arrows across interior edges of D are in fact green, namely that θ is a positive solution to (1)–(4). This concludes the proof of Theorem 6.

As indicated in the introduction, this combined with [3] provides a new proof of Theorem 1.

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