MARGULIS SPACETIMES VIA THE ARC COMPLEX

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Abstract. We study *strip deformations* of convex cocompact hyperbolic surfaces, defined by inserting hyperbolic strips along a collection of disjoint geodesic arcs properly embedded in the surface. We prove that any deformation of the surface that uniformly lengthens all closed geodesics can be realized as a strip deformation, in an essentially unique way. The infinitesimal version of this result gives a parameterization, by the arc complex, of the moduli space of Margulis spacetimes with fixed convex cocompact linear holonomy. As an application, we provide a new proof of the tameness of such Margulis spacetimes M by establishing the Crooked Plane Conjecture, which states that M admits a fundamental domain bounded by piecewise linear surfaces called crooked planes. The noninfinitesimal version gives an analogous theory for complete anti-de Sitter 3-manifolds.

Contents

1. Introduction

The understanding of moduli spaces using simple combinatorial models is a major theme in geometry. While coarse models, like the curve complex

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or pants complex, are used to great effect in the study of the various metrics and compactifications of Teichmüller spaces (see [MM, R, BM, BMNS] for instance), parameterizations and cellulations can provide insight at both macroscopic and microscopic scales. Prominent examples are the two (mutually homeomorphic) cell decompositions of the decorated Teichmüller space of a punctured surface described by Harer [Har] and Penner [P1] (see also [Haz, GL] for generalizations), which have interesting applications to mapping class groups (see [P2]). In this paper we give a parameterization, comparable to Harer's or Penner's, of the moduli space of certain Lorentzian 3-manifolds called Margulis spacetimes.

A *Margulis spacetime* is a quotient of the 3-dimensional Minkowski space $\mathbb{R}^{2,1}$ by a free group Γ acting properly discontinuously by isometries. The first examples were constructed by Margulis [Ma1, Ma2] in 1983, as counterexamples to Milnor's suggestion [Mi] to remove the cocompactness assumption in the Auslander conjecture [Au]. Since then many authors, most prominently Charette, Drumm, Goldman, Labourie, and Margulis, have studied their geometry, topology, and deformation theory: see [D, DG1, DG2, ChaG, GM, GLM1, CDG1, CDG2, ChoG], as well as [DGK1]. Any Margulis spacetime is determined by a noncompact hyperbolic surface S, with $\pi_1(S) = \Gamma$, and an infinitesimal deformation of S called a *proper deformation.* The subset of proper deformations forms a symmetric cone, which we call the *admissible cone*, in the tangent space to the Fricke–Teichmüller space of (classes of) complete hyperbolic structures of the same type as S on the underlying topological surface. In the case that S is convex cocompact, fundamental work of Goldman–Labourie–Margulis [GLM1] shows that the admissible cone is open with two opposite, convex components, consisting of the infinitesimal deformations of S that uniformly expand or uniformly contract the marked length spectrum of S.

In this paper we study a simple geometric construction, called a *strip deformation*, which produces uniformly expanding deformations of S: it is defined by cutting S along finitely many disjoint, properly embedded geodesic arcs, and then gluing in a *hyperbolic strip*, i.e. the region between two ultraparallel geodesic lines in $\mathbb{H}^2$, at each arc. An *infinitesimal strip deformation* (Definition 1.4) is the derivative of a path of strip deformations along some fixed arcs as the widths of the strips decrease linearly to zero. It is easy to see that, as soon as the supporting arcs decompose the surface into disks, an infinitesimal strip deformation lengthens all closed geodesics of S uniformly (this was observed by Thurston [T1] and proved in more detail by Papadopoulos–Théret [PT]); thus it is a proper deformation. Our main result (Theorem 1.7) states that all proper deformations of S can be realized as infinitesimal strip deformations, in an essentially unique way: after making some choices about the geometry of the strips, the map from the complex of arc systems on S to the projectivization of the admissible cone, taking any weighted system of arcs to the corresponding infinitesimal strip deformation, is a homeomorphism.

We note that infinitesimal strip deformations are also used by Goldman–Labourie–Margulis–Minsky in [GLMM]. They construct modified infinitesimal strip deformations along geodesic arcs that accumulate on a geodesic lamination, in order to describe infinitesimal deformations of a surface for which all lengths increase, but not uniformly.

As an application of our main theorem, we give a new proof of the *tameness* of Margulis spacetimes, under the assumption that the associated hyperbolic surface is convex cocompact. This result was recently established, independently, by Choi–Goldman [ChoG] and by the authors [DGK1]. Here we actually prove the stronger result, named the *Crooked Plane Conjecture* by Drumm–Goldman [DG1], that any Margulis spacetime admits a fundamental domain bounded by *crooked planes*, piecewise linear surfaces introduced by Drumm [D]. This follows from our main theorem by observing that a strip deformation encodes precise directions for building fundamental domains in $\mathbb{R}^{2,1}$ bounded by crooked planes (Section 7.4). In the case that the free group Γ has rank two, the Crooked Plane Conjecture was verified by Charette–Drumm–Goldman [CDG3]. In particular, when the surface S is a once-holed torus, they found a tiling of the admissible cone according to which triples of isotopy classes of crooked planes embed disjointly; this picture is generalized by our parameterization via strip deformations.

We now state precisely our main results, both in the setting of Margulis spacetimes just discussed, and in the related setting of complete anti-de Sitter 3-manifolds (Section 1.4).

1.1. Margulis spacetimes.

The 3-dimensional Minkowski space $\mathbb{R}^{2,1}$ is the affine space $\mathbb{R}^3$ endowed with the parallel Lorentzian structure induced by a quadratic form of signature $(2,1)$; its isometry group is $\mathrm{O}(2,1) \ltimes \mathbb{R}^3$, acting affinely. Let G be the group $\mathrm{PGL}_2(\mathbb{R})$, acting on the real hyperbolic plane $\mathbb{H}^2$ by isometries in the usual way, and on the Lie algebra $\mathfrak{g} = \mathfrak{pgl}_2(\mathbb{R})$ by the adjoint action. We shall identify $\mathbb{R}^{2,1}$ with the Lie algebra $\mathfrak{g}$ endowed with the Lorentzian structure induced by half its Killing form. The group of orientation-preserving isometries of $\mathbb{R}^{2,1}$ identifies with $G \ltimes \mathfrak{g}$, acting on $\mathfrak{g}$ by $(g,w) \cdot v = \mathrm{Ad}(g)v + w$. Its subgroup preserving the time orientation is $G_0 \ltimes \mathfrak{g}$, where $G_0 = \mathrm{PSL}_2(\mathbb{R})$ is the identity component of G.

By [FG] and [Me], if a discrete group Γ acts properly discontinuously and freely by isometries on $\mathbb{R}^{2,1}$, and if Γ is not virtually solvable, then Γ is a free group and its action on $\mathbb{R}^{2,1}$ is orientation-preserving (see e.g. [Ab]) and induces an embedding of Γ into $G \ltimes \mathfrak{g}$ with image

$$\Gamma^{\rho,u} = \{(\rho(\gamma), u(\gamma)) \mid \gamma \in \Gamma\} \subset G \ltimes \mathfrak{g}, \tag{1.1}$$

where $\rho \in \mathrm{Hom}(\Gamma, G)$ is an injective and discrete representation and $u : \Gamma \to \mathfrak{g}$ a ρ-cocycle, i.e. $u(\gamma_1\gamma_2) = u(\gamma_1) + \mathrm{Ad}(\rho(\gamma_1))\, u(\gamma_2)$ for all $\gamma_1, \gamma_2 \in \Gamma$. By definition, a Margulis spacetime is a manifold $M = \Gamma^{\rho,u} \backslash \mathbb{R}^{2,1}$ determined by such a proper group action. Properness is invariant under conjugation by $G \ltimes \mathfrak{g}$. We shall consider conjugate proper actions to be equivalent; in other words, we shall consider Margulis spacetimes to be equivalent if there exists a marked isometry between them. In particular, we will be interested in holonomies ρ up to conjugacy, i.e. as classes in $\mathrm{Hom}(\Gamma, G)/G$, and in ρ-cocycles u up to addition of a coboundary, i.e. as classes in the cohomology group $H^1_\rho(\Gamma, \mathfrak{g}) := H^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho})$.

Note that for a Margulis spacetime $\Gamma^{\rho,u}\backslash\mathbb{R}^{2,1}$, the representation ρ is the holonomy of a noncompact hyperbolic surface $S = \rho(\Gamma)\backslash\mathbb{H}^2$, and the ρ-cocycle $u : \Gamma \to \mathbb{R}^{2,1} = \mathfrak{g}$ can be interpreted as an infinitesimal deformation of this holonomy, obtained as the derivative at $t = 0$ of some smooth path $(\rho_t)_{t\geq 0} \subset \mathrm{Hom}(\Gamma, G)$ with $\rho_0 = \rho$, in the sense that

$$\rho_t(\gamma) = e^{tu(\gamma)+o(t)}\rho(\gamma)$$

for all $\gamma \in \Gamma$ (see [DGK1, § 2.3] for instance). Thus the moduli space of Margulis spacetimes projects to the space of noncompact hyperbolic surfaces; describing the fiber above S amounts to identifying the *proper deformations* u of ρ, i.e. the infinitesimal deformations u of ρ for which the group $\Gamma^{\rho,u}$ acts properly discontinuously on $\mathbb{R}^{2,1}$.

A properness criterion was given by Goldman–Labourie–Margulis [GLM1]: suitably interpreted [GM], it states that for a convex cocompact representation $\rho \in \mathrm{Hom}(\Gamma, G)$ and a ρ-cocycle $u : \Gamma \to \mathfrak{g}$, the group $\Gamma^{\rho,u}$ acts properly discontinuously on $\mathbb{R}^{2,1}$ if and only if the infinitesimal deformation u "uniformly lengthens all closed geodesics", i.e.

$$\inf_{\substack{\gamma\in\Gamma,\\ \lambda_\gamma(\rho)>0}} \frac{\mathrm{d}\lambda_\gamma(u)}{\lambda_\gamma(\rho)} > 0, \tag{1.2}$$

or "uniformly contracts all closed geodesics", i.e. (1.2) holds for $-u$ instead of u. Here $\lambda_\gamma : \mathrm{Hom}(\Gamma, G) \to \mathbb{R}_+$ is the function (see (2.1)) assigning to any representation τ the hyperbolic translation length of $\tau(\gamma)$. That the injective and discrete representation ρ is convex cocompact means that Γ is finitely generated and that $\rho(\Gamma)$ does not contain any parabolic element; equivalently, $S = \rho(\Gamma)\backslash\mathbb{H}^2$ is the union of a compact convex set (called the convex core), whose preimage in $\mathbb{H}^2$ is the smallest nonempty, closed, $\rho(\Gamma)$-invariant, convex subset of $\mathbb{H}^2$, and of finitely many ends of infinite volume (called the funnels). In [DGK1] we gave a new proof of the Goldman–Labourie–Margulis criterion, as well as another equivalent properness criterion in terms of expanding (or contracting) equivariant vector fields on $\mathbb{H}^2$. These criteria are to be extended to arbitrary injective and discrete ρ (for finitely generated Γ) in [GLM2, DGK3], allowing $\rho(\Gamma)$ to have parabolic elements.

We now fix a convex cocompact hyperbolic surface S (possibly nonorientable) with fundamental group $\Gamma = \pi_1(S)$ and holonomy representation $\rho \in \mathrm{Hom}(\Gamma, G)$. We shall use the following terminology.

Definition 1.1. The *Fricke–Teichmüller space* $\mathfrak{F} \subset \mathrm{Hom}(\Gamma, G)/G$ of ρ is the set of conjugacy classes of holonomies of convex cocompact hyperbolic structures on the topological surface underlying $S \simeq \rho(\Gamma)\backslash\mathbb{H}^2$. Its tangent space $T_{[\rho]}\mathfrak{F}$ identifies with the first cohomology group $H^1_\rho(\Gamma, \mathfrak{g})$.

Definition 1.2. The *positive admissible cone* in $T_{[\rho]}\mathfrak{F} \simeq H^1_\rho(\Gamma, \mathfrak{g})$ is the subset of classes of ρ-cocycles u satisfying (1.2). The *admissible cone* is the union of the positive admissible cone and of its opposite. The projectivization of the admissible cone, a subset of $\mathbb{P}(H^1_\rho(\Gamma, \mathfrak{g}))$, will be denoted $\mathsf{adm}(\rho)$.

The positive admissible cone is a convex cone in the finite-dimensional vector space $T_{[\rho]}\mathfrak{F} \simeq H^1_\rho(\Gamma, \mathfrak{g})$, and is open by [GLM1] (see also the proof of Proposition 3.1.(2) in Section 3.1 for another argument).

We now describe the fundamental objects of the paper, namely strip deformations, which will be used to parameterize $\mathsf{adm}(\rho)$ (Theorem 1.7).

1.2. **The arc complex and strip deformations.** We call *arc* of S any nontrivial isotopy class of embedded lines in S for which each end exits in a funnel; we shall denote by $\mathscr{A}$ the set of arcs of S. A *geodesic arc* is a geodesic representative of an arc. The following notion was first introduced by Thurston [T1, proof of Lem. 3.4].

Definition 1.3. A *strip deformation* of the hyperbolic surface S along a geodesic arc $\underline{\alpha}$ is a new hyperbolic surface that is obtained from S by cutting along $\underline{\alpha}$ and gluing in (without any shearing) a *strip*, the region in $\mathbb{H}^2$ bounded by two ultraparallel geodesics. A strip deformation of S along a collection of pairwise disjoint and nonisotopic geodesic arcs $\underline{\alpha}_0, \dots, \underline{\alpha}_k$ is a hyperbolic surface obtained by successively performing this operation for each geodesic arc $\underline{\alpha}_i$, where $0 \leq i \leq k$. This is independent of the order of the operations since the $\underline{\alpha}_i$ are disjoint.

We shall also say that the holonomy representation of the resulting surface (defined up to conjugation) is a *strip deformation* of the holonomy representation ρ of S.

The nonshearing condition in the first sentence of Definition 1.3 means that the strip at the arc $\underline{\alpha}$ is inserted so that the two endpoints of the most narrow cross section of the strip are identified with the two preimages of a single point $p_\alpha \in \underline{\alpha}$ (see Figure 1). This point $p_\alpha \in \underline{\alpha}$ is called the *waist* of the strip. The thickness of the strip at its narrowest cross section is called the *width* of the strip. In the above definition, the waist and width of each strip may be chosen arbitrarily.

We shall also use the infinitesimal version of this construction:

Definition 1.4. An *infinitesimal strip deformation* of S is the class in $H^1_\rho(\Gamma, \mathfrak{g})$ of a ρ-cocycle $u : \Gamma \to \mathfrak{g}$ obtained as the derivative at $t = 0$ of a path $t \mapsto \rho_t \in \mathrm{Hom}(\Gamma, G)$ of strip deformations of $\rho_0 = \rho$, along fixed geodesic arcs $\underline{\alpha}_0, \dots, \underline{\alpha}_k$, with fixed waists, and such that the widths of the strips, measured at the waists, are of the form $m_i t$ for some fixed numbers $m_0, \dots, m_k > 0$; these numbers are called the *widths* of the infinitesimal strip deformation.

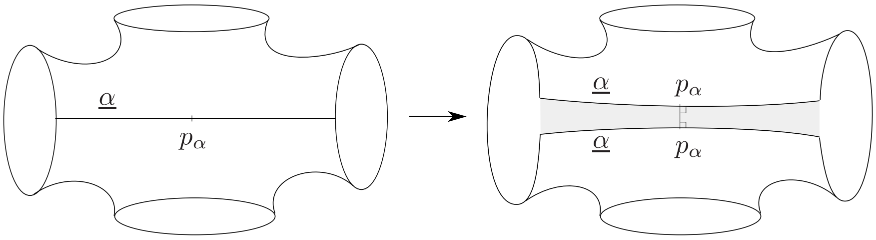

Figure 1. A strip deformation along one arc in a four-holed sphere

Our parameterization of the admissible cone by strip deformations will depend on certain choices:

Definition 1.5. A *strip template on S* is the choice, for each arc $\alpha \in \mathscr{A}$, of

- a geodesic representative $\underline{\alpha}$ of α,
- a point $p_\alpha \in \underline{\alpha}$ (the *waist*),
- a positive number $m_\alpha > 0$ (the *width*),

such that the $\underline{\alpha}$ intersect minimally, meaning that the representatives $\underline{\alpha}_1$ and $\underline{\alpha}_2$ of two arcs α_1 and α_2 always have smallest possible intersection number (including ideal intersection points).

Strip templates on S exist: for instance, take all representatives $\underline{\alpha}$ to intersect the boundary of the convex core orthogonally. We now fix an arbitrary strip template $(\underline{\alpha}, p_\alpha, m_\alpha)_{\alpha \in \mathscr{A}}$.

For any arc $\alpha \in \mathscr{A}$, we define $\boldsymbol{f}(\alpha) \in H^1_\rho(\Gamma, \mathfrak{g})$ to be the infinitesimal strip deformation of ρ along $\underline{\alpha}$ with waist p_α and width m_α. Recall that the *arc complex* $\overline{X}$ of S is the simplicial complex with vertex set $\mathscr{A}$ and with one k-dimensional simplex for each collection of $k+1$ pairwise homotopically disjoint arcs. Top-dimensional cells of $\overline{X}$ correspond to so-called *hyperideal triangulations* of S (see Section 2.2). The map $\alpha \mapsto \boldsymbol{f}(\alpha)$ extends by barycentric interpolation to a map $\boldsymbol{f} : \overline{X} \to H^1_\rho(\Gamma, \mathfrak{g})$, which we call the *infinitesimal strip map* associated with the strip template $(\underline{\alpha}, p_\alpha, m_\alpha)_{\alpha \in \mathscr{A}}$. By postcomposing with the projectivization map $H^1_\rho(\Gamma, \mathfrak{g}) \to \mathbb{P}(H^1_\rho(\Gamma, \mathfrak{g}))$, we obtain the *projectivized infinitesimal strip map*

$$f : \overline{X} \longrightarrow \mathbb{P}\big(H^1_\rho(\Gamma, \mathfrak{g})\big)$$

associated with $(\underline{\alpha}, p_\alpha, m_\alpha)_{\alpha \in \mathscr{A}}$. We shall consider the following subset of $\overline{X}$.

Definition 1.6. The space of *arc systems* of S is the subset X of the arc complex $\overline{X}$ obtained by removing all cells corresponding to collections of arcs that do *not* subdivide the surface S into topological disks.

For instance, no vertex of $\overline{X}$ is in X, but the interior of any top-dimensional cell is (see Section 6 for more examples). By work of Harer [Har] or Penner [P1] on the decorated Teichmüller space, X is homeomorphic to an open ball of dimension $3|\chi| - 1 = \dim(\mathfrak{F}) - 1$, where χ is the Euler characteristic of S. Our main result is that any point of the positive admissible cone is realized as an infinitesimal strip deformation, in a *unique* way given our choice of strip template $(\underline{\alpha}, p_\alpha, m_\alpha)_{\alpha\in\mathscr{A}}$:

Theorem 1.7. *The map* f *restricts to a homeomorphism between* X *and the projectivized admissible cone* $\mathsf{adm}(\rho)$*.*

It is natural to wonder about the image of $\boldsymbol{f}$ in $H^1_\rho(\Gamma, \mathfrak{g})$, before projectivization. Since $\mathsf{adm}(\rho) = f(X)$ is convex, it seems reasonable to hope that $\boldsymbol{f}(X)$ should appear as the boundary of a convex object in $H^1_\rho(\Gamma, \mathfrak{g})$: thus the following conjecture would provide a concrete realization of the space X of arc systems as part of the boundary of the convex hull of a natural discrete subset in a finite-dimensional vector space. We do not know if such realizations exist in general.

Conjecture 1.8. There exists a strip template on S with widths $m_\alpha = 1$ for all $\alpha \in \mathscr{A}$ such that $\boldsymbol{f}(X)$ is a convex hypersurface in $H^1_\rho(\Gamma, \mathfrak{g})$.

1.3. **Fundamental domains for Margulis spacetimes.** In 1992, Drumm [D] introduced piecewise linear surfaces in the 3-dimensional Minkowski space $\mathbb{R}^{2,1}$, called *crooked planes* (see [ChaG]). This enabled him to construct Margulis spacetimes with arbitrary linear holonomy. The *Crooked Plane Conjecture* of Drumm–Goldman [DG1] states that *any* Margulis spacetime should arise from this construction. Charette–Drumm–Goldman [CDG1, CDG2, CDG3] proved this conjecture in the special case that the fundamental group is a free group of rank two. Here we give a proof of the Crooked Plane Conjecture in the general case that the linear holonomy is convex cocompact.

Theorem 1.9. *Any discrete subgroup of* $\mathrm{O}(2,1) \ltimes \mathbb{R}^3$ *acting properly discontinuously and freely on* $\mathbb{R}^{2,1}$*, with convex cocompact linear part, admits a fundamental domain in* $\mathbb{R}^{2,1}$ *bounded by finitely many crooked planes.*

This is an easy consequence of Theorem 1.7: the idea is to interpret infinitesimal strip deformations as motions of crooked planes making them disjoint in $\mathbb{R}^{2,1}$ (see Section 7.4). Theorem 1.9 provides a new proof of the tameness of Margulis spacetimes with convex cocompact linear holonomy, independent from the original proofs given in [ChoG, DGK1].

Theorems 1.7 and 1.9 will be generalized to the case that the linear holonomy has parabolic elements in [DGK3].

1.4. **Strip deformations and anti-de Sitter 3-manifolds.** In [DGK1] we showed that, in a precise sense, Margulis spacetimes behave like "infinitesimal analogues" or "renormalized limits" of complete AdS manifolds, which are quotients of the negatively-curved *anti-de Sitter space* AdS^3. Following this point of view further, we now derive analogues of Theorems 1.7 and 1.9 for AdS manifolds.

The anti-de Sitter space $\mathrm{AdS}^3 = \mathrm{PO}(2,2)/\mathrm{O}(2,1)$ is a model space for Lorentzian manifolds of constant negative curvature. It can be realized as the set of negative points in $\mathbb{P}^3(\mathbb{R})$ with respect to a quadratic form of signature $(2,2)$; its isometry group is $\mathrm{PO}(2,2)$. Equivalently, AdS^3 can be realized as the identity component $G_0 = \mathrm{PSL}_2(\mathbb{R})$ of $G = \mathrm{PGL}_2(\mathbb{R})$, endowed with the biinvariant Lorentzian structure induced by half the Killing form of $\mathfrak{g} = \mathfrak{pgl}_2(\mathbb{R}) = \mathfrak{psl}_2(\mathbb{R})$; the group of orientation-preserving isometries then identifies with

$$(G \times G)_+ := \{(g_1, g_2) \in G \times G \mid g_1 g_2 \in G_0\},$$

acting on G_0 by right and left multiplication: $(g_1, g_2) \cdot g = g_2 g g_1^{-1}$.

By [KR], any torsion-free discrete subgroup of $(G \times G)_+$ acting properly discontinuously on AdS^3 is, up to switching the two factors of $G \times G$, of the form

$$\Gamma^{\rho,j} = \{(\rho(\gamma), j(\gamma)) \mid \gamma \in \Gamma\} \subset G \times G$$

where Γ is a discrete group and $\rho, j \in \mathrm{Hom}(\Gamma, G)$ are two representations with j injective and discrete. Suppose that Γ is finitely generated. By [Ka, GK], a necessary and sufficient condition for the action of $\Gamma^{\rho,j}$ on AdS^3 to be properly discontinuous is that (up to switching the two factors) j be injective and discrete and ρ be "uniformly shorter" than j, in the sense that there exists a (j,ρ)-equivariant Lipschitz map $\mathbb{H}^2 \to \mathbb{H}^2$ with Lipschitz constant < 1; in the case that ρ is convex cocompact, this is equivalent to

$$\inf_{\substack{\gamma \in \Gamma, \\ \lambda_\gamma(\rho) > 0}} \frac{\lambda_\gamma(j)}{\lambda_\gamma(\rho)} > 1, \tag{1.3}$$

where $\lambda_\gamma : \mathrm{Hom}(\Gamma, G) \to \mathbb{R}_+$ is the hyperbolic translation length function of γ as above, see (2.1). One should view (1.2) as the derivative of (1.3) as j tends to ρ with derivative u. If Γ is the fundamental group of a compact surface and ρ is injective and discrete, then (1.3) is never satisfied [T1].

Suppose that ρ is convex cocompact, of infinite covolume, and let S be the hyperbolic surface $\rho(\Gamma)\backslash\mathbb{H}^2$, with holonomy ρ. Let $\mathsf{Adm}^+(\rho)$ be the subset of the Fricke–Teichmüller space $\mathfrak{F}$ of ρ (Definition 1.1) consisting of classes of convex cocompact representations $j \in \mathrm{Hom}(\Gamma, G)$ that are "uniformly longer" than ρ in the sense of (1.3). Let $(\underline{\alpha}, p_\alpha, m_\alpha)_{\alpha \in \mathscr{A}}$ be a strip template on S (Definition 1.5). For any $\alpha \in \mathscr{A}$, let $F(\alpha) \in \mathfrak{F}$ be the class of the strip deformation of ρ along $\underline{\alpha}$ with waist $p_\alpha \in \underline{\alpha}$ and width $m_\alpha > 0$. Since the vertices of a cell of the arc complex $\overline{X}$ correspond to disjoint arcs, the cut-and-paste operations along them do not interfere and the map $\alpha \mapsto F(\alpha)$ naturally extends to a map $F : \overline{\mathrm{C}X} \to \mathfrak{F}$, where $\overline{\mathrm{C}X}$ is the abstract cone over the arc complex $\overline{X}$, with the property that

$$\boldsymbol{f}(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(tx) \in T_{[\rho]}\mathfrak{F} \simeq H^1_\rho(\Gamma, \mathfrak{g}) \tag{1.4}$$

for all $x \in \overline{X}$. Recall that $\overline{\mathrm{C}X}$ is the quotient of $\overline{X} \times \mathbb{R}_+$ by the equivalence relation $(x, 0) \sim (x', 0)$ for all $x, x' \in \overline{X}$; we abbreviate (x,t) as tx. Let $\mathrm{C}X \subset \overline{\mathrm{C}X}$ be the abstract *open* cone over X, equal to the image of $X \times \mathbb{R}_+^*$. We prove the following "macroscopic" version of Theorem 1.7.

Theorem 1.10. *For convex cocompact ρ, the map F restricts to a homeomorphism between* $\mathrm{C}X$ *and* $\mathsf{Adm}^+(\rho)$.

In other words, any "uniformly lengthening" deformation of ρ can be realized as a strip deformation, and the realization is unique once the geodesic representatives $\underline{\alpha}$, waists p_α, and widths m_α are fixed for all arcs $\alpha \in \mathscr{A}$.

Note that the situation is very different when Γ is the fundamental group of a compact surface: as mentioned above, in this case j is Fuchsian and ρ is necessarily non-Fuchsian [T1], up to switching the two factors. As proved independently in [GKW] and [DT], the subset $\mathsf{Adm}^+(\rho)$ of the Fricke–Teichmüller space (i.e. the classical Teichmüller space in the orientable case) consisting of representations j "uniformly longer" than ρ is always nonempty. It would be interesting to obtain a parameterization of $\mathsf{Adm}^+(\rho)$ by some simple combinatorial object in this situation as well. See the recent paper [T] for an approach via harmonic maps.

By analogy with the Minkowski setting, it is natural to ask whether a free, properly discontinuous action on AdS^3 admits a fundamental domain bounded by nice polyhedral surfaces. In Section 8, we introduce piecewise geodesic surfaces in AdS^3 that we call *AdS crooked planes*, and prove:

Theorem 1.11. *Let $\rho, j \in \mathrm{Hom}(\Gamma, G)$ be the holonomy representations of two convex cocompact hyperbolic structures on a fixed surface and assume that $\Gamma^{\rho,j}$ acts properly discontinuously on* AdS^3. *Then $\Gamma^{\rho,j}$ admits a fundamental domain bounded by finitely many* AdS *crooked planes.*

Theorem 1.11 provides another proof of the tameness (obtained in [DGK1, GK]) of complete AdS 3-manifolds of finite type, in this special case.

In contrast with Theorem 1.11, in [DGK2] we construct examples of pairs (ρ, j) with j convex cocompact and ρ noninjective or nondiscrete such that the group $\Gamma^{\rho,j}$ acts properly discontinuously on AdS^3 but does *not* admit any fundamental domain bounded by disjoint crooked planes. It would be interesting to determine exactly which proper actions admit such fundamental domains. The examples of [DGK2] build on a disjointness criterion that we establish there for AdS crooked planes (see Proposition 8.2 of the current paper for a sufficient condition).

1.5. **Organization of the paper.**

Section 2 is devoted to some basic estimates for infinitesimal strip deformations. These allow us, in Section 3, to reduce the proofs of Theorems 1.7 and 1.10 to Claim 3.2, about the behavior of the strip map at faces of codimension zero and one of the arc complex. We prove Claim 3.2 in Section 5, after introducing some formalism in Section 4. In Section 6 we give some basic examples of the tiling of the admissible cone produced by Theorem 1.7. Finally, Sections 7 and 8 are devoted to the proofs of Theorems 1.9 (the Crooked Plane Conjecture) and 1.11 (its anti-de Sitter counterpart) using strip deformations. In Appendix A we make some remarks about the choice of strip template involved in the definition of the map f, in relation with Conjecture 1.8; this appendix is not needed anywhere in the paper.

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2. Metric estimates for (infinitesimal) strip deformations

Let S be a convex cocompact hyperbolic surface of infinite volume, with fundamental group $\Gamma = \pi_1(S)$ (a free group) and holonomy representation $\rho \in \mathrm{Hom}(\Gamma, G)$. Let $\mathfrak{F} \subset \mathrm{Hom}(\Gamma, G)/G$ be the corresponding Fricke–Teichmüller space (Definition 1.1), whose tangent space $T_{[\rho]}\mathfrak{F}$ identifies with the cohomology group $H^1_\rho(\Gamma, \mathfrak{g})$. For any $\gamma \in \Gamma$ and any $\tau \in \mathrm{Hom}(\Gamma, G)$ we set

$$\lambda_\gamma(\tau) := \inf_{p \in \mathbb{H}^2} d(p, \tau(\gamma) \cdot p), \tag{2.1}$$

where d is the hyperbolic metric on $\mathbb{H}^2$: this is the translation length of $\tau(\gamma)$ if $\tau(\gamma) \in G$ is hyperbolic, and 0 otherwise. We thus obtain a function $\lambda_\gamma : \mathfrak{F} \to \mathbb{R}_+$, whose differential is denoted $\mathrm{d}\lambda_\gamma : T\mathfrak{F} \to \mathbb{R}$.

Let $\overline{X}$ be the arc complex of S, with vertex set $\mathscr{A}$. As in Section 1.2, we fix a strip template $(\underline{\alpha}, p_\alpha, m_\alpha)_{\alpha \in \mathscr{A}}$ on S, which defines a strip map $F : \overline{CX} \to \mathfrak{F}$ and an infinitesimal strip map $\boldsymbol{f} : \overline{X} \to H^1_\rho(\Gamma, \mathfrak{g})$. The goal of this section is to provide estimates for the effect of F and $\boldsymbol{f}$ on the curve lengths λ_γ.

2.1. Variation of length of geodesics under strip deformations.

We shall use the following terminology.

Definition 2.1. For $x \in \overline{X}$, the *support* $|x| \subset S$ of x is the union of the geodesic arcs $\underline{\alpha}$ corresponding to the vertices of the smallest cell of $\overline{X}$ containing x. For $\underline{\alpha} \subset |x|$, let $w_{x,\alpha}$ be the width (as measured at the narrowest point) of the strip to be inserted along $\underline{\alpha}$ to produce $F(x)$ from S. The *strip width function* $W_x : |x| \to \mathbb{R}_+$ takes $p \in \underline{\alpha} \subset |x|$ to

$$W_x(p) := w_{x,\alpha} \cosh d(p, p_\alpha), \tag{2.2}$$

where the distance $d(p, p_\alpha)$ is measured along $\underline{\alpha}$.

Note that $w_{x,\alpha}$ is m_α times the weight of α in the barycentric expression for $x \in \overline{X}$, and $W_x(p)$ is the length of the path crossing the strip at p at constant distance from the waist segment. For $p \in \mathbb{H}^2$, we denote by $\measuredangle_p$ the measure of angles of geodesics at p, valued in $[0, \pi/2]$.

Lemma 2.2. *For any $\gamma \in \Gamma \smallsetminus \{e\}$ and any $x \in \overline{X}$,*

$$\mathrm{d}\lambda_\gamma\big(\boldsymbol{f}(x)\big) = \sum_{p \in \underline{\gamma} \cap |x|} W_x(p) \, \sin \measuredangle_p(\underline{\gamma}, |x|) \, \geq 0, \tag{2.3}$$

where $\underline{\gamma}$ *is the geodesic representative of* γ *on* S*.*

Formula (2.3) is analogous to the cosine formula expressing the effect of an earthquake on the length of a closed geodesic (see [Ke]), and is proved similarly. One difference is that (2.3) involves the sine instead of the cosine: a strip deformation should be thought of as an analogue of an earthquake where instead of sliding against itself, the surface is pushed apart in a direction orthogonal to each geodesic arc of the support. In (2.3), unlike in the formula for earthquakes, the contribution of each intersection point p depends, not only on the angle, but also (via W_x) on p itself.

Proof of Lemma 2.2. Up to passing to a double covering, we may assume that S is orientable and $\rho \in \mathrm{Hom}(\Gamma, G)$ takes values in $G_0 = \mathrm{PSL}_2(\mathbb{R})$. By linearity, it is sufficient to prove (2.3) when x is a vertex $\alpha \in \mathscr{A}$ of the arc complex $\overline{X}$ and the corresponding strip width $w_{x,\alpha}$ is 1. For $t \in \mathbb{R}$, we set

$$a_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix},\ b_t := \begin{pmatrix} \cosh\frac{t}{2} & \sinh\frac{t}{2} \\ \sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix},\ r_t := \begin{pmatrix} \cos\frac{t}{2} & \sin\frac{t}{2} \\ -\sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix},$$

where all matrices are understood to be elements of $G_0 = \mathrm{PSL}_2(\mathbb{R})$. The isometries a_t are translations along the line $(0, \infty)$ in the upper half-plane model of $\mathbb{H}^2$; the b_t are translations along $(-1, 1)$; the r_t are rotations around the point $\sqrt{-1}$. Up to conjugation we may assume that $\rho(\gamma) = a_{\lambda_\gamma(\rho)}$. Suppose the oriented geodesic loop $\underline{\gamma}$ crosses the geodesic representative $\underline{\alpha}$ at points $q_1, \dots, q_k$, in this order. For $1 \leq i \leq k$, we use the following notation (see Figure 2):

- $\ell_i > 0$ is the distance between q_{i-1} and q_i along $\underline{\gamma}$, with the convention that $q_0 = q_k$,
- $d_i \in \mathbb{R}$ is the signed distance from q_i to the waist p_α along $\underline{\alpha}$, for the orientation of $\underline{\alpha}$ towards the left of $\underline{\gamma}$ at q_i,
- $\theta_i \in (0, \pi)$ is the angle, at q_i, between the oriented geodesics $\underline{\gamma}$ and $\underline{\alpha}$, for the orientation of $\underline{\alpha}$ towards the left of $\underline{\gamma}$ at q_i.

For any $t \in \mathbb{R}_+$, the element $F(tx) \in \mathrm{Hom}(\Gamma, G_0)/G \subset \mathfrak{F}$ lifts to a homomorphism $\Gamma \to G_0$ sending γ to

$$a_{\ell_1}\big(r_{\theta_1}a_{d_1}\,b_t\,a_{-d_1}r_{-\theta_1}\big)\,a_{\ell_2}(\dots)\,a_{\ell_k}\big(r_{\theta_k}a_{d_k}\,b_t\,a_{-d_k}r_{-\theta_k}\big) \in G_0. \tag{2.4}$$

Ignoring nondiagonal entries, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \big(r_{\theta_i}a_{d_i}\,b_t\,a_{-d_i}r_{-\theta_i}\big) = \frac{\sin\theta_i \cdot \cosh d_i}{2} \begin{pmatrix} 1 & * \\ * & -1 \end{pmatrix}.$$

Therefore, if we set $\ell := \ell_1 + \dots + \ell_k = \lambda_\gamma(\rho)$, then (2.4) is equal to

$$\begin{aligned} & \rho(\gamma) + t\left(\textstyle\sum_{i=1}^k \frac{\sin\theta_i \cdot \cosh d_i}{2}\, a_{\ell_1+\dots+\ell_i} \begin{pmatrix} 1 & * \\ * & -1 \end{pmatrix} a_{\ell_{i+1}+\dots+\ell_k}\right) + o(t) \\ = & \begin{pmatrix} e^{\ell/2} & 0 \\ 0 & e^{-\ell/2} \end{pmatrix} + \tfrac{t}{2}\left(\textstyle\sum_{i=1}^k \sin\theta_i \cdot \cosh d_i \begin{pmatrix} e^{\ell/2} & * \\ * & -e^{-\ell/2} \end{pmatrix}\right) + o(t). \end{aligned}$$

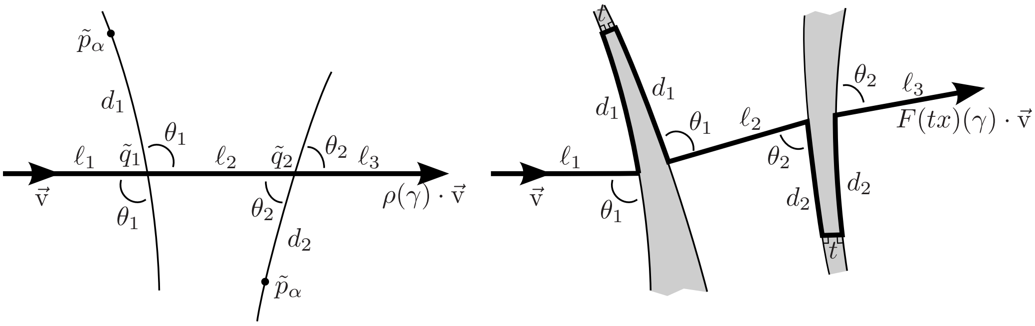


FIGURE 2. Left panel: a unit vector $\vec{v} \in T^1(\mathbb{H}^2)$ tangent to the translation axis $\mathcal{A}_{\rho(\gamma)}$ of $\rho(\gamma)$, and its image under $\rho(\gamma)$ (here $k = 2$). The points $\tilde{q}_1, \tilde{q}_2, \tilde{p}_\alpha$ are lifts of q_1, q_2, p_α, respectively. Right panel: the same vector $\vec{v}$ and its image under the lift (2.4) of $F(tx)(\gamma)$ to G_0. Recall that G_0 acts simply transitively on $T^1(\mathbb{H}^2)$.

By differentiating the formula $\lambda(g) = 2\operatorname{arccosh}(|\mathrm{tr}(g)|/2)$ for hyperbolic $g \in G_0 = \mathrm{PSL}_2(\mathbb{R})$ (where $\lambda(g)$ is the translation length of g in $\mathbb{H}^2$), we find

$$\mathrm{d}\lambda_\gamma\big(\boldsymbol{f}(x)\big) = \sum_{i=1}^{k} \sin\theta_i \cdot \cosh d_i = \sum_{i=1}^{k} W_x(q_i)\, \sin\measuredangle_{q_i}(\underline{\gamma}, |x|). \quad \square$$

2.2. **Angles at the boundary of the convex core.** Let $\Delta \subset \mathscr{A}$ be a hyperideal triangulation of S, i.e. a set of $3|\chi|$ arcs corresponding to a top-dimensional face of the arc complex $\overline{X}$ (here $\chi \in -\mathbb{N}$ is the Euler characteristic of S). Then Δ divides S into $2|\chi|$ connected components (hyperideal triangles). Let ∂S denote the geodesic boundary of the convex core of S.

Proposition 2.3. *For any choice of minimally intersecting geodesic representatives $(\underline{\alpha})_{\alpha\in\mathscr{A}}$ of the arcs of S, there exists $\theta_0 > 0$ such that all the $\underline{\alpha}$ intersect ∂S at an angle $\geq \theta_0$ (measured in $[0, \pi/2]$). Moreover, θ_0 can be taken to depend continuously on the holonomy ρ and on the choice of geodesic representatives of the arcs of any fixed hyperideal triangulation Δ of S.*

This is a consequence of the minimal intersection numbers of the $\underline{\alpha}$, as we now explain.

Proof. Fix a hyperideal triangulation Δ of S, consider one component $\underline{\eta}$ of ∂S, and choose one geodesic arc $\underline{\beta}$ of Δ that exits $\underline{\eta}$ at one end. In the universal cover $\mathbb{H}^2$, a lift $\tilde{\eta}$ of $\underline{\eta}$ is intersected by a collection $\{\tilde{\beta}_i\}_{i\in\mathbb{Z}}$ of naturally ordered, pairwise disjoint lifts of $\underline{\beta}$. Each $\tilde{\beta}_i$ escapes through two components ($\tilde{\eta}$ and another one) of the lift of ∂S; let β_i^* denote the geodesic arc orthogonal to these two components. The β_i^* are lifts of the geodesic arc β^* of S which is in the same class as $\underline{\beta}$ but intersects ∂S at right angles.

Consider another geodesic arc $\underline{\alpha} \neq \underline{\beta}$ of S, which also crosses $\underline{\eta}$. Let α^* be the geodesic representative of α that is orthogonal to ∂S. In $\mathbb{H}^2$, a lift of α^* that intersects $\tilde{\eta}$ lies entirely between β_i^* and β_{i+1}^*, for some $i \in \mathbb{Z}$. By minimality of the intersection numbers, the corresponding lift of $\underline{\alpha}$ lies entirely between $\tilde{\beta}_i$ and $\tilde{\beta}_{i+1}$. Since the angles at which the $\tilde{\beta}_i$ intersect $\tilde{\eta}$ are all the same, the angle at which $\underline{\alpha}$ intersects $\underline{\eta}$ is bounded away from 0, independently of α. We conclude by repeating for all boundary components $\underline{\eta}$. $\square$

2.3. **The unit-peripheral normalization.** Given a choice $(\underline{\alpha}, p_\alpha)_{\alpha\in\mathscr{A}}$ of geodesic representatives and waists for the arcs of S, we now discuss a specific choice of widths $(m_\alpha)_{\alpha\in\mathscr{A}}$, which we call the unit-peripheral normalization.

Remark 2.4. For any systems $(m_\alpha)_{\alpha\in\mathscr{A}}$ and $(m'_\alpha)_{\alpha\in\mathscr{A}}$ of widths, there is a cell-preserving, cellwise projective homeomorphism $h : \overline{X} \to \overline{X}$ such that

$$f_m = f_{m'} \circ h,$$

where $f_m : \overline{X} \to \mathbb{P}(H^1_\rho(\Gamma, \mathfrak{g}))$ (resp. $f_{m'} : \overline{X} \to \mathbb{P}(H^1_\rho(\Gamma, \mathfrak{g}))$) is the projectivized infinitesimal strip map defined in Section 1.2 with respect to the strip template $(\underline{\alpha}, p_\alpha, m_\alpha)_{\alpha\in\mathscr{A}}$ (resp. $(\underline{\alpha}, p_\alpha, m'_\alpha)_{\alpha\in\mathscr{A}}$). Such a map h preserves X.

Choosing all the widths m_α so that

$$\sum_{p\in|x|\cap\partial S} W_x(p)\,\sin\measuredangle_p(\partial S, |x|) = 1 \tag{2.5}$$

for all $x \in \overline{X}$ (or equivalently for all vertices $x = \alpha \in \mathscr{A}$) corresponds, by Lemma 2.2, to asking all infinitesimal strip deformations associated with the choice of $(\underline{\alpha}, p_\alpha, m_\alpha)_{\alpha\in\mathscr{A}}$ to increase the total length of ∂S at unit rate.

Definition 2.5. We call (2.5) the *unit-peripheral normalization.*

Remark 2.6. In the unit-peripheral normalization,

1. $\boldsymbol{f}(\overline{X})$ is contained in some affine hyperplane of $T_{[\rho]}\mathfrak{F} = H^1_\rho(\Gamma, \mathfrak{g})$ (namely $\sum_{i=1}^m \mathrm{d}\lambda_{\gamma_i} = 1$ where $\gamma_1, \dots, \gamma_m \in \Gamma$ correspond to the connected components of ∂S);
2. by (2.3), the bound θ_0 of Proposition 2.3 satisfies
$$W_x(p) \leq 1/\sin\theta_0$$
for all $x \in \overline{X}$ and $p \in \partial S$. This inequality actually holds for all p in the convex core of S, by convexity of cosh in the definition (2.2) of W_x.

In Section 2.4 we shall bound the length variation of closed geodesics under infinitesimal strip deformations using Lemma 2.2 and the following.

Proposition 2.7. *In the unit-peripheral normalization, there exists $K > 0$ (depending on ρ and on the choice $(\underline{\alpha}, p_\alpha)_{\alpha\in\mathscr{A}}$ of geodesic representatives and waists) such that for any $\gamma \in \Gamma \smallsetminus \{e\}$ and any $x \in \overline{X}$,*

$$\sum_{p\in\underline{\gamma}\cap|x|} W_x(p) \leq K\,\measuredangle(\underline{\gamma}, |x|)\;\lambda_\gamma(\rho),$$

where $\underline{\gamma}$ *is the closed geodesic of* $S = \rho\backslash\mathbb{H}^2$ *corresponding to* γ *and*

$$\measuredangle(\underline{\gamma}, |x|) := \max_{p\in\underline{\gamma}\cap|x|} \measuredangle_p(\underline{\gamma}, |x|). \tag{2.6}$$

The idea behind Proposition 2.7 is that the support $|x|$ may intersect the convex core along very long segments (making the finite set $\underline{\gamma}\cap|x|$ very large), but the strip width function W_x dies off exponentially quickly away from the endpoints inside these segments, and so the sum of the values of W_x can be controlled by the length of γ. The factor $\measuredangle(\underline{\gamma}, |x|)$ gives some extra leverage that will only be needed in the noninfinitesimal case of Proposition 3.1.(3).

The proof of Proposition 2.7 builds on the following observation, which is standard hyperbolic trigonometry.

Lemma 2.8. *For any* $\varepsilon > 0$*, there exists* $\kappa > 0$ *with the following property: for any geodesic lines* ℓ *and* ℓ' *of* $\mathbb{H}^2$ *and any* $p, q \in \ell$*,*

(1) if ℓ *and* ℓ' *meet at the point* p*, then* $d(q, \ell') \leq \varepsilon$ *whenever* $d(p, q) \leq |\log \measuredangle_p(\ell, \ell')| - \kappa$*,*

(2) if ℓ *and* ℓ' *are disjoint and* $d(p, \ell') \leq 1$*, then* $d(q, \ell') \leq \varepsilon$ *whenever* $d(p, q) \leq |\log d(p, \ell')| - \kappa$*.*

Proof of Lemma 2.8. (1) By the law of sines,

$$d(q, \ell') \leq \sinh d(q, \ell') = \sinh d(p, q) \cdot \sin \measuredangle_p(\ell, \ell') \leq e^{d(p,q)}\, \measuredangle_p(\ell, \ell'),$$

which is $\leq \varepsilon$ whenever $d(p, q) \leq \log(\varepsilon/\measuredangle_p(\ell, \ell'))$. We may take $\kappa = |\log \varepsilon|$.

(2) Let r be the point of ℓ which is closest to ℓ'. The quadrilateral determined by p, r, and their projections to ℓ' is a Lambert quadrilateral (right angles everywhere except possibly at p), hence by a classical result

$$\sinh d(p, \ell') = \cosh d(p, r)\, \sinh d(\ell, \ell').$$

A similar formula holds for q. Using the convexity of sinh and the fact that the function $\cosh(x)/e^x$ is decreasing, we obtain

$$d(q, \ell') \leq \sinh d(q, \ell') = \sinh d(p, \ell') \frac{\cosh d(q, r)}{\cosh d(p, r)} \leq d(p, \ell')\, \sinh(1)\, e^{d(p,q)},$$

which is $\leq \varepsilon$ whenever $d(p, q) \leq \log(\varepsilon/(d(p, \ell') \sinh(1)))$. We may take $\kappa = |\log(\varepsilon/\sinh(1))|$. ☐

Proof of Proposition 2.7. Fix $\gamma \in \Gamma \smallsetminus \{e\}$ and let $m := \#(\underline{\gamma} \cap |x|) \in \mathbb{N}$. Any lift $\tilde{\gamma}$ of $\underline{\gamma}$ to $\mathbb{H}^2$ intersects the preimage of $|x|$ in a collection of points $\{p_i\}_{i\in\mathbb{Z}}$, naturally ordered along $\tilde{\gamma}$, so that $\rho(\gamma)$ takes p_i to p_{i+m} for all $i \in \mathbb{Z}$. Let $\ell_i \subset \mathbb{H}^2$ be the lifted geodesic arc of $|x|$ that contains p_i, and let $\tilde{W}_x$ be the lift of the function W_x to $\mathbb{H}^2$. Recall that the support $|x|$ consists of at most $3N$ geodesic arcs, where $N := |\chi|$ is the absolute value of the Euler characteristic of S. It is enough to find a constant $K' \geq 0$, independent of γ and x, such that for any $i \in \mathbb{Z}$,

$$\tilde{W}_x(p_i) \leq K' \measuredangle(\underline{\gamma}, |x|)\, d(p_i, p_{i+6N}); \tag{2.7}$$

indeed, the result will follow (with $K = 6NK'$) by adding up for $1 \leq i \leq m$.

Let $\Omega \subset \mathbb{H}^2$ be the preimage of the convex core of S. We first observe the existence of a constant $D > 0$, depending only on the constant θ_0 of Proposition 2.3 and on the boundary lengths of the convex core of S, such that if ℓ_i and ℓ_{i+6N} exit the same boundary component $\tilde{\eta}$ of Ω, then $d(p_i, p_{i+6N}) \geq D$. Indeed, $|x|$ has at most $6N$ half-arcs exiting any boundary component of Ω; therefore, if ℓ_i and ℓ_{i+6N} both exit $\tilde{\eta}$, then $\ell_j = \rho(\gamma') \cdot \ell_i$ for some integer $i < j \leq i + 6N$ and some element $\gamma' \in \Gamma$ stabilizing $\tilde{\eta}$. By Proposition 2.3, this implies that the shortest distance from ℓ_i to ℓ_j is bounded from below, independently of γ, i, j; therefore, so is $d(p_i, p_{i+6N})$. We may and shall assume $D \leq 1$.

Let us prove the existence of $K' \geq 0$ such that (2.7) holds for all $i \in \mathbb{Z}$ with $d(p_i, p_{i+6N}) \geq D$. By Lemma 2.8.(1), for any $\varepsilon > 0$, there exists $\kappa > 0$ such that ℓ_i remains ε-close to $\tilde{\gamma}$ for at least

$$|\log(\measuredangle(\underline{\gamma}, |x|))| - \kappa$$

units of length to the left and right of p_i. If ε is small enough (depending only on the constant θ_0 of Proposition 2.3), then ℓ_i cannot exit Ω on this interval around p_i: otherwise $\tilde{\gamma}$ would exit it as well, by Proposition 2.3. But on this interval, the maximum of $\tilde{W}_x|_{\ell_i}$ is at least $\tilde{W}_x(p_i)\, e^{|\log(\measuredangle(\underline{\gamma},|x|))|-\kappa}/2$, by definition (2.2) of W_x. Using Remark 2.6.(2), it follows that

$$\tilde{W}_x(p_i) \leq \measuredangle(\underline{\gamma}, |x|) \frac{2e^{\kappa}}{\sin \theta_0},$$

and so (2.7) holds with $K' = 2e^{\kappa}/(D \sin \theta_0)$ when $d(p_i, p_{i+6N}) \geq D$.

We now treat the case of integers $i \in \mathbb{Z}$ such that $d(p_i, p_{i+6N}) < D$. By the law of sines, $\sinh d(p_i, \ell_{i+6N}) = \sin \measuredangle(\underline{\gamma}, \ell_{i+6N}) \sinh d(p_i, p_{i+6N})$, hence, by convexity of sinh,

$$d(p_i, \ell_{i+6N}) \leq \sinh d(p_i, \ell_{i+6N}) \leq \delta_{\gamma,i} := \sinh(1)\ \measuredangle(\underline{\gamma}, |x|)\ d(p_i, p_{i+6N}).$$

(Recall that $D \leq 1$.) By Lemma 2.8.(2), for any $\varepsilon > 0$ there exists $\kappa > 0$ such that ℓ_i remains ε-close to ℓ_{i+6N} for at least

$$|\log \delta_{\gamma,i}| - \kappa$$

units of length to the left and right of p_i. If ε is small enough (depending again only on θ_0), then ℓ_i cannot exit Ω on this interval around p_i: otherwise ℓ_{i+6N} would exit the same boundary component of Ω, by Proposition 2.3, which would contradict the definition of D. But on this interval, the maximum of $\tilde{W}_x|_{\ell_i}$ is at least $\tilde{W}_x(p_i)\, e^{|\log \delta_{\gamma,i}|-\kappa}/2$, by definition of W_x. Using Remark 2.6.(2), it follows that

$$\tilde{W}_x(p_i) \leq \delta_{\gamma,i} \frac{2e^{\kappa}}{\sin \theta_0},$$

and so (2.7) holds with $K' = 2 \sinh(1)\ e^{\kappa}/\sin \theta_0$ when $d(p_i, p_{i+6N}) < D$. $\square$

2.4. **Boundedness of the map $\boldsymbol{f}$.** Lemma 2.2 and Proposition 2.7 imply that in the unit-peripheral normalization (2.5),

$$0 \leq \frac{\mathrm{d}\lambda_\gamma\big(\boldsymbol{f}(x)\big)}{\lambda_\gamma(\rho)} \leq K\, \measuredangle(\underline{\gamma}, |x|)^2 \tag{2.8}$$

for all $\gamma \in \Gamma\smallsetminus\{e\}$ and $x \in \overline{X}$, where $\underline{\gamma}$ is the geodesic representative of γ on S and $\measuredangle(\underline{\gamma}, |x|)$ the maximal intersection angle as in (2.6). The square on the right-hand side of (2.8) can be interpreted by saying that small intersection angles between $\underline{\gamma}$ and $|x|$ weaken the effect of the strip deformation $\boldsymbol{f}(x)$ on the length of γ in two different ways: first, by spreading out the intersection points along $\underline{\gamma}$ (Proposition 2.7); second, by lessening the effect of each (Lemma 2.2).

Proposition 2.9. *In the unit-peripheral normalization, the closure of $\boldsymbol{f}(\overline{X})$ in $H^1_\rho(\Gamma, \mathfrak{g})$ is compact and does not contain* 0*; the closure of the projectivization $f(\overline{X})$ is compact in some affine chart of $\mathbb{P}(H^1_\rho(\Gamma, \mathfrak{g}))$.*

Proof. By Remark 2.6.(1), the set $\boldsymbol{f}(\overline{X})$ is contained in some affine hyperplane that does not contain 0. There is a family $\{\gamma_1, \dots, \gamma_{3N}\} \subset \Gamma\smallsetminus\{e\}$ such that the $\mathrm{d}\lambda_{\gamma_i}$ form a dual basis of $T_{[\rho]}\mathfrak{F}$. We then apply (2.8) to the γ_i. ☐

The set $f(\overline{X}) \subset \mathbb{P}(H^1_\rho(\Gamma, \mathfrak{g}))$ does not depend on the choice of widths m_α, by Remark 2.4.

2.5. **Lengthening the boundary of S.** To conclude this section, we show that a strip deformation with large weight (i.e. belonging to $F(t\overline{X})$ for some large $t > 0$) lengthens the boundary of S by a large amount, regardless of the supporting arcs. This will be needed in Section 3.1 in the proof that the map F is proper (Proposition 3.1.(3)).

Lemma 2.10. *In the unit-peripheral normalization, there exists $K' \geq 0$ (depending on ρ and on the choice of geodesic representatives $(\underline{\alpha})_{\alpha\in\mathscr{A}}$) such that for any $t > 0$ and any $x \in \overline{X}$, the total boundary length of the convex core for $F(tx) \in \mathfrak{F}$ is at least $2\log t - K'$.*

As can be seen in the proof by considering one very long arc, $\log t$ is actually the optimal order of magnitude.

Proof. Up to passing to a double covering, we may assume that S is orientable and $\rho \in \mathrm{Hom}(\Gamma, G)$ takes values in $G_0 = \mathrm{PSL}_2(\mathbb{R})$. Let $x \in \overline{X}$. By definition (2.5) of the unit-peripheral normalization, we can find a point $p \in |x| \cap \partial S$ such that

$$W_x(p) \geq \frac{1}{6N},$$

where $3N = \dim(\mathfrak{F})$ as before. Let $\underline{\alpha}$ be the arc of $|x|$ through p. In the upper half-plane model of $\mathbb{H}^2$, we may assume that p lifts to $\sqrt{-1}$ and $\underline{\alpha}$ to $(0, \infty)$. Taking the basepoint of $\pi_1(S)$ at p, the holonomy of the boundary loop γ through p (suitably oriented) then has the form

$$\rho(\gamma) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

with $a, b, c > 0$ and $ac - b^2 = 1$, which we see as an element of $G_0 = \mathrm{PSL}_2(\mathbb{R})$. By Proposition 2.3, we can bound b below by some $b_0 > 0$ independent of p and x. Recall from Definition 2.1 that $W_x(p) = w_{x,\alpha} \cosh d(p, p_\alpha)$, where the distance is measured along $\underline{\alpha}$. If only one end of $\underline{\alpha}$ exits through the boundary loop γ, then inserting a strip of width $tw_{x,\alpha}$, as in the deformation $F(tx)$, corresponds to multiplying $\rho(\gamma)$ on the right by

$$g := \begin{pmatrix} e^{\frac{d(p,p_\alpha)}{2}} & 0 \\ 0 & e^{-\frac{d(p,p_\alpha)}{2}} \end{pmatrix} \begin{pmatrix} \cosh \frac{tw_{x,\alpha}}{2} & \sinh \frac{tw_{x,\alpha}}{2} \\ \sinh \frac{tw_{x,\alpha}}{2} & \cosh \frac{tw_{x,\alpha}}{2} \end{pmatrix} \begin{pmatrix} e^{-\frac{d(p,p_\alpha)}{2}} & 0 \\ 0 & e^{\frac{d(p,p_\alpha)}{2}} \end{pmatrix},$$

which increases $\lambda(\rho(\gamma))$. If the other end of $\underline{\alpha}$, or some other arcs in the support of $|x|$, also exit through γ, then inserting strips to produce $F(tx)$ increases $\lambda(\rho(\gamma))$ even more, so we may use $\lambda(\rho(\gamma)g)$ as a lower bound for the length of the boundary of the convex core corresponding to $F(tx)$. A computation shows

$$\begin{aligned} \mathrm{tr}(\rho(\gamma)g) \;&\geq\; b\left(e^{d(p,p_\alpha)} + e^{-d(p,p_\alpha)}\right) \sinh \frac{tw_{x,\alpha}}{2} \\ &\geq\; t\, b\, w_{x,\alpha} \cosh d(p, p_\alpha) = t\, b\, W_x(p) \;\geq\; t\, \frac{b_0}{6N}, \end{aligned}$$

and so

$$\lambda_\gamma(F(tx)) \geq 2 \operatorname{arccosh}\left(\frac{\mathrm{tr}(\rho(\gamma)g)}{2}\right) \geq 2 \log t - K',$$

where $K' := 2|\log(12N/b_0)| \geq 0$ does not depend on t nor x. □

3. Reduction of the proof of Theorems 1.7 and 1.10

We continue with the notation of Section 2. Let $X \subset \overline{X}$ be the space of arc systems on S (Definition 1.6). By work of Harer [Har] or Penner [P1] on the decorated Teichmüller space, X is an open ball of dimension $3|\chi| - 1$, where $\chi \in -\mathbb{N}$ is the Euler characteristic of S. In order to prove Theorems 1.7 and 1.10, it is sufficient to prove the following proposition.

Proposition 3.1. *Let $\rho \in \mathrm{Hom}(\Gamma, G)$ be a convex cocompact representation, $(\underline{\alpha}, p_\alpha, m_\alpha)_{\alpha \in \mathscr{A}}$ a strip template on $S = \rho(\Gamma)\backslash\mathbb{H}^2$ (Definition 1.5), and $F : \overline{CX} \to \mathfrak{F}$ and $f : \overline{X} \to \mathbb{P}(H^1_\rho(\Gamma, \mathfrak{g}))$ the corresponding strip maps of Section 1.2. Then*

1. *The restriction of f to X (resp. of F to $\mathrm{C}X$) takes values in the projectivized admissible cone $\mathsf{adm}(\rho)$ (resp. in $\mathsf{Adm}^+(\rho)$);*
2. *$\mathsf{adm}(\rho)$ is an open ball of dimension $3|\chi| - 1$, and $\mathsf{Adm}^+(\rho)$ is open in $\mathfrak{F}$ and homotopically trivial;*
3. *The restrictions $f : X \to \mathsf{adm}(\rho)$ and $F : \mathrm{C}X \to \mathsf{Adm}^+(\rho)$ are proper;*

(4) The restrictions $f : X \to \mathsf{adm}(\rho)$ *and* $F : \mathrm{C}X \to \mathsf{Adm}^+(\rho)$ *are local homeomorphims.*

Indeed, (3) and (4) imply that the restrictions $f : X \to \mathsf{adm}(\rho)$ and $F : \mathrm{C}X \to \mathsf{Adm}^+(\rho)$ are coverings, and (2) implies that these coverings are trivial. We will prove (1), (2), (3) in Section 3.1, while in Section 3.2 we will reduce (4) to a basic claim (Claim 3.2) about the behavior of the infinitesimal strip map $\boldsymbol{f}$ at faces of codimension 0 and 1 in the arc complex, to be proved in Section 5.

3.1. **Range and properness of f and F.** In this section we prove statements (1), (2), and (3) of Proposition 3.1.

Proof of Proposition 3.1.(1). (See also [PT].) The inclusion $f(X) \subset \mathsf{adm}(\rho)$ means that any infinitesimal strip deformation $\boldsymbol{f}(x)$ of ρ, performed on pairwise disjoint geodesic arcs $\underline{\alpha}_0, \dots, \underline{\alpha}_k$ that subdivide $S = \rho(\Gamma)\backslash\mathbb{H}^2$ into topological disks, lengthens every curve at a *uniform* rate relative to its length. Lemma 2.2 gives lengthening, but not uniform lengthening a priori. To see that the latter is true, note that, by compactness, there exist $R > 0$ and $\theta \in (0, \pi/2]$ such that any closed geodesic $\underline{\gamma}$ of S must cross one of the $\underline{\alpha}_i$ at least once every R units of length, at an angle $\geq \theta$ (because $\underline{\gamma}$ cannot exit the convex core). If we denote by γ the element of Γ corresponding to $\underline{\gamma}$, then Lemma 2.2 implies

$$\frac{\mathrm{d}\lambda_\gamma\big(\boldsymbol{f}(x)\big)}{\lambda_\gamma(\rho)} \geq \frac{\#\left(\underline{\gamma}\cap\bigcup_{1\leq i\leq k}\underline{\alpha}_i\right)}{\lambda_\gamma(\rho)}\, w\,\sin\theta \geq \frac{w\,\sin\theta}{R} =: \varepsilon > 0, \tag{3.1}$$

where $w > 0$ is defined to be the minimum, for $1 \leq i \leq k$, of the strip widths w_{x,α_i} of Definition 2.1. This proves that any infinitesimal strip deformation $\boldsymbol{f}(x)$, for $x \in X$, satisfies (1.2), hence $f(X) \subset \mathsf{adm}(\rho)$.

To see that $F(\mathrm{C}X) \subset \mathsf{Adm}^+(\rho)$, it is sufficient to establish that $F(X) \subset \mathsf{Adm}^+(\rho)$ (where we identify X with $1X \subset \mathrm{C}X$), for we may adjust the widths m_α as we wish. Note that the bounds θ and R above can be taken to hold uniformly when ρ and the geodesic representatives $\underline{\alpha}_i$ vary in a compact subset of the deformation space. We can thus argue by integration of the infinitesimal inequality (3.1), using (1.4). More precisely, for any $t_0 \geq 0$ and $x \in X$, the vector $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0} F(tx) \in T_{F(t_0x)}\mathfrak{F}$ is realized as an infinitesimal strip deformation of $F(t_0x)$, for instance along geodesic arcs that bound the strips used to produce $F(t_0x)$ from ρ. Bounds θ and R as above hold for these geodesic arcs, independently of t_0 and x, as long as $t_0 \leq 1$. Therefore, (3.1) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \frac{\lambda_\gamma(F(tx))}{\lambda_\gamma(F(t_0x))} \geq \varepsilon$$

for all $t_0 \in [0, 1]$, all $\gamma \in \Gamma \smallsetminus \{e\}$, and all $x \in X$, and so $\lambda_\gamma(F(x)) \geq e^{\varepsilon}\lambda_\gamma(\rho)$ for all $\gamma \in \Gamma$ and all $x \in X$, proving $F(X) \in \mathsf{Adm}^+(\rho)$. □

Proof of Proposition 3.1.(2). For the openness of $\mathsf{Adm}^+(\rho)$ in the Fricke–Teichmüller space $\mathfrak{F}$, see [Ka] or [GK]. Alternatively, one can note that for any $[j] \in \mathfrak{F}$ and $\varepsilon > 0$, there exists a neighborhood B_ε of $[j]$ in $\mathfrak{F}$ such that the convex cores of all hyperbolic metrics on S with holonomies in B_ε are mutually $(1+\varepsilon)$-bi-Lipschitz. Thus

$$\frac{1}{1+\varepsilon} \leq \frac{\lambda_\gamma(j')}{\lambda_\gamma(j)} \leq 1+\varepsilon$$

for all $\gamma \in \Gamma \smallsetminus \{e\}$ and $[j'] \in B_\varepsilon$. In particular, if $j \in \mathrm{Hom}(\Gamma, G)$ satisfies (1.3), then for ε small enough, any $j' \in \mathrm{Hom}(\Gamma, G)$ with $[j'] \in B_\varepsilon$ also satisfies (1.3), and so $B_\varepsilon \subset \mathsf{Adm}^+(\rho)$, proving the openness of $\mathsf{Adm}^+(\rho)$.

The fact that $\mathsf{adm}(\rho)$ is open in $\mathbb{P}(T_{[\rho]}\mathfrak{F}) \simeq \mathbb{P}(H^1_\rho(\Gamma, \mathfrak{g}))$ follows from [GLM1] or [DGK1]. Alternatively, we may argue as above: by (1.2) and linearity of the maps $\mathrm{d}\lambda_\gamma$ for $\gamma \in \Gamma$, it is enough to check that the neighborhood B_ε of $[j]$, corresponding to mutually $(1+\varepsilon)$-bi-Lipschitz convex cores, can be taken to contain a ball of radius $\geq C\varepsilon$ as $\varepsilon \to 0$, for some $C > 0$ independent of ε and some smooth Riemannian metric on a neighborhood U of $[\rho]$ in $\mathfrak{F}$. This in turn follows from the existence of a *smooth* local trivialization of the natural bundle of hyperbolic surfaces above U, and compactness of the convex core.

The positive admissible cone of ρ is also a *convex* subset of $H^1_\rho(\Gamma, \mathfrak{g})$ (since the defining inequality (1.2) is a convex condition), hence its projectivization $\mathsf{adm}(\rho)$ is an open ball of dimension $3|\chi| - 1$.

Finally, we check that $\mathsf{Adm}^+(\rho)$ is homotopically trivial. Fix $k \in \mathbb{N}$ and consider a continuous map σ from the sphere $\mathbb{S}^k$ to $\mathsf{Adm}^+(\rho)$. We want to deform σ to a constant map, inside the set of continuous maps from $\mathbb{S}^k$ to $\mathsf{Adm}^+(\rho)$. Choose a hyperideal triangulation Δ of S. For any $t \geq 0$, let $\sigma_t : \mathbb{S}^k \to \mathfrak{F}$ be the postcomposition of σ with a strip deformation of width t along all arcs of Δ simultaneously (taking for instance geodesic representatives that exit the boundary of the convex core perpendicularly). Then $\sigma_t(\mathbb{S}^k) \subset \mathsf{Adm}^+(\rho)$ by Proposition 3.1.(1), and for large t the convex core of any hyperbolic metric corresponding to some $\sigma_t(q) \in \mathfrak{F}$, for $q \in \mathbb{S}^k$, looks in fact like a collection of near-ideal triangles connected by long, thin isthmi of length $t + O(1)$, according to the combinatorics of the dual trivalent graph of Δ (see Figure 3). Here the error term $O(1)$ is uniform in $q \in \mathbb{S}^k$. The thickness of each isthm (i.e. the smallest length of a geodesic arc across the convex core in the isotopy class of the strip) is $O(e^{-t/2})$. These thicknesses form coordinates for the Fricke–Teichmüller space $\mathfrak{F}$ of S. There exists $\varepsilon > 0$, depending on ρ, such that when these coordinates are all $\leq \varepsilon$, then the metric is in $\mathsf{Adm}^+(\rho)$. Choose t large enough to make all isthm thicknesses $\leq \varepsilon$ in all the metrics $\sigma_t(q)$ for q ranging in $\mathbb{S}^k$, then interpolate linearly to thicknesses $(\varepsilon, ..., \varepsilon)$. This proves that σ is homotopically trivial. ☐

In particular, $\mathsf{Adm}^+(\rho)$ is connected and simply connected. Theorem 1.10 will imply that it is actually a ball.

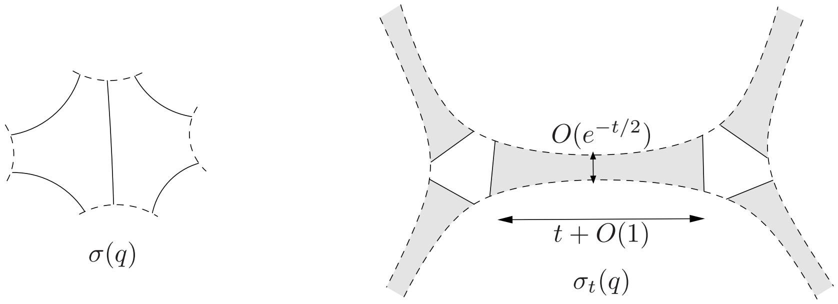


Figure 3. The convex core of a convex cocompact hyperbolic surface (here $\sigma(q)$ for $q \in \mathbb{S}^k$), before and after inserting strips of width t for large $t > 0$. The strips are shaded.

Proof of Proposition 3.1.(3). To establish the properness of the restriction $f : X \to \mathsf{adm}(\rho)$, we consider a sequence (x_n) escaping to infinity in X, such that $(f(x_n))$ converges to some class $[u] \in \mathbb{P}(H^1_\rho(\Gamma, \mathfrak{g}))$; we must show that $[u]$ does not lie in $\mathsf{adm}(\rho)$. By Remark 2.4, up to replacing the sequence (x_n) with $(h(x_n))$ for some cell-preserving, cellwise projective homeomorphism $h : \overline{X} \to \overline{X}$, so that $|x_n| = |h(x_n)|$ for all n, we may assume that we are working in the unit-peripheral normalization.

Suppose that, up to passing to a subsequence, the supports $|x_n|$ stabilize; then (x_n) converges to some point $x \in \overline{X} \smallsetminus X$, up to passing again to a subsequence. By construction, the restriction of f to any cell of the arc complex $\overline{X}$ is continuous: in particular, $[u] = f(x)$. Since $x \notin X$, the support $|x|$ fails to decompose the surface into disks, hence the infinitesimal deformation $\boldsymbol{f}(x)$ fails to lengthen all curves (Lemma 2.2), and $[u] \notin \mathsf{adm}(\rho)$.

Otherwise, the supports $|x_n|$ diverge. Up to passing to a subsequence, we may assume that they admit a Hausdorff limit $\Lambda \subset S$, which consists of a nonempty compact lamination together with some isolated leaves escaping in the funnels. Moreover, since the closure of $\boldsymbol{f}(\overline{X})$ in $H^1_\rho(\Gamma, \mathfrak{g})$ is compact and does not contain 0 (Proposition 2.9), we may assume that the sequence $(\boldsymbol{f}(x_n))$ converges to some infinitesimal deformation u in the projective class $[u]$. For any $\varepsilon > 0$ we can find a simple closed geodesic $\underline{\gamma}$ forming angles $\leq \varepsilon/2$ with Λ, hence $\leq \varepsilon$ with $|x_n|$ for large n. By (2.8), the corresponding element $\gamma \in \Gamma$ satisfies

$$\frac{\mathrm{d}\lambda_\gamma(u)}{\lambda_\gamma(\rho)} \leq \varepsilon^2 K.$$

Since this holds for arbitrarily small ε, we see that u does not satisfy (1.2), i.e. $[u] \notin \mathsf{adm}(\rho)$. Thus f is proper.

We now show that the restriction $F : \mathrm{C}X \to \mathsf{Adm}^+(\rho)$ is proper. As in the infinitesimal case, up to applying a cellwise linear homeomorphism $\mathrm{C}X \to \mathrm{C}X$, we may assume that we are working in the unit-peripheral normalization. Consider a sequence $(t_n x_n)$ escaping to infinity in $\mathrm{C}X$, with $t_n \in \mathbb{R}^*_+$ and $x_n \in X$ for all n; we must show that $F(t_n x_n)$ escapes to infinity in $\mathsf{Adm}^+(\rho)$. If $t_n \to +\infty$, then $F(t_n x_n)$ escapes to infinity in $\mathfrak{F}$, because the length of the boundary of the convex core of the corresponding hyperbolic metric on S goes to infinity by Lemma 2.10. Up to passing to a subsequence, we may therefore assume that (t_n) is bounded and that $(F(t_n x_n))$ is bounded in $\mathfrak{F}$. We then argue as in the infinitesimal case. If the supports $|x_n|$ stabilize, then, up to passing to a subsequence, $(t_n x_n)$ converges to tx for some $t \geq 0$ and $x \in \overline{X} \smallsetminus X$; by continuity of F on each cell of $\overline{X}$, the sequence $(F(t_n x_n))$ converges to $F(tx)$. Since $x \notin X$ we have $F(tx) \notin \mathsf{Adm}^+(\rho)$: indeed, the support $|x|$ is disjoint from some simple closed geodesic $\underline{\gamma}$, hence the corresponding element $\gamma \in \Gamma$ satisfies $\lambda_\gamma(F(tx)) = \lambda_\gamma(\rho)$. If the supports $|x_n|$ diverge, then for any $\varepsilon > 0$ we can find a simple closed geodesic $\underline{\gamma}$ forming angles $\leq \varepsilon$ with $|x_n|$ for all large enough n. Proposition 2.7 then implies

$$\sum_{p \in \underline{\gamma} \cap |x_n|} W_{x_n}(p) \leq K\varepsilon\, \lambda_\gamma(\rho)$$

for the corresponding element $\gamma \in \Gamma$. Consider the representative of γ in the metric $F(t_n x_n)$ that agrees with $\underline{\gamma}$ outside of the strips and also includes (nongeodesic) segments crossing each strip at constant distance from the waist. The length of this representative is exactly

$$\lambda_\gamma(\rho) + t_n \sum_{p \in \underline{\gamma} \cap |x_n|} W_{x_n}(p).$$

Thus the length of γ in the metric $F(t_n x_n)$ is $\leq (1 + K t_n \varepsilon)\, \lambda_\gamma(\rho)$, and so any limit $[\rho'] \in \mathfrak{F}$ of a subsequence of $(F(t_n x_n))$ satisfies $\lambda_\gamma(\rho') \leq (1 + K t \varepsilon)\, \lambda_\gamma(\rho)$. This holds for arbitrarily small ε, hence $[\rho'] \notin \mathsf{Adm}^+(\rho)$. ☐

3.2. **Reduction of Proposition 3.1.(4).** The following claim is a stepping stone to the proof of Proposition 3.1.(4); it will be proved in Section 5. The numbering of the statements (0), (1) refers to the codimension of the faces.

Claim 3.2. *In the setting of Proposition 3.1, let Δ, Δ' be two hyperideal triangulations of S differing by just one diagonal switch. Then*

(0) *The points $f(\alpha)$, for $\alpha \in \mathscr{A}$ ranging over the $3|\chi|$ edges of Δ, are the vertices of a nondegenerate, top-dimensional simplex in $\mathbb{P}(H^1_\rho(\Gamma, \mathfrak{g}))$, denoted $f(\Delta)$;*
(1) *The simplices $f(\Delta)$ and $f(\Delta')$ have disjoint interiors in $\mathbb{P}(H^1_\rho(\Gamma, \mathfrak{g}))$;*
(2) *There exists a choice of geodesic arcs $\underline{\alpha}$ and waists p_α for which $f(\Delta) \cup f(\Delta')$ is convex in $\mathbb{P}(H^1_\rho(\Gamma, \mathfrak{g}))$.*

Here, we say that a closed subset of $\mathbb{P}(H^1_\rho(\Gamma, \mathfrak{g}))$ is *convex* if it is convex and compact in some affine chart of $\mathbb{P}(H^1_\rho(\Gamma, \mathfrak{g}))$. (Note that the whole image $f(X)$ has compact closure in some affine chart, by Proposition 2.9.)

In order to deduce Proposition 3.1.(4) from Claim 3.2, we shall use the following observation.

Remark 3.3. The space of strip templates on S (Definition 1.5) is connected. Moreover, if $\mathscr{A}' \subset \mathscr{A}$ is finite, then any choice of minimally intersecting geodesic representatives $(\underline{\alpha})_{\alpha\in\mathscr{A}'}$ can be extended to a strip template $(\underline{\alpha}, p_\alpha, m_\alpha)_{\alpha\in\mathscr{A}}$ on S.

Indeed, one system of minimally intersecting geodesic representatives of the arcs is given by the geodesics orthogonal to the boundary of the convex core, and any other system is obtained from this one by pushing the endpoints at infinity of the geodesics forward by an isotopy of the circles at infinity of the funnels. The value of this isotopy can be prescribed on any finite subset.

Proof of Proposition 3.1.(4) assuming Claim 3.2. Let us prove that f is a local homeomorphism at any point $x \in X$. The simplicial structure of $\overline{X}$ defines a partition of X into *strata*, where the stratum of x is the unique open simplex containing x. If x belongs to a top-dimensional stratum of X, then local homeomorphicity is Claim 3.2.(0). If x belongs to a codimension-1 stratum, then local homeomorphicity is Claim 3.2.(1).

Before treating the important case that x belongs to a codimension-2 stratum, we first recall some useful classical terminology. For any stratum σ of X, the union of the simplices of $\overline{X}$ containing the closure $\overline{\sigma}$ of σ is the suspension of $\overline{\sigma}$ with a simplicial sphere L_σ, called the *link* of σ, naturally identified with a subcomplex of $\overline{X}$. (That L_σ is a sphere follows e.g. from [Har] or [P1].) The infinitesimal strip map $\boldsymbol{f} : \mathrm{C}\overline{X} \to H^1_\rho(\Gamma, \mathfrak{g})$ induces a cellwise affine map

$$\boldsymbol{f}_\sigma :\ \mathrm{C}L_\sigma \longrightarrow H^1_\rho(\Gamma,\mathfrak{g})/\mathrm{span}(\boldsymbol{f}(\sigma)) \smallsetminus \{0\} \simeq \mathbb{R}^{\mathrm{codim}(\sigma)} \smallsetminus \{0\},$$

called the *link map* of $\boldsymbol{f}$ at σ. It also induces a *(positively) projectivized link map*

$$f_\sigma :\ L_\sigma \longrightarrow \mathbb{P}^+\big(H^1_\rho(\Gamma,\mathfrak{g})/\mathrm{span}(\boldsymbol{f}(\sigma))\big) \simeq \mathbb{S}^{\mathrm{codim}(\sigma)-1}.$$

By Claim 3.2.(0), to prove that f is a local homeomorphism at x, it is sufficient to prove that f_σ is a homeomorphism, where σ is the stratum of x.

We now turn to codimension-2 strata σ of X. Consider an affine chart of $\mathbb{P}(H^1_\rho(\Gamma,\mathfrak{g}))$ containing $f(X)$ (Proposition 2.9) and equip it with a Euclidean metric, so that the simplices $f(\Delta)$ in this chart become endowed with dihedral angles at their codimension-2 faces. By Claim 3.2.(0), all dihedral angles are in $(0,\pi)$. By Claim 3.2.(1), the projectivized link map f_σ is a (piecewise projective) covering map from a circle to a circle, of some integer degree. We only need to show that this degree is ± 1, i.e. that the sum of the dihedral angles around σ is 2π.

Codimension-2 faces σ come in two types. The first type corresponds to decompositions of S into two hyperideal quadrilaterals and $2|\chi| - 4$ hyperideal triangles. Each quadrilateral can be independently divided into triangles in two ways, differing by a diagonal exchange. Thus L_σ is the suspension of two pairs of points, i.e. a cyclic graph of length 4. The four corresponding dihedral angles can only sum to 2π, not 4π. (In fact, this is a “direct sum” of two copies of the codimension one case.)

The second type of codimension-2 strata of X corresponds to decompositions of S into one hyperideal pentagon and $2|\chi| - 3$ hyperideal triangles. The link of such a stratum σ is a pentagon (a so-called *pentagon move* between triangulations, see Figure 4). Since the five angles lie in $(0, \pi)$, their sum is either 2π or 4π, i.e. the degree of f_σ is ± 1 or ± 2. By Remark 3.3, the degree remains constant as we change the strip template on S (positions of the geodesic arcs and waists), because one cannot pass continuously from 2π to 4π. However, by Claim 3.2.(2) there is at least one strip template for which the degree is ± 1. Indeed, by convexity of $f(\Delta) \cup f(\Delta')$, one pair of consecutive angles has sum $\leq \pi$: the remaining three numbers cannot add to $\geq 3\pi$, so a total of 4π is impossible. The degree is ± 1 for this strip template, hence for all strip templates, and f_σ is a homeomorphism. Thus f is a local homeomorphism at x.

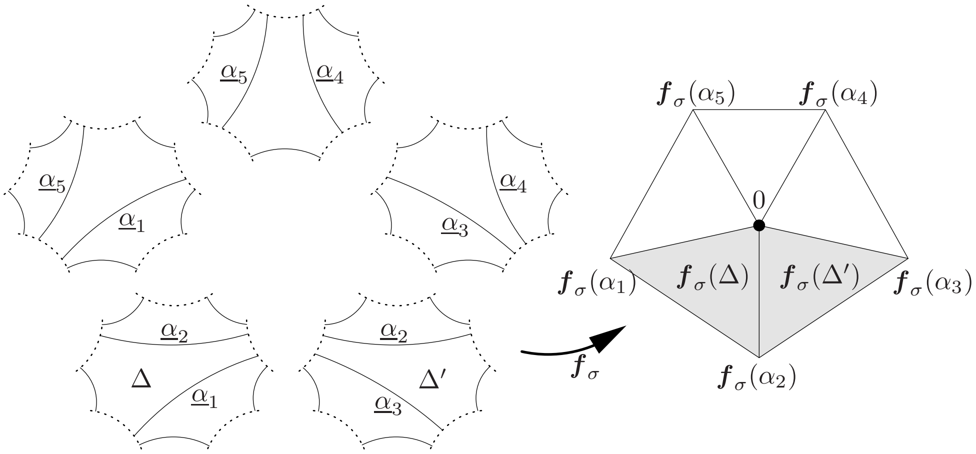


FIGURE 4. A pentagon move and its image under $\boldsymbol{f}_\sigma$, where σ is a cellulation containing one pentagon and many triangles. In the source (left panel), the geodesic arcs $\underline{\alpha}_i$ cross various boundary components of the convex core of S (dashed) at near-right angles, creating hyperideal triangulations such as Δ or Δ'.

For x in a stratum σ of X of codimension $d \geq 3$, we argue by induction on d. The projectivized link map f_σ is a map from a $(d-1)$-sphere to a $(d-1)$-sphere $\mathbb{S}^{d-1}$. It is a local homeomorphism by induction, hence a covering. But any connected covering of $\mathbb{S}^{d-1}$ is a homeomorphism since $\mathbb{S}^{d-1}$ is simply connected when $d \geq 3$. Therefore f is a local homeomorphism at x.

Note that, as a consequence, the infinitesimal strip map $\boldsymbol{f}$ also defines a local homeomorphism from $\mathrm{C}X$ to the admissible cone.

The local homeomorphicity of the restriction $F : \mathrm{C}X \to \mathsf{Adm}^+(\rho)$ follows from the result for $\boldsymbol{f}$ and from the fact that an infinitesimal perturbation of the widths of an actual (noninfinitesimal) strip deformation is the same as an infinitesimal deformation performed on the deformed surface (as in the proof of Proposition 3.1.(1)). More precisely, let us prove that F is a local homeomorphism at a point $tx \in \mathrm{C}X$, where $t \in \mathbb{R}_+^*$ and $x \in X$. Let $\overline{\sigma}_1, \dots, \overline{\sigma}_k$ be the top-dimensional closed cells of $\overline{X}$ containing x, and let σ be the stratum of x, with closure $\overline{\sigma} = \overline{\sigma}_1 \cap \dots \cap \overline{\sigma}_k$. Our chosen strip template on S naturally yields geodesic representatives, waists, and widths for the arcs spanning the $\overline{\sigma}_i$ on the deformed hyperbolic surface S_{tx} defined by $F(tx) \in \mathfrak{F}$. Extend this to a strip template on S_{tx} and let $\boldsymbol{f}^{tx} : \overline{\mathrm{C}X} \to T_{F(tx)}\mathfrak{F}$ be the associated infinitesimal strip map. The restriction of $\boldsymbol{f}^{tx}$ to the closed cone $\overline{\mathrm{C}\sigma_i}$ over $\overline{\sigma}_i$ does not depend on the choice of extension. The restriction $F|_{\overline{\mathrm{C}\sigma_i}}$ is a smooth map whose derivative $\mathrm{d}_{tx}(F|_{\overline{\mathrm{C}\sigma_i}})$ is the only linear extension of $\boldsymbol{f}^{tx}|_{\overline{\mathrm{C}\sigma_i}}$ to the cone $\overline{\mathrm{C}\sigma_i} + \mathrm{span}_{\mathbb{R}}(\sigma)$. In particular, $\mathrm{d}_{tx}(F|_{\overline{\mathrm{C}\sigma_i}})$ is nonsingular by Claim 3.2.(0), hence F is a smooth immersion on $\overline{\mathrm{C}\sigma_i}$, in particular a local diffeomorphism on the interior $\mathrm{C}\sigma_i$. Since F is an immersion on each $\overline{\mathrm{C}\sigma_i}$, it defines a link map $(\mathrm{d}_{tx}F)_\sigma : L_\sigma \to \mathbb{P}^+\left(T_{F(tx)}\mathfrak{F}/\mathrm{d}_{tx}F(\mathrm{span}_{\mathbb{R}}(\sigma))\right)$, where L_σ is the link of σ as above, and it only remains to show that $(\mathrm{d}_{tx}F)_\sigma$ is a homeomorphism. Since $\mathrm{d}_{tx}F(\mathrm{span}_{\mathbb{R}}(\sigma)) = \mathrm{span}_{\mathbb{R}}(\boldsymbol{f}^{tx}(\sigma))$, the map $(\mathrm{d}_{tx}F)_\sigma$ actually agrees with the link map $\boldsymbol{f}^{tx}_\sigma$ for the infinitesimal strip map $\boldsymbol{f}^{tx}$, which is known to be a homeomorphism by the infinitesimal result. $\square$

4. Formalism for infinitesimal strip deformations

In Section 3 we explained how Theorems 1.7 and 1.10 reduce to Claim 3.2. Before proving this claim in Section 5, we introduce some notation and formalism that will also be useful in Section 7.

4.1. **Killing vector fields on $\mathbb{H}^2$.**

We identify the 3-dimensional Minkowski space $\mathbb{R}^{2,1}$ with the Lie algebra $\mathfrak{g}$, equipped with the symmetric bilinear form $\langle\cdot,\cdot\rangle$ equal to half the Killing form. Note that $\mathfrak{g}$ also naturally identifies with the space of *Killing vector fields* on $\mathbb{H}^2$, i.e. vector fields whose flow preserves the hyperbolic metric: an element $V \in \mathfrak{g}$ defines the Killing field

$$p \longmapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (e^{tV} \cdot p) \ \in T_p\mathbb{H}^2,$$

and any Killing field is of this form for a unique $V \in \mathfrak{g}$. We shall write $V(p) \in T_p\mathbb{H}^2$ for the vector at p of the Killing field $V \in \mathfrak{g}$. Recall that $V \in \mathfrak{g}$ is said to be *hyperbolic* (resp. *parabolic*, resp. *elliptic*) if the one-parameter subgroup of G generated by e^V is hyperbolic (resp. parabolic, resp. elliptic). We view the hyperbolic plane $\mathbb{H}^2$ as a hyperboloid in $\mathbb{R}^{2,1}$ and its boundary at infinity $\partial_\infty\mathbb{H}^2$ as the projectivized light cone:

$$\begin{aligned} \mathbb{H}^2 &= \big\{v \in \mathbb{R}^{2,1} \mid v_1^2 + v_2^2 - v_3^2 = -1,\ v_3 > 0\big\}, \\ \partial_\infty\mathbb{H}^2 &= \big\{[v] \in \mathbb{P}(\mathbb{R}^{2,1}) \mid v_1^2 + v_2^2 - v_3^2 = 0\big\}. \end{aligned} \tag{4.1}$$

The geodesic lines of $\mathbb{H}^2$ are the intersections of the hyperboloid with the linear planes of $\mathbb{R}^{2,1}$. For any $p \in \mathbb{H}^2$, the tangent space $T_p\mathbb{H}^2$ naturally identifies with the linear subspace $p^\perp \subset \mathbb{R}^{2,1}$ of vectors orthogonal to p; this linear subspace is the translate back to the origin of the affine plane tangent at p to the hyperboloid $\mathbb{H}^2 \subset \mathbb{R}^{2,1}$. For any $V \in \mathfrak{g} \simeq \mathbb{R}^{2,1}$ (seen as a Killing field on $\mathbb{H}^2$),

$$V(p) = V \wedge p \ \in T_p\mathbb{H}^2 = p^\perp \subset \mathbb{R}^{2,1} \simeq \mathfrak{g}, \tag{4.2}$$

where $\wedge$ is the natural Minkowski cross product on $\mathbb{R}^{2,1}$:

$$(v_1, v_2, v_3) \wedge (w_1, w_2, w_3) := (-v_2w_3 + v_3w_2\,,\ -v_3w_1 + v_1w_3\,,\ v_1w_2 - v_2w_1).$$

Note that for any $v, w \in \mathbb{R}^{2,1}$ the vector $v \wedge w \in \mathbb{R}^{2,1}$ is orthogonal to both v and w, and $(v, w, v \wedge w)$ is positively oriented (i.e. satisfies the "right-hand rule") if v and w are not collinear. Here is an easy consequence of (4.2).

Lemma 4.1. *An element $V \in \mathbb{R}^{2,1} \simeq \mathfrak{g}$, seen as a Killing vector field on $\mathbb{H}^2$, may be described as follows:*

1. *If V is timelike (i.e. $\langle V, V\rangle < 0$), then it is elliptic: it is an infinitesimal rotation of velocity $\pm\sqrt{|\langle V, V\rangle|}$ centered at $\frac{\pm V}{\sqrt{|\langle V,V\rangle|}} \in \mathbb{H}^2 \subset \mathbb{R}^{2,1}$. The velocity is positive if V is future-pointing and negative otherwise.*
2. *If V is lightlike (i.e. $\langle V, V\rangle = 0$ and $V \neq 0$), then it is parabolic with fixed point $[V] \in \partial_\infty\mathbb{H}^2 \subset \mathbb{P}(\mathbb{R}^{2,1})$.*
3. *If V is spacelike (i.e. $\langle V, V\rangle > 0$), then it is hyperbolic: it is an infinitesimal translation of velocity $\sqrt{\langle V, V\rangle}$ with axis $\ell = V^\perp \cap \mathbb{H}^2 \subset \mathbb{R}^{2,1}$. If $v^+, v^- \in V^\perp$ are future-pointing lightlike vectors representing respectively the attracting and repelling fixed points of V in $\partial_\infty\mathbb{H}^2 \subset \mathbb{P}(\mathbb{R}^{2,1})$, then the triple (v^+, V, v^-) is positively oriented (i.e. satisfies the right-hand rule).*

(3') *Let ℓ' be a geodesic of $\mathbb{H}^2$ whose endpoints in $\partial_\infty\mathbb{H}^2 \subset \mathbb{P}(\mathbb{R}^{2,1})$ are represented by future-pointing lightlike vectors $w_1, w_2 \in \mathbb{R}^{2,1}$. Then V is an infinitesimal translation along an axis* orthogonal *to ℓ' if and only if V is spacelike and belongs to* span(w_1, w_2). *Endow ℓ' with the transverse orientation placing $[w_1]$ on the left; then V translates in the positive (resp. negative) direction if and only if $V \in \mathbb{R}_+^* w_1 + \mathbb{R}_-^* w_2$ (resp. $V \in \mathbb{R}_-^* w_1 + \mathbb{R}_+^* w_2$).*

4.2. **Bookkeeping for cocycles.** We now introduce a formalism for describing cocycles that will be useful in the proofs of Claim 3.2 (Section 5) and Theorem 1.9 (Section 7.4). The basic idea, following Thurston's description of earthquakes [T2], is to consider deformations (to be named φ) of the hyperbolic surface S that are locally isometric everywhere except along some fault lines, where they are discontinuous. Each such deformation is characterized up to equivalence by a map (to be named ψ) describing the relative motion of two pieces of the surface adjacent to a fault line.

Consider a *geodesic cellulation* $\underline{\Delta}$ of the convex cocompact hyperbolic surface $S = \rho(\Gamma)\backslash\mathbb{H}^2$, consisting of vertices $\mathscr{V}$, geodesic edges $\mathscr{E}$, and finite-sided polygons $\mathscr{T}$ with disjoint interiors, such that the intersection of two edges (resp. polygons), if nonempty, is a vertex (resp. an edge or a vertex) in their boundary. The elements of $\mathscr{E}$ may be geodesic segments, properly embedded geodesic rays, or properly embedded geodesic lines; the elements of $\mathscr{T}$ may have infinite area. A particular case of interest is when $\mathscr{E}$ consists of the geodesic representatives of the supporting arcs of a strip deformation $\boldsymbol{f}(x)$; in this case all polygons are hyperideal and $\mathscr{V} = \emptyset$. Let $\widetilde{\Delta}$ be the the preimage of $\underline{\Delta}$ in the universal cover $\mathbb{H}^2$. The vertices $\widetilde{\mathscr{V}}$, edges $\widetilde{\mathscr{E}}$, and polygons $\widetilde{\mathscr{T}}$ of the cellulation $\widetilde{\Delta}$ are the respective preimages of $\mathscr{V}, \mathscr{E}$, and $\mathscr{T}$. In what follows, we refer to the elements of $\widetilde{\mathscr{T}}$ as the *tiles*. We denote by $\pm\widetilde{\mathscr{E}}$ the set of edges of $\widetilde{\mathscr{E}}$ endowed with a transverse orientation. For $e \in \pm\widetilde{\mathscr{E}}$, we denote by $-e$ the same edge with the opposite transverse orientation.

Let us first recall a description of infinitesimal earthquake transformations. For simplicity, we assume that the fault locus is a finite disjoint union of properly embedded geodesic lines. We may build a geodesic cellulation $\underline{\Delta}$ such that the union of all edges in $\mathscr{E}$ contains the fault locus. Now consider an assignment $\varphi : \widetilde{\mathscr{T}} \to \mathfrak{g}$ of infinitesimal motions to the tiles $\widetilde{\mathscr{T}}$ of the lifted cellulation $\widetilde{\Delta}$ (using the interpretation of Section 4.1). Define a map $\psi : \pm\widetilde{\mathscr{E}} \to \mathfrak{g}$ by assigning to any transversely oriented edge $e \in \pm\widetilde{\mathscr{E}}$ the *relative motion* along that edge: $\psi(e) = \varphi(\delta') - \varphi(\delta) \in \mathfrak{g}$ where $\delta, \delta' \in \widetilde{\mathscr{T}}$ are the tiles adjacent to e, with e transversely oriented from δ to δ'. The map φ defines an infinitesimal left earthquake transformation of S if ψ is ρ-equivariant, i.e.

$$\psi(\rho(\gamma) \cdot e) = \mathrm{Ad}(\rho(\gamma))\, \psi(e) \tag{4.3}$$

for all $\gamma \in \Gamma$ and $e \in \pm\widetilde{\mathscr{E}}$, and if for any $e \in \pm\widetilde{\mathscr{E}}$ whose projection lies in the fault locus, $\psi(e)$ is an infinitesimal translation to the left along e (and $\psi(e) = 0$ if e does not project to the fault locus). It is a simple observation that the infinitesimal deformation of S described by φ depends only on the relative motion map ψ; that is, φ may be recovered from ψ up to a global isometry (see below).

We now generalize this description of infinitesimal earthquakes and work with a larger class of deformations, for which the relative motion between adjacent tiles is allowed to be any infinitesimal motion. Consider a ρ-equivariant map $\psi : \pm\widetilde{\mathscr{E}} \to \mathfrak{g}$ satisfying the following *consistency conditions*:

- $\psi(-e) = -\psi(e)$ for all $e \in \pm\widetilde{\mathscr{E}}$;
- the total motion around any vertex is zero: if $e_1, \dots, e_k \in \pm\widetilde{\mathscr{E}}$ are the transversely oriented edges crossed (in the positive direction) by a loop encircling a vertex $v \in \widetilde{\mathscr{V}}$, then $\sum_{i=1}^k \psi(e_i) = 0$.

Under these conditions, ψ defines a cohomology class $[u] \in H^1_\rho(\Gamma, \mathfrak{g})$ as follows. Choose a base tile $\delta_0 \in \widetilde{\mathscr{T}}$ and an element $v_0 \in \mathfrak{g}$. Then ψ determines a map $\varphi : \widetilde{\mathscr{T}} \to \mathfrak{g}$ by integration: given a tile $\delta \in \widetilde{\mathscr{T}}$, consider a path $p : [0,1] \to \mathbb{H}^2$ with initial endpoint $p(0)$ in the interior of δ_0 and final endpoint $p(1)$ in the interior of δ, such that $p(t)$ avoids the vertices $\widetilde{\mathscr{V}}$ of $\widetilde{\Delta}$; we set

$$\varphi(\delta) := v_0 + \sum \psi(e),$$

where the sum is over all transversely oriented edges $e \in \pm\widetilde{\mathscr{E}}$ crossed (in the positive direction) by the path $p(t)$. This does not depend on the choice of p, by the consistency conditions above. For any tile $\delta \in \widetilde{\mathscr{T}}$,

$$u(\gamma) := \varphi(\rho(\gamma) \cdot \delta) - \mathrm{Ad}(\rho(\gamma))\, \varphi(\delta)$$

defines a ρ-cocycle $u : \Gamma \to \mathfrak{g}$, independent of δ: we shall say that φ is (ρ, u)-*equivariant*. The cohomology class of u depends only on ψ, not on the choice of δ_0 and v_0: indeed, the map integrating ψ with initial data $\delta_0' \in \widetilde{\mathscr{T}}$ and $v_0' \in \mathfrak{g}$ differs from φ by the constant vector $w_0 := v_0' - \varphi(\delta_0')$, and therefore the cocycle it determines differs from u by the coboundary $u_{w_0} = (\gamma \mapsto w_0 - \mathrm{Ad}(\rho(\gamma))\, w_0)$. The map φ assigns infinitesimal motions to the tiles in $\widetilde{\mathscr{T}}$; by construction, $\psi(e) = \varphi(\delta') - \varphi(\delta)$ for any tiles δ, δ' adjacent to an edge $e \in \pm\widetilde{\mathscr{E}}$ transversely oriented from δ to δ'.

Let $\Psi(\pm\widetilde{\mathscr{E}}, \mathfrak{g})$ be the space of ρ-equivariant maps $\psi : \pm\widetilde{\mathscr{E}} \to \mathfrak{g}$ satisfying the two consistency conditions above. The integration process $\psi \mapsto [u]$ we have just described defines an $\mathbb{R}$-linear map

$$\mathscr{L} : \Psi\big(\pm\widetilde{\mathscr{E}}, \mathfrak{g}\big) \longrightarrow H^1_\rho(\Gamma, \mathfrak{g}). \tag{4.4}$$

Note that each tile has trivial stabilizer in Γ, because it is finite-sided and Γ is torsion-free. Therefore the map $\mathscr{L}$ is onto, i.e. any infinitesimal deformation $[u] \in H^1_\rho(\Gamma, \mathfrak{g})$ of ρ is achieved by some assignment ψ of relative motions. Indeed, choose a representative in $\widetilde{\mathscr{T}}$ for each element of $\mathscr{T}$, and choose arbitrary values for φ on these representatives. We can extend this to a (ρ, u)-equivariant map $\varphi : \widetilde{\mathscr{T}} \to \mathfrak{g}$, and define $\psi(e) := \varphi(\delta') - \varphi(\delta)$ for any tiles δ, δ' adjacent to an edge $e \in \pm\widetilde{\mathscr{E}}$ transversely oriented from δ to δ'. This map ψ satisfies the consistency conditions above.

4.3. **Infinitesimal strip deformations.**

In our main case of interest, the cellulation $\underline{\Delta}$ is a geodesic hyperideal triangulation and the edges $\mathscr{E}$ of $\underline{\Delta}$ are the geodesic representatives of the supporting arcs of an infinitesimal strip deformation.

Remark 4.2. In our geodesic hyperideal triangulations, we do not a priori require the extended edges exiting a boundary component to meet in a single point in $\mathbb{P}^2(\mathbb{R}) \smallsetminus \mathbb{H}^2$.

Recall that $(\underline{\alpha}, p_\alpha, m_\alpha)_{\alpha \in \mathscr{A}}$ is the choice of strip template on S defining the infinitesimal strip map $\boldsymbol{f}$ (see Section 1.2).

Definition 4.3. Let $\alpha \in \mathscr{A}$ be an arc of S with $\underline{\alpha} \in \mathscr{E}$. The *relative motion map* $\psi_\alpha \in \Psi(\pm\widetilde{\mathscr{E}}, \mathfrak{g})$ of the infinitesimal strip deformation $\boldsymbol{f}(\alpha) \in H^1_\rho(\Gamma, \mathfrak{g})$ is defined as follows:

- for any transversely oriented lift $\tilde{\alpha} \in \pm\widetilde{\mathscr{E}}$ of $\underline{\alpha}$, the element $\psi_\alpha(\tilde{\alpha}) \in \mathfrak{g}$ is the infinitesimal translation of velocity m_α along the axis orthogonal to $\tilde{\alpha}$ at (the lift of) p_α, in the positive direction;
- $\psi_\alpha(e) = 0$ for any other transversely oriented edge $e \in \pm\widetilde{\mathscr{E}}$.

Recall the map $\mathscr{L}$ from (4.4). The following immediate observation is essential for the proof of Claim 3.2 in Sections 5.1 and 5.2.

Observation 4.4. *The relative motion map* ψ_α realizes *the infinitesimal strip deformation* $\boldsymbol{f}(\alpha)$*, in the sense that* $\mathscr{L}(\psi_\alpha) = \boldsymbol{f}(\alpha)$*.*

In Section 5.2, it will be necessary to work simultaneously with two different geodesic hyperideal triangulations $\underline{\Delta}$ and $\underline{\Delta}'$. Consider a common refinement $\underline{\Delta}''$ of $\underline{\Delta}, \underline{\Delta}'$. For $\underline{\Delta}', \underline{\Delta}''$, we use notation $\mathscr{E}', \mathscr{E}''$ and $\mathscr{L}', \mathscr{L}''$ similar to Section 4.2. Then there are natural inclusion maps

$$\mathscr{I} : \Psi(\pm\widetilde{\mathscr{E}}, \mathfrak{g}) \hookrightarrow \Psi(\pm\widetilde{\mathscr{E}}'', \mathfrak{g}) \quad \text{and} \quad \mathscr{I}' : \Psi(\pm\widetilde{\mathscr{E}}', \mathfrak{g}) \hookrightarrow \Psi(\pm\widetilde{\mathscr{E}}'', \mathfrak{g})$$

defined as follows: for any $\psi \in \Psi(\pm\widetilde{\mathscr{E}}, \mathfrak{g})$ and $e'' \in \pm\widetilde{\mathscr{E}}''$, set

- $\mathscr{I}(\psi)(e'') := \varepsilon\, \psi(e)$ if e'' is contained in an edge $e \in \pm\widetilde{\mathscr{E}}$, with $\varepsilon = 1$ (resp. $\varepsilon = -1$) if the transverse orientations of e'' and e agree (resp. disagree),
- $\mathscr{I}(e'') := 0$ otherwise,

and similarly for $\mathscr{I}'$. By using these inclusion maps we may compare relative motion maps defined on the two different triangulations $\underline{\Delta}$ and $\underline{\Delta}'$. We consider $\psi \in \Psi(\pm\widetilde{\mathscr{E}}, \mathfrak{g})$ and $\psi' \in \Psi(\pm\widetilde{\mathscr{E}}', \mathfrak{g})$ equivalent if $\mathscr{I}(\psi) = \mathscr{I}'(\psi')$. Here are two simple observations:

Observations 4.5. *(1)* $\mathscr{L}'' \circ \mathscr{I} = \mathscr{L}$ *and* $\mathscr{L}'' \circ \mathscr{I}' = \mathscr{L}'$*.*
(2) Let $\alpha \in \mathscr{A}$ *be an arc of* S *with* $\underline{\alpha} \in \mathscr{E} \cap \mathscr{E}' \subset \mathscr{E}''$*. Let* $\psi_\alpha \in \Psi(\pm\widetilde{\mathscr{E}}, \mathfrak{g})$ *and* $\psi'_\alpha \in \Psi(\pm\widetilde{\mathscr{E}}', \mathfrak{g})$ *be the two relative motion maps realizing* $\boldsymbol{f}(\alpha)$ *as in Observation 4.4, so that* $\mathscr{L}(\psi_\alpha) = \mathscr{L}'(\psi'_\alpha) = \boldsymbol{f}(\alpha)$*. Then* $\mathscr{I}(\psi_\alpha) = \mathscr{I}'(\psi'_\alpha)$*.*

Observation 4.5.(2) means that for any arc $\alpha \in \mathscr{A}$ the map ψ_α is well defined, up to equivalence, independently of the geodesic hyperideal triangulation $\underline{\Delta}$ containing $\underline{\alpha}$.

We will also need to compose (i.e. add) infinitesimal strip deformations supported on arcs that intersect. Suppose that α and α' are two arcs of S with geodesic representatives $\underline{\alpha}$ and $\underline{\alpha}'$ contained in $\underline{\Delta}$ and $\underline{\Delta}'$, respectively. We define the sum $\psi_\alpha + \psi_{\alpha'}$ to be

$$\psi_\alpha + \psi_{\alpha'} := \mathscr{I}(\psi_\alpha) + \mathscr{I}'(\psi_{\alpha'}) \in \Psi(\pm\widetilde{\mathscr{E}}'', \mathfrak{g}).$$

Then $\mathscr{L}$ commutes with this operation: by Observation 4.5.(1),

$$\mathscr{L}''(\psi_\alpha + \psi_{\alpha'}) = \mathscr{L}(\psi_\alpha) + \mathscr{L}'(\psi_{\alpha'}).$$

5. Behavior of the strip map at faces of codimension 0 and 1

We now prove Claim 3.2 (hence Proposition 3.1 and Theorems 1.7 and 1.10), using the formalism of Section 4.

5.1. **Proof of Claim 3.2.(0).** Let $\underline{\Delta}$ be the geodesic hyperideal triangulation of $S = \rho(\Gamma)\backslash\mathbb{H}^2$ whose edges $\mathscr{E}$ are the geodesic representatives $\underline{\alpha}$ of the arcs α of Δ given by our choice of strip template on S. We continue with the notation of Sections 4.2 and 4.3. Let us prove that the infinitesimal strip deformations $\boldsymbol{f}(\alpha) \in H^1_\rho(\Gamma, \mathfrak{g})$, for $\underline{\alpha} \in \mathscr{E}$, span all of $H^1_\rho(\Gamma, \mathfrak{g})$. Since $\dim H^1_\rho(\Gamma, \mathfrak{g}) = \#\mathscr{E}$, it is equivalent to show that the $\boldsymbol{f}(\alpha)$ are linearly independent. Suppose that

$$\sum_{\underline{\alpha} \in \mathscr{E}} c_\alpha \, \boldsymbol{f}(\alpha) = 0 \tag{5.1}$$

for some $(c_\alpha) \in \mathbb{R}^{\mathscr{E}}$, and let us prove that $c_\alpha = 0$ for all $\underline{\alpha} \in \mathscr{E}$. By Observation 4.4 and linearity of $\mathscr{L}$, the left-hand side of (5.1) is realized by the ρ-equivariant relative motion of the tiles $\psi := \sum_{\underline{\alpha} \in \mathscr{E}} c_\alpha \psi_\alpha : \pm\widetilde{\mathscr{E}} \to \mathfrak{g}$ such that for any transversely oriented lift $\tilde{\alpha} \in \pm\widetilde{\mathscr{E}}$ of any $\underline{\alpha} \in \mathscr{E}$,

$$\psi(\tilde{\alpha}) = c_\alpha \, \psi_\alpha(\tilde{\alpha}) \tag{5.2}$$

(see Definition 4.3). Since $\mathscr{L}(\psi) = 0$ by (5.1), there is a map $\varphi : \widetilde{\mathscr{T}} \to \mathfrak{g}$ integrating ψ that is $(\rho, 0)$-equivariant (i.e. ρ-equivariant in the sense of (4.3)). Indeed, choose an arbitrary base tile δ_0 and an arbitrary motion $v_0 \in \mathfrak{g}$ of that tile. The map $\varphi' : \widetilde{\mathscr{T}} \to \mathfrak{g}$ determined by ψ and this initial data, as in Section 4.2, is (ρ, u_{w_0})-equivariant for some *ρ-coboundary* $u_{w_0} = (\gamma \mapsto w_0 - \mathrm{Ad}(\rho(\gamma))\, w_0)$. The map $\varphi := \varphi' - w_0 : \widetilde{\mathscr{T}} \to \mathfrak{g}$ integrates ψ and is $(\rho, 0)$-equivariant.

Consider an edge $\tilde{\alpha} \in \widetilde{\mathscr{E}}$, with adjacent tiles $\delta, \delta' \in \widetilde{\mathscr{T}}$. The vectors $v := \varphi(\delta)$ and $v' := \varphi(\delta')$ encode the infinitesimal motions of the respective tiles δ, δ'. The vector $v \in \mathfrak{g}$ may be decomposed as $v = v_t + v_\ell$, where $v_t \in \mathrm{span}(\tilde{\alpha}) \subset \mathbb{R}^{2,1}$ is called the *transverse motion* and $v_\ell \in \mathrm{span}(\tilde{\alpha})^\perp$ the *longitudinal motion*. By Lemma 4.1.(3), the longitudinal motion v_ℓ is an infinitesimal translation with axis $\tilde{\alpha}$. Similarly, we decompose v' as $v' = v'_t + v'_\ell$. By (5.2) and Lemma 4.1.(3'), if we endow $\tilde{\alpha}$ with the transverse orientation from δ to δ', then $\psi(\tilde{\alpha}) = v' - v \in \mathrm{span}(\tilde{\alpha})$, which means that v and v' impart the same longitudinal motion to $\tilde{\alpha}$, i.e. $v_\ell = v'_\ell$. Thus $\tilde{\alpha}$ receives a well-defined amount $\sqrt{\langle v_\ell, v_\ell \rangle} = \sqrt{\langle v'_\ell, v'_\ell \rangle}$ of longitudinal motion from φ, equal to the longitudinal motion of the tile on either side of the edge; this amount is invariant under the action of $\rho(\Gamma)$ because φ is $(\rho, 0)$-equivariant. It is sufficient to prove that all longitudinal motions of edges of $\widetilde{\mathscr{E}}$ are zero, because then $\varphi = 0$ and $\psi = 0$, and so the linear dependence (5.1) is trivial. Indeed, the three linear forms on $\mathbb{R}^{2,1}$ that vanish on the three edges bounding a tile form a dual basis of $\mathbb{R}^{2,1}$ (because the edges, extended in $\mathbb{P}(\mathbb{R}^{2,1})$, have no common intersection point), hence a Killing field that imparts zero longitudinal motion to all three edges must be zero. We now assume by contradiction that not all longitudinal motions are zero.

Choose for $\tilde{\alpha}$ an edge receiving maximal longitudinal motion, i.e. such that $v_\ell = v'_\ell$ has maximal Lorentzian norm $\sqrt{\langle v_\ell, v_\ell \rangle}$ among all edges. Let A, B, C, D, E, F (resp. A, B, C', D', E', F') be the endpoints in $\partial_\infty \mathbb{H}^2 \subset \mathbb{P}(\mathbb{R}^{2,1})$ of all the edges of δ (resp. δ'), cyclically ordered (see Figure 5, left). For convenience, we refer to an edge by its two endpoints, so that e.g.

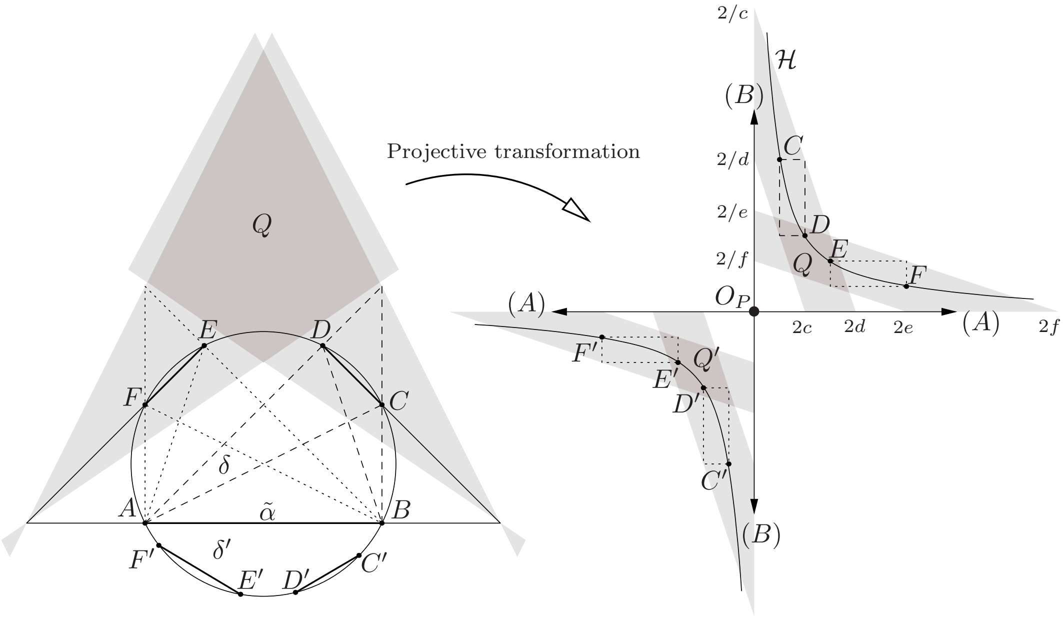


FIGURE 5. Left: View of $\mathbb{P}(\mathbb{R}^{2,1})$ in a spacelike affine slice. Right: View of $\mathbb{P}(\mathbb{R}^{2,1})$ in the timelike affine slice parallel to $\operatorname{span}(A,B)$ passing through v and v'. Note that $\partial_\infty \mathbb{H}^2$ is an ellipse in the left panel, and a hyperbola $\mathcal{H}$ in the right panel.

$\tilde{\alpha} = AB := \mathbb{H}^2 \cap \operatorname{span}(A,B)$. The fact that the longitudinal motion of AB is at least that of CD means that the image $[v] \in \mathbb{P}(\mathbb{R}^{2,1})$ of v lies in a bigon of $\mathbb{P}(\mathbb{R}^{2,1})$ bounded by two projective lines through the point $AB \cap CD$, namely the line through $AB \cap CD$ and $AC \cap BD$ and the line through $AB \cap CD$ and $AD \cap BC$. Of the two regions of $\mathbb{P}(\mathbb{R}^{2,1})$ that these lines determine, the correct one is the one containing CD (if $[v]$ is on CD then the longitudinal motion of CD is zero). We refer to Figure 5 (left panel), in which the relevant region is shaded. Similarly, the fact that the longitudinal motion of AB is at least that of EF means that $[v]$ lies in a region of $\mathbb{P}(\mathbb{R}^{2,1})$ bounded by two projective lines through the point $AB \cap EF$, namely the line through $AB \cap EF$ and $AE \cap BF$ and the line through $AB \cap EF$ and $AF \cap BE$. These two conditions determine a quadrilateral Q of $\mathbb{P}(\mathbb{R}^{2,1})$ to which $[v]$ must belong. Similarly, $[v'] \in \mathbb{P}(\mathbb{R}^{2,1})$ must belong to another quadrilateral Q' of $\mathbb{P}(\mathbb{R}^{2,1})$, corresponding to the fact that the longitudinal motion of AB is at least that of $C'D'$ and of $E'F'$.

Let P be the affine plane of $\mathbb{R}^{2,1}$ parallel to $\operatorname{span}(A,B)$ passing through v and v'; it does not contain the origin since we have assumed that $v_\ell = v'_\ell$ is nonzero. In the affine chart of $\mathbb{P}(\mathbb{R}^{2,1})$ obtained by slicing $\mathbb{R}^{2,1}$ along P, points may be labeled spacelike, lightlike, or timelike according to their relative position with respect to the orthogonal projection O_P of the origin of $\mathbb{R}^{2,1}$ to P. In this affine chart the points A and B are at infinity, the points C, D, E, F lie in this order on one branch of a hyperbola $\mathcal{H}$ whose asymptotes of directions A (horizontal) and B (vertical) intersect at O_P, and the points C', D', E', F' lie in this order on the other branch of $\mathcal{H}$ (see Figure 5, right). The two asymptotes are lightlike and divide the chart into four quadrants: two of them timelike (namely those containing $\mathcal{H}$) and two of them spacelike. We claim that Q and Q' lie in *opposite*, *timelike* quadrants; this will contradict the fact that the vector of relative motion $\psi(\tilde{\alpha}) = v' - v \in P$ is spacelike. Indeed, by construction the quadrilateral Q is the intersection of two infinite Euclidean bands: the first band is the union of all translates, along the direction of the line (CD), of the rectangle circumscribed to the segment $[CD]$ with sides parallel to the asymptotes; the second band is the union of all translates, along the direction of the line (EF), of the rectangle circumscribed to the segment $[EF]$. Without loss of generality, we may assume that C, D, E, F have respective coordinates $(c, 1/c), (d, 1/d), (e, 1/e), (f, 1/f)$ where $0 < c < d < e < f < +\infty$. Then the four boundary lines of the two Euclidean bands intersect the horizontal axis at distance $2c < 2d < 2e < 2f$ from O_P, and the vertical axis at distance $2/c > 2/d > 2/e > 2/f$ from O_P. The quadrilateral Q, which is the intersection of the two bands, therefore lies entirely in the timelike quadrant that contains the branch of $\mathcal{H}$ on which C, D, E, F lie. Similarly, Q' lies entirely in the opposite quadrant. Therefore the vector $\psi(\tilde{\alpha}) = v' - v \in P$ is timelike, a contradiction. Claim 3.2.(0) is proved.

5.2. **Proof of Claim 3.2.(1)–(2).** The two hyperideal triangulations Δ and Δ' have all but one arc in common. Let α (resp. α') be the arc of Δ (resp. Δ') that is not an arc of Δ' (resp. Δ). By Claim 3.2.(0), the sets $\{\boldsymbol{f}(\beta) \,|\, \beta \text{ an arc of } \Delta\}$ and $\{\boldsymbol{f}(\beta) \,|\, \beta \text{ an arc of } \Delta'\}$ are both bases of $H^1_\rho(\Gamma, \mathfrak{g})$. Therefore there is, up to scale, exactly one linear relation of the form

$$c_\alpha\, \boldsymbol{f}(\alpha) + c_{\alpha'}\, \boldsymbol{f}(\alpha') + \sum_{\substack{\beta \text{ arc of both} \\ \Delta \text{ and } \Delta'}} c_\beta\, \boldsymbol{f}(\beta) = 0 \quad \in H^1_\rho(\Gamma, \mathfrak{g}), \tag{5.3}$$

where $c_\alpha, c_{\alpha'}, c_\beta \in \mathbb{R}$. Claim 3.2.(1) is equivalent to the inequality $c_\alpha c_{\alpha'} > 0$. Given the nondegeneracy guaranteed by Claim 3.2.(0) and using Remark 3.3, this inequality will hold for all strip templates on S if it holds for one particular choice of minimally intersecting geodesic representatives, waists, and widths for the arcs of Δ and Δ'. We exhibit such a choice satisfying slightly more, namely

$$\begin{cases} c_\alpha & > & 0, \\ c_{\alpha'} & > & 0, \\ c_\beta & \leq & 0 \quad \text{for all other arcs } \beta \text{ of } \Delta \text{ and } \Delta'. \end{cases} \tag{5.4}$$

By Proposition 2.9, the last inequality will also imply Claim 3.2.(2).

The arcs α, α' are the diagonals of a quadrilateral bounded by four arcs $\beta_1, \beta_2, \beta_3, \beta_4$, with α separating β_1, β_2 from β_3, β_4 and α' separating β_2, β_3

from β_4, β_1. In the following, we show how to choose geodesic representatives, waists, and widths for the arcs of Δ and Δ' so that (5.3) becomes

$$\boldsymbol{f}(\alpha) + \boldsymbol{f}(\alpha') - \sum_{\beta \in \{\beta_1, \beta_2, \beta_3, \beta_4\}} \boldsymbol{f}(\beta) = 0 \tag{5.5}$$

(which implies (5.4)). In particular, all coefficients c_β in (5.3) vanish except for $\beta = \beta_1, \ldots, \beta_4$.

Let $\underline{\beta}_1, \ldots, \underline{\beta}_4 \subset S$ be any geodesic representatives of $\beta_1, \ldots, \beta_4$, respectively, and let R be the hyperideal quadrilateral bounded by these four edges. Let $\tilde{\beta}_1, \ldots, \tilde{\beta}_4 \subset \mathbb{H}^2$ be lifts of these edges bounding a lift $\widetilde{R}$ of R. The quadrilateral $\widetilde{R}$ is the intersection of $\mathbb{H}^2$ with the cone spanned by four spacelike vectors $v_1, \ldots, v_4 \in \mathbb{R}^{2,1}$, where we index the v_i so that

$$\tilde{\beta}_i = \mathbb{H}^2 \cap (\mathbb{R}_+ v_i + \mathbb{R}_+ v_{i+1})$$

for all $1 \leq i \leq 4$, with indices to be interpreted cyclically modulo 4 throughout the section (i.e. $v_5 = v_1$): see Figure 6. We now choose the geodesic representatives $\underline{\alpha}$ and $\underline{\alpha}'$ so that their lifts $\tilde{\alpha}$ and $\tilde{\alpha}'$ inside $\widetilde{R}$ satisfy

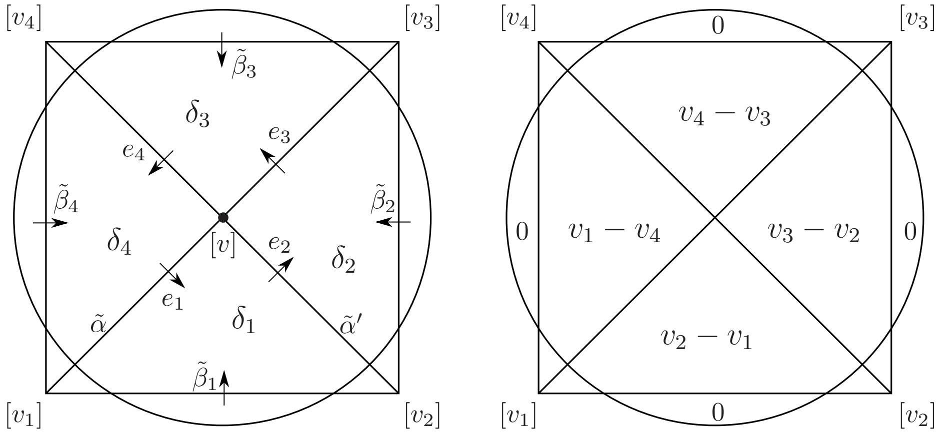


FIGURE 6. Left: View of $\mathbb{P}(\mathbb{R}^{2,1})$ in an affine slice normal to v. The quadrilateral $\widetilde{R}$ is divided into four small tiles $\delta_1, \delta_2, \delta_3, \delta_4$. The set of edges $\mathscr{E}''$ consists of $\mathscr{E} \cap \mathscr{E}'$ together with the four geodesic rays e_i formed by dividing $\tilde{\alpha}$ and $\tilde{\alpha}'$ in half at their intersection point $\{v\}$. Arrows show transverse orientations. Right: A similar picture with the value of φ at each tile.

$$\tilde{\alpha} = \mathbb{H}^2 \cap (\mathbb{R}_+ v_1 + \mathbb{R}_+ v_3) \qquad \text{and} \qquad \tilde{\alpha}' = \mathbb{H}^2 \cap (\mathbb{R}_+ v_2 + \mathbb{R}_+ v_4).$$

This configuration is achieved, for instance, if all the chosen geodesic representatives of the arcs of S are perpendicular to the boundary of the convex core: then v_1, v_2, v_3, v_4 are dual to the relevant boundary components of the preimage of the convex core in $\mathbb{H}^2$.

Let $\underline{\Delta}, \underline{\Delta}'$ be geodesic hyperideal triangulations of S corresponding to Δ, Δ', respectively, and containing our chosen geodesic representatives $\underline{\alpha}$, $\underline{\alpha}', \underline{\beta}_i$ from above. Let $\underline{\Delta}''$ be the smallest geodesic cellulation refining $\underline{\Delta}$ and $\underline{\Delta}'$, with polygon set $\widetilde{\mathscr{T}}''$ and edge set $\widetilde{\mathscr{E}}''$ in $\mathbb{H}^2$. In the remainder of the proof,

- we exhibit a $(\rho, 0)$-equivariant map $\varphi : \widetilde{\mathscr{T}}'' \to \mathfrak{g}$;
- we compute the corresponding map $\psi : \pm\widetilde{\mathscr{E}}'' \to \mathfrak{g}$ (see Section 4.2);
- we check, using Lemma 4.1.(3'), that

$$\psi = \psi_\alpha + \psi_{\alpha'} - \sum_{\beta \in \{\beta_1, \beta_2, \beta_3, \beta_4\}} \psi_\beta \tag{5.6}$$

for some appropriate choice of waists and widths, where we interpret $\psi_\alpha, \psi_{\alpha'}, \psi_\beta$ (Definition 4.3) as elements of $\Psi(\pm\widetilde{\mathscr{E}}'', \mathfrak{g})$ as in Section 4.3.

This is enough to establish (5.5) because the map $\mathscr{L}$ is linear, we can apply Observation 4.4, and $\mathscr{L}(\psi) = 0$ by $(\rho, 0)$-equivariance of φ.

We first note that the vertex set $\mathscr{V}''$ of $\underline{\Delta}''$ is just the one point $\{\underline{v}\} = \underline{\alpha} \cap \underline{\alpha}'$. The set $\mathscr{T}''$ of polygons consists of the $2|\chi| - 2$ hyperideal triangles in $\mathscr{T} \cap \mathscr{T}'$ and of four "small" (nonhyperideal) triangles arranged around $\underline{v}$. The set $\mathscr{E}''$ of edges consists of $\mathscr{E} \cap \mathscr{E}'$ together with the four geodesic rays formed by cutting $\underline{\alpha}$ and $\underline{\alpha}'$ in half at $\underline{v}$. Let $\widetilde{\mathscr{V}}'', \widetilde{\mathscr{E}}'', \widetilde{\mathscr{T}}''$ be the respective preimages in $\mathbb{H}^2$ of $\mathscr{V}'', \mathscr{E}'', \mathscr{T}''$. There are four "small" tiles $\delta_1, \delta_2, \delta_3, \delta_4 \in \widetilde{\mathscr{T}}''$ that partition $\widetilde{R}$:

$$\delta_i := \mathbb{H}^2 \cap (\mathbb{R}_+ v_i + \mathbb{R}_+ v_{i+1} + \mathbb{R}_+ v).$$

For any i, the tile δ_i is bounded by the infinite edge $\tilde{\beta}_i$ together with the two half-infinite edges e_i and e_{i+1}, where

$$e_i := \mathbb{H}^2 \cap (\mathbb{R}_+ v_i + \mathbb{R}_+ v)$$

(see Figure 6). We have $\tilde{\alpha} = e_1 \cup e_3$ and $\tilde{\alpha}' = e_2 \cup e_4$. By multiplying the $v_i \in \mathbb{R}^{2,1}$ by positive scalars, we may arrange that

$$v_1 + v_3 = v_2 + v_4. \tag{5.7}$$

Now, define

$$\varphi(\delta_i) := v_{i+1} - v_i$$

for all i, and extend this to a ρ-equivariant (in the sense of (4.3)) map $\varphi : \widetilde{\mathscr{T}}'' \to \mathfrak{g}$, with value 0 outside the $\rho(\Gamma)$-orbits of the δ_i. Let $\psi : \pm\widetilde{\mathscr{E}}'' \to \mathfrak{g}$ be the corresponding ρ-equivariant map describing the relative motion of the tiles.

We first assume that $\beta_1, \beta_2, \beta_3, \beta_4$ are pairwise distinct. Endow each $\tilde{\beta}_i$ with the transverse orientation placing δ_i on the positive side; this makes $\tilde{\beta}_i$ into an element of $\pm\widetilde{\mathscr{E}}''$. Then

$$\psi(\tilde{\beta}_i) = \varphi(\delta_i) - 0 = v_{i+1} - v_i$$

is, by Lemma 4.1.(3'), an infinitesimal translation along a geodesic of $\mathbb{H}^2$ orthogonal to $\tilde{\beta}_i$ at a point $\tilde{p}_i$, for which the translation direction is *negative* with respect to the transverse orientation. We choose the waist $p_{\beta_i} \in \underline{\beta_i}$ to be the projection to $S = \rho(\Gamma)\backslash\mathbb{H}^2$ of $\tilde{p}_i$ and the width $m_{\beta_i} = \sqrt{\langle\psi(\tilde{\beta}_i), \psi(\tilde{\beta}_i)\rangle}$ to be the velocity of the infinitesimal translation $\psi(\tilde{\beta}_i)$. Then

$$\psi(\tilde{\beta}_i) = -\psi_{\beta_i}(\tilde{\beta}_i)$$

by definition of ψ_{β_i}. Next, we transversely orient the ray e_i from δ_{i-1} to δ_i (see Figure 6). The key point is the following: by (5.7),

$$\psi(e_i) = \varphi(\delta_i) - \varphi(\delta_{i-1}) = v_{i+1} - 2v_i + v_{i-1} = v_{i+2} - v_i.$$

By Lemma 4.1.(3'), this means that $\psi(e_i)$ is an infinitesimal translation along a geodesic of $\mathbb{H}^2$ orthogonal to e_i at some point $\tilde{q}_i$, and that the direction of translation is *positive* with respect to the transverse orientation. Note that $\psi(e_i) = \psi(-e_{i+2})$ (in particular $\tilde{q}_i = \tilde{q}_{i+2}$). We choose the waist $p_\alpha \in \underline{\alpha}$ to be the projection to S of $\tilde{q}_1 = \tilde{q}_3$, and the width $m_\alpha > 0$ to be the velocity of the infinitesimal translation $\psi(\pm e_1) = \psi(\mp e_3)$. Similarly, we choose the waist and width for α' to be defined by $\psi(\pm e_2) = \psi(\mp e_4)$. Then

$$\begin{aligned} \psi(e_1) &= \psi_\alpha(e_1), & \psi(e_2) &= \psi_{\alpha'}(e_2), \\ \psi(e_3) &= \psi_\alpha(e_3), & \psi(e_4) &= \psi_{\alpha'}(e_4). \end{aligned}$$

Since $\psi, \psi_\alpha, \psi_{\alpha'}, \psi_{\beta_i}$ all take value 0 outside the $\rho(\Gamma)$-orbits of the $\tilde{\beta}_i$ and e_i, we conclude that (5.6) holds. This establishes (5.5), hence (5.4), hence Claim 3.2.(1)–(2), in the case that $\beta_1, \beta_2, \beta_3, \beta_4$ are pairwise distinct.

In the case that some of the β_i are equal, we still define φ as above. For $1 \leq i \leq 4$, if β_i is not equal to any other β_j, then we choose the waist p_{β_i} and the width m_{β_i} as above. If $\beta_i = \beta_j$ for some $1 \leq i < j \leq 4$, then $\rho(\gamma) \cdot \tilde{\beta}_i = -\tilde{\beta}_j$ for some $\gamma \in \Gamma$, and

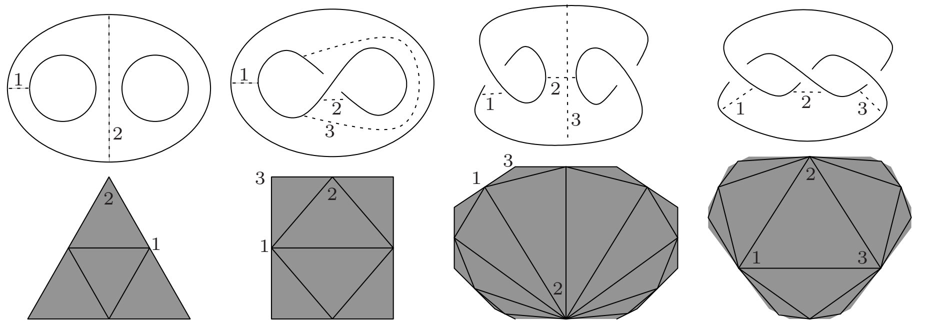
$$\psi(-\tilde{\beta}_j) = \varphi(\rho(\gamma) \cdot \delta_i) - \varphi(\delta_j) = \mathrm{Ad}(\rho(\gamma))\, \varphi(\delta_i) - \varphi(\delta_j)$$

is the sum of two infinitesimal translations orthogonal to $\tilde{\beta}_j$, both positive with respect to the transverse orientation of $\tilde{\beta}_j$. Therefore, using Lemma 4.1.(3'), we see that $\psi(-\tilde{\beta}_j)$ is again a positive infinitesimal translation orthogonal to $\tilde{\beta}_j$. We choose $\psi_{\beta_i} = \psi_{\beta_j}$ to have waist and width defined by $\psi(-\tilde{\beta}_j)$, so that $\psi_{\beta_j}(\tilde{\beta}_j) = -\psi(\tilde{\beta}_j)$. Then (5.6) holds as above. This completes the proof of Claim 3.2.(1)–(2).

Proposition 3.1 is proved, as well as Theorems 1.7 and 1.10.

6. Examples

Four noncompact surfaces (two of them orientable) have a 2-dimensional arc complex $\overline{X}$. They are represented in Figure 7. Here we summarize some elementary facts about $\overline{X}$, and how $\overline{X}$ relates to the geometry of $\mathsf{adm}(\rho)$ when ρ is the holonomy representation of a convex cocompact hyperbolic



(a) **(b)** **(c)** **(d)**

FIGURE 7. Four surfaces of small complexity (top) and their arc complexes, mapped under f to the closure of $\mathsf{adm}(\rho)$ in an affine chart of $\mathbb{P}(H^1_\rho(\Gamma,\mathfrak{g}))$ (bottom). Some arcs are labeled by Arab numerals.

structure on the surface. Margulis spacetimes whose associated hyperbolic surface has one of these four topological types were studied by Charette–Drumm–Goldman: in [CDG1, CDG3], they gave a similar tiling of $\mathsf{adm}(\rho)$ according to which isotopy classes of crooked planes embed disjointly in the Margulis spacetime.

(a) Thrice-holed sphere: The arc complex $\overline{X}$ has 6 vertices, 9 edges, 4 faces. Its image $f(\overline{X})$ is a triangle whose sides stand in natural bijection with the three boundary components of the convex core of $S = \rho(\Gamma)\backslash\mathbb{H}^2$: an infinitesimal deformation u of ρ lies in a side of the triangle if and only if it fixes the length of the corresponding boundary component, to first order. The set $\mathsf{adm}(\rho) = f(X)$ is the interior of the triangle.

(b) Twice-holed projective plane: The arc complex $\overline{X}$ has 8 vertices, 13 edges, 6 faces. Its image $f(\overline{X})$ is a quadrilateral. The horizontal sides of the quadrilateral correspond to infinitesimal deformations u that fix the length of a boundary component. The vertical sides correspond to infinitesimal deformations that fix the length of one of the two simple closed curves running through the half-twist. The set $\mathsf{adm}(\rho) = f(X)$ is the interior of the quadrilateral.

(c) Once-holed Klein bottle: The arc complex $\overline{X}$ is infinite, with one vertex of infinite degree and all other vertices of degree either 2 or 5. The closure of $f(\overline{X})$ is an infinite-sided polygon with sides indexed in $\mathbb{Z} \cup \{\infty\}$. The exceptional side has only one point in $f(\overline{X})$, and corresponds to infinitesimal deformations that fix the length of the only nonperipheral, two-sided simple closed curve γ, which goes through the two half-twists. The group $\mathbb{Z}$ naturally acts on the arc complex $\overline{X}$, via Dehn twists along γ. All nonexceptional sides are contained in $f(\overline{X})$ and correspond to infinitesimal deformations that fix the length of some curve, all these curves being related by some power of the Dehn twist along γ. The set $\mathsf{adm}(\rho) = f(X)$ is the interior of the polygon.

(d) Once-holed torus: The arc complex $\overline{X}$ is infinite, with all vertices of infinite degree; it is known as the *Farey triangulation.* The arcs are parameterized by $\mathbb{P}^1(\mathbb{Q})$. The closure of $f(\overline{X})$ contains infinitely many segments in its boundary. These segments, also indexed by $\mathbb{P}^1(\mathbb{Q})$, are in natural correpondence with the simple closed curves. Only one point of each side belongs to $f(\overline{X})$: namely, the strip deformation along a single arc, which lengthens all curves except the one curve disjoint from that arc. The group $\mathrm{GL}_2(\mathbb{Z})$ acts on $\overline{X}$, transitively on the vertices, via the mapping class group of the once-holed torus. We refer to [GLMM] or [Gu] for more details about $\mathsf{adm}(\rho)$ and its closure in this case.

Remark 6.1. In Examples **(c)** and **(d)**, where the surface has only one boundary loop $\underline{\gamma}$, the closure of $f(\overline{X})$ in $\mathbb{P}(H^1_\rho(\Gamma, \mathfrak{g}))$ does *not* meet the projective line corresponding to infinitesimal deformations that fix the length of $\underline{\gamma}$. This is implied by Proposition 2.9.

7. Fundamental domains in Minkowski 3-space

In this section, we deduce Theorem 1.9 (the Crooked Plane Conjecture, assuming convex cocompact linear holonomy) from Theorem 1.7 (the parameterization by the arc complex of Margulis spacetimes with fixed convex cocompact linear holonomy). To begin, we review the construction of crooked planes in Minkowski space, originally due to Drumm [D].

7.1. Crooked planes in $\mathbb{R}^{2,1}$.

A *crooked plane* in $\mathbb{R}^{2,1}$, as defined in [D], is the union of

- a *stem*, which is the union of all causal (i.e. timelike or lightlike) lines of a given timelike plane that pass through a given point, called the *center*;
- two *wings*, which are two disjoint open lightlike half-planes whose respective boundaries are the two (lightlike) boundary lines of the stem.

Let us fix some notation. We see $\mathbb{H}^2$ as a hyperboloid in $\mathbb{R}^{2,1}$ as in (4.1). For any future-pointing lightlike vector $v_0 \in \mathbb{R}^{2,1}$, we denote by $\mathcal{W}(v_0)$ the *left wing associated with* v_0: by definition, this is the connected component of $v_0^\perp \smallsetminus \mathbb{R}v_0$ consisting of (spacelike) vectors w that lie "to the left of v_0 seen from $\mathbb{H}^2$", i.e. such that (v, v_0, w) is positively oriented for any $v \in \mathbb{H}^2 \subset \mathbb{R}^{2,1}$. For any geodesic line ℓ of $\mathbb{H}^2$, with endpoints in $\partial_\infty\mathbb{H}^2 \subset \mathbb{P}(\mathbb{R}^{2,1})$ represented by future-pointing lightlike vectors $v^+, v^- \in \mathbb{R}^{2,1}$, we denote by $\mathcal{C}(\ell)$ the *left crooked plane centered at* $0 \in \mathbb{R}^{2,1}$ *associated with* ℓ: by definition, this is the union of the stem

$$\mathcal{S}(\ell) := \{w \in \mathrm{span}(\ell) \subset \mathbb{R}^{2,1} \mid \langle w, w\rangle \leq 0\}$$

and of the wings $\mathcal{W}(v^+)$ and $\mathcal{W}(v^-)$ (see Figure 8). A general *left crooked plane* is just a translate $\mathcal{C}(\ell) + v$ of such a set $\mathcal{C}(\ell)$ by some vector $v \in \mathbb{R}^{2,1}$.

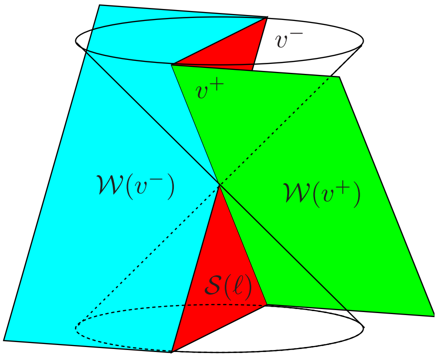


Figure 8. The left crooked plane $\mathcal{C}(\ell)$ in $\mathbb{R}^{2,1}$

The images of left crooked planes under the orientation-reversing linear map $w \mapsto -w$ are called *right crooked planes*; we will not work directly with them here.

Thinking of $\mathbb{R}^{2,1} \simeq \mathfrak{g}$ as the set of Killing vector fields on $\mathbb{H}^2$ as in Section 4.1 and using Lemma 4.1, we can describe $\mathcal{C}(\ell)$ as follows:

- the interior of the stem $\mathcal{S}(\ell)$ is the set of elliptic Killing fields on $\mathbb{H}^2$ whose fixed point belongs to ℓ;
- the lightlike line $\mathbb{R}v^+$, in the boundary of the stem $\mathcal{S}(\ell)$, is $\{0\}$ union the set of parabolic Killing fields with fixed point $[v^+] \in \partial_\infty \mathbb{H}^2$, and similarly for v^-;
- the wing $\mathcal{W}(v^+)$ is the set of hyperbolic Killing fields with attracting fixed point $[v^+] \in \partial_\infty \mathbb{H}^2$, and similarly for v^-.

In other words, $\mathcal{C}(\ell) \smallsetminus \{0\}$ is the set of Killing fields on $\mathbb{H}^2$ with a nonrepelling fixed point in $\overline{\ell}$, where $\overline{\ell}$ is the closure of ℓ in $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$.

Any crooked plane divides $\mathbb{R}^{2,1}$ into two connected components. Given a transverse orientation of ℓ, the *positive crooked half-space* $\mathcal{H}^+(\ell)$ (resp. the *negative crooked half-space* $\mathcal{H}^-(\ell)$) is the connected component of $\mathbb{R}^{2,1} \smallsetminus \mathcal{C}(\ell)$ consisting of nonzero Killing fields on $\mathbb{H}^2$ with a nonrepelling fixed point in $(\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2) \smallsetminus \overline{\ell}$ lying on the positive (resp. negative) side of $\overline{\ell}$.

7.2. Disjointness of crooked half-spaces in $\mathbb{R}^{2,1}$.

In order to build fundamental domains in $\mathbb{R}^{2,1}$ for proper actions of free groups, it is important to understand when two crooked planes are disjoint. A complete disjointness criterion for crooked planes was given by Drumm–Goldman in [DG2]. More recently, the geometry of crooked planes and crooked half-spaces was studied in [BCDG]. We now recall a sufficient condition due to Drumm.

Let ℓ be a transversely oriented geodesic line of $\mathbb{H}^2$ and let $v^+, v^- \in \mathbb{R}^{2,1}$ be future-pointing lightlike vectors representing the endpoints of ℓ in $\partial_\infty \mathbb{H}^2$, with $[v^+]$ lying to the left for the transverse orientation. We shall use the following terminology.

Definition 7.1. The open cone $\mathrm{SQ}(\ell) := \mathbb{R}_+^* v^+ - \mathbb{R}_+^* v^-$ of $\mathrm{span}(v^-, v^+) = \mathrm{span}(\ell)$ is called the *stem quadrant* of the transversely oriented geodesic ℓ.

By Lemma 4.1.(3'), the stem quadrant $\mathrm{SQ}(\ell)$ consists of all infinitesimal translations of $\mathbb{H}^2$ whose axis is orthogonal to ℓ and oriented in the positive direction. The following sufficient condition for disjointness of crooked planes was first proved by Drumm:

Proposition 7.2 (Drumm [D]). *Let ℓ, ℓ' be two geodesics of $\mathbb{H}^2$ with disjoint closures in $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$, transversely oriented away from each other. For any $v \in \mathrm{SQ}(\ell)$ and $v' \in \mathrm{SQ}(\ell')$,*

$$\overline{\mathcal{H}^+(\ell)} + v \subset \mathcal{H}^-(\ell') + v';$$

in particular, the crooked planes $\mathcal{C}(\ell) + v$ and $\mathcal{C}(\ell') + v'$ are disjoint.

Suppose ℓ, ℓ' are as in the hypotheses of the above proposition. For $w \in \mathbb{R}^{2,1}$, we actually have $\mathcal{C}(\ell) \cap (\mathcal{C}(\ell') + w) = \emptyset$ if and only if $w \in \mathrm{SQ}(\ell') - \mathrm{SQ}(\ell)$ [DG2, BCDG]. Thus the space of directions in which one can translate $\mathcal{C}(\ell')$ to make it disjoint from $\mathcal{C}(\ell)$ is a convex open cone of $\mathbb{R}^{2,1}$ with a quadrilateral basis.

It is also clear from the definitions in terms of nonrepelling fixed points of Killing fields that $\overline{\mathcal{H}^+(\ell)} \subset \mathcal{H}^-(\ell') \cup \{0\}$. Therefore Proposition 7.2 is a consequence of the following lemma, applied to (ℓ, v) and (ℓ', v').

Lemma 7.3. *For any transversely oriented geodesic ℓ of $\mathbb{H}^2$ and any $v \in \mathrm{SQ}(\ell)$,*

$$\overline{\mathcal{H}^+(\ell)} + v \subset \overline{\mathcal{H}^+(\ell)} \smallsetminus \{0\}.$$

Proof. Let L^+ be the closure of the connected component of $(\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2) \smallsetminus \overline{\ell}$ lying on the positive side of $\overline{\ell}$ for the transverse orientation, and let L^- be its complement in $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$. Consider a nonzero Killing vector field $V \in \mathbb{R}^{2,1}$ (resp. $V' \in \mathbb{R}^{2,1}$) on $\mathbb{H}^2$ with a nonrepelling fixed point in L^+ (resp. L^-). The lemma says that if $v \in \mathbb{R}^{2,1}$ is a hyperbolic Killing field with translation axis orthogonal to ℓ, oriented towards L^+, then $V + v \notin \{0, V'\}$.

Let α be the geodesic line of $\mathbb{H}^2$ whose closure contains the nonrepelling fixed points of V' and of V; orient it from the former to the latter. For $p \in \alpha$, let $\mathrm{pr}_\alpha : T_p\mathbb{H}^2 \to \mathbb{R}$ be the linear form giving the signed length of the projection to α. By definition of a Killing vector field, for any $Y \in \mathbb{R}^{2,1}$ the function $p \mapsto \mathrm{pr}_\alpha(Y(p))$ is constant on α; we call its value the *component of Y along α*. The Killing field V (resp. V') has nonnegative (resp. nonpositive) component along α, because α is oriented towards (resp. away from) the nonrepelling fixed point of V (resp. V'). On the other hand, v has positive component along α: indeed, if the oriented translation axis β of v does not meet α, then this component is $\sqrt{\langle v, v\rangle} \cosh d(\alpha, \beta) > 0$; otherwise, β meets α at an angle $\theta \in [0, \pi/2)$ and the component is $\sqrt{\langle v, v\rangle} \cos\theta > 0$. Thus $V + v$ has positive component along α, while V' has nonpositive component, which implies $V + v \notin \{0, V'\}$. $\square$

7.3. **Drumm's strategy.** In the early 1990s, Drumm [D] introduced a strategy to produce proper affine deformations u of ρ. We now briefly recall it; see [ChaG] for more details.

Begin with a convex cocompact representation $\rho \in \mathrm{Hom}(\Gamma, G)$. Then $\rho(\Gamma)$ is a Schottky group, playing ping pong on $\mathbb{H}^2$: there is a fundamental domain $\mathcal{F}$ in $\mathbb{H}^2$ for the action of $\rho(\Gamma)$ that is bounded by finitely many pairwise disjoint geodesics $\ell_1, \ell'_1, \dots, \ell_r, \ell'_r$, and there is a free generating subset $\{\gamma_1, \dots, \gamma_r\}$ of Γ such that $\ell'_i = \rho(\gamma_i) \cdot \ell_i$ for all i. The corresponding left crooked planes centered at the origin in $\mathbb{R}^{2,1}$ satisfy $\mathcal{C}(\ell'_i) = \rho(\gamma_i) \cdot \mathcal{C}(\ell_i)$. Now orient transversely each geodesic ℓ_i or ℓ'_i away from $\mathcal{F}$ and translate the corresponding crooked plane $\mathcal{C}(\ell_i)$ or $\mathcal{C}(\ell'_i)$ by a vector v_i or v'_i in the corresponding stem quadrant $\mathrm{SQ}(\ell_i)$ or $\mathrm{SQ}(\ell'_i)$. By Proposition 7.2, the resulting crooked planes are pairwise disjoint and bound a closed region $\mathcal{R}$ in $\mathbb{R}^{2,1}$. The Minkowski isometries that identify opposite pairs of crooked planes generate an affine deformation $\Gamma^{\rho,u}$ of $\rho(\Gamma)$, where $u(\gamma_i) = v'_i - \rho(\gamma_i) \cdot v_i$ for all $1 \leq i \leq r$. (In other words, u comes from propagating the movement of the original crooked planes equivariantly by translating, not only each crooked plane, but the whole closed positive crooked half-space it bounds.)

Remark 7.4. If S is nonorientable, then some elements $\rho(\gamma_i)$ belong to $\mathrm{SO}(2,1) \smallsetminus \mathrm{SO}(2,1)_0$ (corresponding to one-sided loops in S); the associated affine isometries $(\rho(\gamma_i), u(\gamma_i))$ preserve the orientation but reverse the time orientation of $\mathbb{R}^{2,1}$.

By construction, $\mathcal{R}$ is a fundamental domain for the action of $\Gamma^{\rho,u}$ on the union $\Gamma^{\rho,u} \cdot \mathcal{R}$ of all translates of $\mathcal{R}$. Drumm proved the following.

Theorem 7.5 (Drumm [D]). *In the setting above,* $\Gamma^{\rho,u} \cdot \mathcal{R} = \mathbb{R}^{2,1}$.

In particular, $\Gamma^{\rho,u}$ acts properly on $\mathbb{R}^{2,1}$ and $\mathcal{R}$ is a fundamental domain for this action.

In summary, given a fundamental domain in $\mathbb{H}^2$ and appropriate motions of the corresponding crooked planes, Drumm's procedure yields a proper cocycle u. The idea of the proof of Theorem 1.9 is that Theorem 1.7, correctly interpreted, gives an inverse to Drumm's procedure: a proper cocycle u is an infinitesimal strip deformation by Theorem 1.7, and this determines disjoint geodesics in $\mathbb{H}^2$ (namely the preimage of the support of the strips) that bound a fundamental domain in $\mathbb{H}^2$, as well as relative motions of the corresponding left crooked planes in $\mathbb{R}^{2,1}$. These yield a fundamental domain in $\mathbb{R}^{2,1}$ bounded by crooked planes.

7.4. **Proof of Theorem 1.9: the Crooked Plane Conjecture.** By [FG] and [Me], any discrete subgroup of $\mathrm{O}(2,1) \ltimes \mathbb{R}^3$ that is not virtually solvable and acts properly discontinuously and freely on $\mathbb{R}^{2,1}$ is of the form $\Gamma^{\rho,u}$ as in (1.1), where Γ is a free group. We assume that ρ is convex cocompact. By [GLM1] or [DGK1], the cocycle u belongs to the admissible cone of ρ (Definition 1.2). Note that replacing u with $-u$ amounts to conjugating the

$\Gamma^{\rho,u}$-action on $\mathbb{R}^{2,1}$ by the orientation-reversing linear map $w \mapsto -w$, which maps left crooked planes to right crooked planes. Therefore, it is sufficient to consider the case that u belongs to the *positive* admissible cone of ρ, and to prove that in this case there exists a fundamental domain in $\mathbb{R}^{2,1}$ for $\Gamma^{\rho,u}$ that is bounded by finitely many *left* crooked planes.

By Theorem 1.7, the cocycle u is an infinitesimal strip deformation supported on some collection $\mathscr{E}$ of geodesic arcs $\underline{\alpha}$ on $S = \rho(\Gamma)\backslash\mathbb{H}^2$, which cut the surface into topological disks. We use the notation and formalism of Section 4. By Observation 4.4, the infinitesimal strip deformation u is described by a (ρ, u)-equivariant assignment $\varphi : \widetilde{\mathscr{T}} \to \mathfrak{g}$ of infinitesimal motions to the tiles, such that for any tiles δ, δ' adjacent to an edge $\tilde{\alpha} \in \pm\widetilde{\mathscr{E}}$ transversely oriented from δ to δ', the Killing vector field $\psi(\tilde{\alpha}) = \varphi(\delta') - \varphi(\delta)$ is an infinitesimal translation of $\mathbb{H}^2$ orthogonal to $\tilde{\alpha}$, in the positive direction; in other words, $\psi(\tilde{\alpha})$ belongs to the stem quadrant $\mathrm{SQ}(\tilde{\alpha})$ (Definition 7.1), by Lemma 4.1.(3'). To any $\tilde{\alpha} \in \widetilde{\mathscr{E}}$ we associate the crooked plane $\mathcal{D}_{\tilde{\alpha}} := \mathcal{C}(\tilde{\alpha}) + v_{\tilde{\alpha}}$ where

$$v_{\tilde{\alpha}} := \frac{\varphi(\delta) + \varphi(\delta')}{2}. \tag{7.1}$$

One could think of $v_{\tilde{\alpha}}$ as the motion of the edge $\tilde{\alpha}$ under the infinitesimal deformation u, which we take to be the average of the motions of the adjacent tiles. Since φ is (ρ, u)-equivariant, the map $\tilde{\alpha} \mapsto \mathcal{D}_{\tilde{\alpha}}$ is $(\rho, (\rho, u))$-equivariant, meaning that $\mathcal{D}_{\rho(\gamma)\cdot\tilde{\alpha}} = \rho(\gamma) \cdot \mathcal{D}_{\tilde{\alpha}} + u(\gamma)$ for all $\tilde{\alpha} \in \widetilde{\mathscr{E}}$ and $\gamma \in \Gamma$.

We claim that the crooked planes $\mathcal{D}_{\tilde{\alpha}}$, for $\tilde{\alpha} \in \widetilde{\mathscr{E}}$, are pairwise disjoint. Indeed, consider two adjacent edges $\tilde{\alpha}, \tilde{\alpha}'$ bounding tiles $\delta, \delta', \delta''$ as in Figure 9. Transversely orient $\tilde{\alpha}$ from δ to δ', and $\tilde{\alpha}'$ from δ' to δ'', so that the positive half-plane of $\tilde{\alpha}$ in $\mathbb{H}^2$ (i.e. the connected component of $\mathbb{H}^2 \smallsetminus \tilde{\alpha}$ lying on the positive side of $\tilde{\alpha}$ for the transverse orientation) contains that of $\tilde{\alpha}'$. Note that

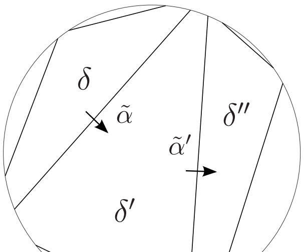


FIGURE 9. Two adjacent edges $\tilde{\alpha}, \tilde{\alpha}$ and tiles $\delta, \delta', \delta''$ as in the proofs of Theorems 1.9 and 1.11

$$v_{\tilde{\alpha}} - \varphi(\delta') = \frac{1}{2}\psi(-\tilde{\alpha}) \qquad \text{and} \qquad v_{\tilde{\alpha}'} - \varphi(\delta') = \frac{1}{2}\psi(\tilde{\alpha}').$$

Since $\psi(-\tilde{\alpha}) \in \mathrm{SQ}(-\tilde{\alpha})$ and $\psi(\tilde{\alpha}') \in \mathrm{SQ}(\tilde{\alpha}')$, Proposition 7.2 implies

$$\overline{\mathcal{H}^+(\tilde{\alpha}')} + v_{\tilde{\alpha}'} - \varphi(\delta') \subset \mathcal{H}^-(-\tilde{\alpha}) + v_{\tilde{\alpha}} - \varphi(\delta'),$$

hence

$$\overline{\mathcal{H}^+(\tilde{\alpha}')} + v_{\tilde{\alpha}'} \subset \mathcal{H}^-(-\tilde{\alpha}) + v_{\tilde{\alpha}} = \mathcal{H}^+(\tilde{\alpha}) + v_{\tilde{\alpha}}. \tag{7.2}$$

In particular, the crooked planes $\mathcal{D}_{\tilde{\alpha}}$ and $\mathcal{D}_{\tilde{\alpha}'}$ are disjoint whenever $\tilde{\alpha}, \tilde{\alpha}'$ border the same tile. Now, let $\tilde{\alpha}, \tilde{\alpha}' \in \widetilde{\mathscr{E}}$ be any distinct edges. Assign transverse orientations so that the positive half-plane of $\tilde{\alpha}$ in $\mathbb{H}^2$ contains that of $\tilde{\alpha}'$. There is a sequence of transversely oriented edges $\tilde{\alpha} = e_0, e_1, \dots, e_N = \tilde{\alpha}' \in \pm\widetilde{\mathscr{E}}$ such that any consecutive edges e_i, e_{i+1} border a common tile and the positive half-plane of e_i in $\mathbb{H}^2$ contains that of e_{i+1}. Applying (7.2) to e_i, e_{i+1}, we see by induction that

$$\overline{\mathcal{H}^+(\tilde{\alpha}')} + v_{\tilde{\alpha}'} \subset \mathcal{H}^+(\tilde{\alpha}) + v_{\tilde{\alpha}}. \tag{7.3}$$

In particular, the crooked planes $\mathcal{D}_{\tilde{\alpha}}$ and $\mathcal{D}_{\tilde{\alpha}'}$ are disjoint.

To conclude, we note that since the arcs supporting the infinitesimal strip deformation u cut the surface $S = \rho(\Gamma)\backslash\mathbb{H}^2$ into topological disks, we may choose geodesic arcs $\tilde{\alpha}_1, \tilde{\alpha}_1', \dots, \tilde{\alpha}_r, \tilde{\alpha}_r'$ from $\widetilde{\mathscr{E}}$ that bound a fundamental domain $\mathcal{F}$ in $\mathbb{H}^2$ for the action of $\rho(\Gamma)$, and a free generating subset $\{\gamma_1, \dots, \gamma_r\}$ of Γ such that $\tilde{\alpha}_i' = \rho(\gamma_i) \cdot \tilde{\alpha}_i$ for all i. By (7.3), the crooked planes $\mathcal{D}_{\tilde{\alpha}_i}$ and $\mathcal{D}_{\tilde{\alpha}_i'}$, for $1 \leq i \leq r$, are pairwise disjoint and bound a closed, connected region $\mathcal{R}$ in $\mathbb{R}^{2,1}$. For any i, the element $(\rho(\gamma_i), u(\gamma_i)) \in \Gamma^{\rho,u}$ identifies $\mathcal{D}_{\tilde{\alpha}_i}$ with $\mathcal{D}_{\tilde{\alpha}_i'}$. Therefore, $\mathcal{R}$ is a fundamental domain for the action of $\Gamma^{\rho,u}$ on $\Gamma^{\rho,u} \cdot \mathcal{R}$. That $\Gamma^{\rho,u} \cdot \mathcal{R} = \mathbb{R}^{2,1}$ then follows from Theorem 7.5, or alternatively from Lemma 7.6 below, which implies that the crooked planes $\mathcal{D}_{\tilde{\alpha}}$, for $\tilde{\alpha} \in \widetilde{\mathscr{E}}$, do not accumulate on any set.

Lemma 7.6. *For any $p \in \mathbb{H}^2$ and any sequence $(\tilde{\alpha}_n) \in \widetilde{\mathscr{E}}^{\mathbb{N}}$ going to infinity,*

$$\inf\left\{\|V_n(p)\| \mid V_n \in \mathcal{D}_{\tilde{\alpha}_n}\right\} \underset{n\to+\infty}{\longrightarrow} +\infty.$$

Proof. It is enough to treat the case that there exists a sequence $(\delta_n)_{n\in\mathbb{N}}$ of distinct elements of $\widetilde{\mathscr{T}}$ (tiles) such that $p \in \delta_0$ and $\tilde{\alpha}_n \in \widetilde{\mathscr{E}}$ is adjacent to δ_n and δ_{n-1} for all $n \geq 1$. We transversely orient $\tilde{\alpha}_n$ towards δ_n. Consider a Killing field $V_n = Y_n + v_{\tilde{\alpha}_n} \in \mathcal{D}_{\tilde{\alpha}_n}$, where $Y_n \in \mathcal{C}(\tilde{\alpha}_n)$. By definition, the Killing field Y_n admits a nonrepelling fixed point q_n in the closure of $\tilde{\alpha}_n$ in $\mathbb{H}^2 \cup \partial_\infty\mathbb{H}^2$. Let ℓ_n be the geodesic line of $\mathbb{H}^2$ whose closure contains p and q_n; orient it from the former to the latter. The component (see the proof of Lemma 7.3) of Y_n along ℓ_n is nonnegative. Moreover, for any $i \leq n$ the line ℓ_n crosses $\tilde{\alpha}_i$ in the positive direction (at a point r_i), and so the Killing field $\psi(\tilde{\alpha}_i)$ has nonnegative component along ℓ_n, equal to $\|\psi(\tilde{\alpha}_i)(r_i)\| \sin \measuredangle_{r_i}(\ell_n, \tilde{\alpha}_i)$. The term $\|\psi(\tilde{\alpha}_i)(r_i)\|$ is at least the width of the infinitesimal strip deformation along $\tilde{\alpha}_i$ and, for $2 \leq i \leq n-1$, the angle $\measuredangle_{r_i}(\ell_n, \tilde{\alpha}_i)$ is bounded below by a positive constant depending only on the geometry of the tiles. Indeed, $\measuredangle_{r_i}(\ell_n, \tilde{\alpha}_i)$ is at least the minimum among the angles of intersection of $\tilde{\alpha}_i$ with each of the four lines connecting one endpoint of $\tilde{\alpha}_{i-1}$ and one endpoint of $\tilde{\alpha}_{i+1}$; up to isometry, there are only finitely many such arrangements appearing. It follows that the component along ℓ_n of the Killing field

$$V_n = Y_n + \varphi(\delta_0) + \psi(\tilde{\alpha}_1) + \psi(\tilde{\alpha}_2) + \cdots + \psi(\tilde{\alpha}_{n-1}) + \frac{1}{2}\psi(\tilde{\alpha}_n)$$

goes to infinity as $n \to +\infty$. This completes the proof. ☐

Remark 7.7. In (7.1) above, we could have taken

$$v_{\tilde{\alpha}} := (1-t)\,\varphi(\delta) + t\,\varphi(\delta')$$

for an arbitrary fixed $t \in (0,1)$, not necessarily $t = 1/2$; the proof would have worked the same way. For each arc supporting the strip deformation, there is an interval's worth of parallel crooked planes (for t varying in $(0,1)$), each embedded in the Margulis spacetime; their union is a *parallel crooked slab*, as defined in [CDG3]. Crooked planes from different parallel crooked slabs never intersect; crooked planes in the same parallel crooked slab are tangent along subsets of their stems.

8. FUNDAMENTAL DOMAINS IN ANTI-DE SITTER 3-SPACE

In this section we introduce piecewise totally geodesic surfaces in AdS^3 analogous to the crooked planes of Section 7.1. We establish a sufficient condition for disjointness similar to Proposition 7.2 and prove Theorem 1.11.

8.1. AdS **crooked planes.**

As in Minkowski space, we define a crooked plane in AdS^3 to be the union of three pieces (see Figure 10):

- a *stem*, defined to be the union of all causal (i.e. timelike or lightlike) geodesics of a given timelike plane of AdS^3 that pass through a given point, called the *center* of the AdS crooked plane;
- two *wings*, defined to be two disjoint open lightlike half-planes of AdS^3 whose respective boundaries are the two (lightlike) boundary lines of the stem.

As in Minkowski space, an AdS crooked plane centered at the identity is determined by a geodesic line ℓ of $\mathbb{H}^2$ and a choice of orientation (left or right): we denote by $\mathsf{C}(\ell)$ the *left* AdS *crooked plane centered at* $e \in G_0$ *associated with* ℓ, which is described explicitly as follows:

- the interior of the stem $\mathsf{S}(\ell)$ of $\mathsf{C}(\ell)$ is the set of elliptic elements $h \in G_0$ whose fixed point belongs to ℓ;
- the boundary of the stem $\mathsf{S}(\ell)$ is $\{e\}$ union the set of parabolic elements $h \in G_0$ fixing one of the two endpoints $[v^+], [v^-]$ of ℓ in $\partial_\infty \mathbb{H}^2$;
- the wings of $\mathsf{C}(\ell)$ are $\mathsf{W}(v^+)$ and $\mathsf{W}(v^-)$, where $\mathsf{W}(v^+)$ is the set of hyperbolic elements $h \in G_0$ with attracting fixed point $[v^+]$, and similarly for v^-.

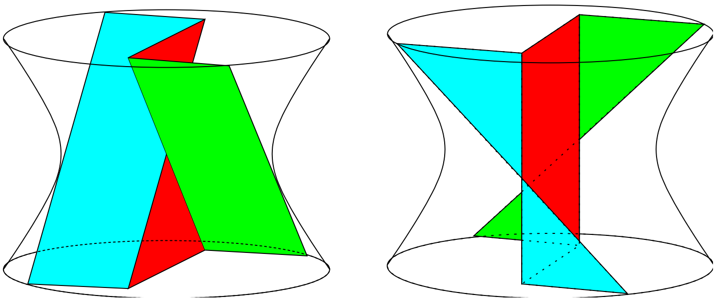


FIGURE 10. A left AdS crooked plane, seen in two different affine charts of $\mathbb{P}^3(\mathbb{R}) \supset \mathrm{AdS}^3$. The stem (red) is a bigon whose closure meets the boundary of AdS^3 in two points. On the left, these two points are at infinity; on the right, the center of the stem is at infinity. Each wing (green or blue) is itself a bigon, bounded by a line contained in the boundary of AdS^3 and a line of the stem.

In other words, $\mathsf{C}(\ell)\smallsetminus\{e\}$ is the set of orientation-preserving isometries of $\mathbb{H}^2$ (i.e. elements of $G_0 \simeq \mathrm{AdS}^3$) with a nonrepelling fixed point in $\overline{\ell}$, where $\overline{\ell}$ is the closure of ℓ in $\mathbb{H}^2 \cup \partial_\infty\mathbb{H}^2$, as in Section 7.1. Note that this is exactly the image under the exponential map of the crooked plane $\mathcal{C}(\ell) \subset \mathbb{R}^{2,1}$ from Section 7.1 (see also [Go]). We also have $\mathsf{C}(g \cdot \ell) = g\mathsf{C}(\ell)g^{-1}$ for all $g \in G_0$.

A general *left* AdS *crooked plane* is a G_0-translate (on either side) of $\mathsf{C}(\ell)$ for some geodesic ℓ of $\mathbb{H}^2$. The images of left AdS crooked planes under the orientation-reversing isometry $g \mapsto g^{-1}$ of AdS^3 are called *right* AdS *crooked planes*; we will not work with them in this paper.

Similarly to the Minkowski setting, an AdS crooked plane divides AdS^3 into two connected components (see Figure 10). Note that by contrast a timelike geodesic plane in AdS^3 *does not* divide AdS^3 into two components: it is one-sided (topologically a Möbius strip). Given a transverse orientation of ℓ, we denote by $\mathsf{H}^+(\ell)$ (resp. $\mathsf{H}^-(\ell)$) the connected component of $\mathrm{AdS}^3 \smallsetminus \mathsf{C}(\ell)$ consisting of nontrivial elements $g \in G_0$ with a nonrepelling fixed point in $(\mathbb{H}^2 \cup \partial_\infty\mathbb{H}^2) \smallsetminus \overline{\ell}$ lying on the positive (resp. negative) side of $\overline{\ell}$.

Remark 8.1. The Minkowski space $\mathbb{R}^{2,1}$ and the anti-de Sitter space AdS^3 can both be embedded into the 3-dimensional *Einstein space* Ein^3. By [Go], the closure in Ein^3 of a Minkowski or AdS crooked plane is a *crooked surface* in the sense of Frances [F]. Note that Drumm's strategy from Section 7.3 has recently been carried out in the Einstein setting in [CFL], although no complete disjointness criterion is known for the moment.

8.2. **Disjointness for left** AdS **crooked planes.** In [DGK2] we give a complete disjointness criterion for AdS crooked planes. Here we only need the following sufficient condition analogous to Proposition 7.2; as before, $\mathrm{SQ}(\ell)$ denotes the stem quadrant of ℓ (Definition 7.1).

Proposition 8.2. *Let ℓ, ℓ' be two disjoint geodesics of $\mathbb{H}^2$, transversely oriented away from each other. For any $g \in \exp(\mathrm{SQ}(\ell))$ and $g' \in \exp(\mathrm{SQ}(\ell'))$,*

$$g\,\overline{\mathsf{H}^+(\ell)} \subset g'\mathsf{H}^-(\ell');$$

in particular, the crooked planes $g\mathsf{C}(\ell)$ and $g'\mathsf{C}(\ell')$ are disjoint.

It is clear from the definitions in terms of nonrepelling fixed points of isometries of $\mathbb{H}^2$ that $\mathsf{H}^+(\ell) \subset \mathsf{H}^-(\ell')$. Therefore Proposition 8.2 is a consequence of the following lemma, applied to (ℓ, g) and (ℓ', g').

Lemma 8.3. *For any transversely oriented geodesic ℓ of $\mathbb{H}^2$ and any element $g \in \exp(\mathrm{SQ}(\ell))$,*

$$g\,\overline{\mathsf{H}^+(\ell)} \subset \mathsf{H}^+(\ell).$$

Note that, in the analogy between the Minkowski and anti-de Sitter settings, Lemma 8.3 is slightly stronger than Lemma 7.3: for $g \in \exp(\mathrm{SQ}(\ell))$ the intersection $g\mathsf{C}(\ell) \cap \mathsf{C}(\ell)$ is empty, whereas for $v \in \mathrm{SQ}(\ell)$ the intersection $(\mathcal{C}(\ell) + v) \cap \mathcal{C}(\ell)$ is the union of two affine subcones of the stem. This is the subject of the following remark.

Remark 8.4. In $\mathbb{R}^{2,1}$, a crooked plane meets any translate of itself. In AdS^3, the crooked plane $\mathsf{C}(\ell)$ does not meet its translates $g\mathsf{C}(\ell)$ for $g \in \exp(\mathrm{SQ}(\ell))$: these are obtained from $\mathsf{C}(\ell)$ by sliding *and tilting* (see Figure 11).

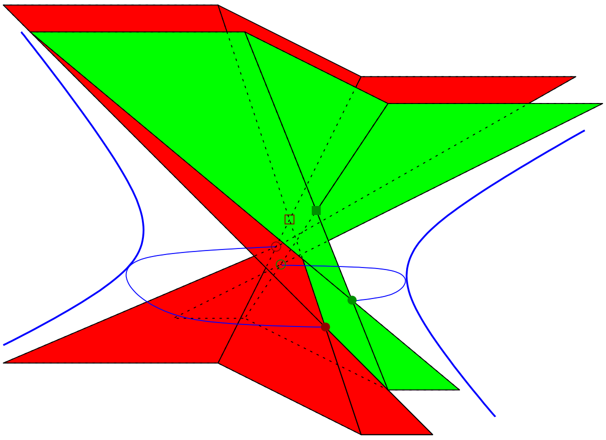


FIGURE 11. Two disjoint AdS crooked planes, green and red, in an affine chart of $\mathbb{P}^3(\mathbb{R})$ truncated above and below. The two centers are marked by square dots. The closures of the stems meet the boundary of AdS^3 in four points marked by round dots. Dual to each center is a copy of $\mathbb{H}^2$, part of whose boundary at infinity of AdS^3 is also shown (blue ellipse arcs).

Proof of Lemma 8.3. Let L^+ be the closure of the connected component of $(\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2) \smallsetminus \overline{\ell}$ lying on the positive side of $\overline{\ell}$ for the transverse orientation, and let L^- be its complement in $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$. Consider an element $h \in G_0$ (resp. $h' \in G_0$) with a nonrepelling fixed point in L^+ (resp. L^-). The lemma says that if $g \in G_0$ is hyperbolic with translation axis orthogonal to ℓ, oriented towards L^+, then $gh \neq h'$.

Note that for any $p \in L^+ \cap \mathbb{H}^2$ and $p' \in L^- \cap \mathbb{H}^2$,

$$d(g \cdot p, p') - d(p, p') \geq \eta := 2 \log \cosh \left(\frac{\lambda(g)}{2} \right) > 0, \tag{8.1}$$

where $\lambda(g) > 0$ is the translation length of g in $\mathbb{H}^2$; see below for a proof. This inequality remains true when p is either a point of $L^+ \cap \mathbb{H}^2$ or a horoball centered in $L^+ \cap \partial_\infty \mathbb{H}^2$, and p' is either a point of $L^- \cap \mathbb{H}^2$ or a horoball centerered in $L^- \cap \partial_\infty \mathbb{H}^2$, with p and p' disjoint: indeed, the (signed) distance function to a given horosphere q of $\mathbb{H}^2$ is a Busemann function, of the form $\lim_{t \to +\infty} d(\cdot, q_t) - t$ where $(q_t)_{t \geq 0}$ is a geodesic ray from q to its center in $\partial_\infty \mathbb{H}^2$; by continuity, the inequality $d(g \cdot q_t, q'_t) - d(q_t, q'_t) \leq -\eta$ passes to the limit for $q_t \in L^+$ and $q_t \in L^-$. Thus, if p (resp. p') is a singleton or horoball of $\mathbb{H}^2$ with $h \cdot p \subset p$ (resp. $h' \cdot p' \subset p'$), then

$$d(gh \cdot p, h' \cdot p') \geq d(g \cdot p, p') \geq d(p, p') + \eta.$$

It follows that gh cannot equal h'. ☐

Proof of (8.1). Note that $p', p, g \cdot p$ project in that order to the oriented translation axis $\mathcal{A}$ of g. Thus, without loss in generality, we may assume that $p \in \ell$. Let p'' be the intersection point of ℓ with the geodesic line through $g \cdot p$ and p'. Choose points $q, q' \in \ell$ so that q, p, p'', q' lie in that order along ℓ (possibly $p = p''$). We refer to Figure 12. By the triangle inequality,

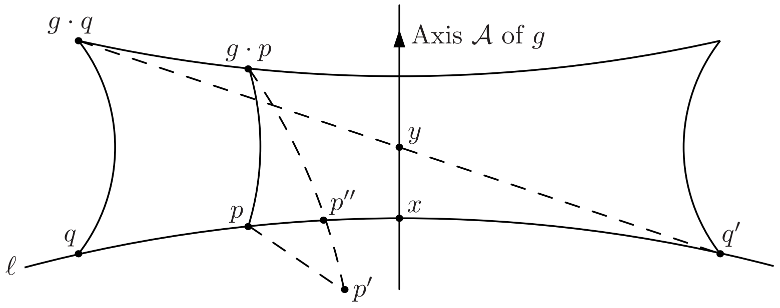


FIGURE 12. Illustration of the proof of (8.1)

$$\begin{aligned}
& d(g \cdot p, p') - d(p, p') \\
\geq \; & d(g \cdot p, p') - d(p, p'') - d(p'', p') = d(g \cdot p, p'') - d(p, p'') \\
\geq \; & d(g \cdot p, q') - d(q', p'') - d(p, p'') = d(g \cdot p, q') - d(p, q') \\
\geq \; & d(g \cdot q, q') - d(g \cdot q, g \cdot p) - d(p, q') = d(g \cdot q, q') - d(q, q').
\end{aligned}$$

Define $\{x\} = \ell \cap \mathcal{A}$ and now take q and q' to be at the same distance $t > 0$ from x, on opposite sides. Let y be the intersection point of $\mathcal{A}$ with the line through $g \cdot q$ and q'. Then $d(x, y) = \lambda(g)/2$, and so the cosine formula in the right-angled triangle yxq' yields

$$d(g \cdot q, q') = 2\, d(y, q') = 2 \operatorname{arccosh}\big(e^{\eta/2} \cosh t\big),$$

where $\eta := 2 \log \cosh(\lambda(g)/2)$. On the other hand, $d(q, q') = 2t$, so it is sufficient to see that $2 \operatorname{arccosh}(e^{\eta/2} \cosh t) - 2t \geq \eta$ for all $t \geq 0$. This follows from the fact that the function $t \mapsto 2 \operatorname{arccosh}(e^{\eta/2} \cosh t) - t$ is decreasing on $\mathbb{R}_+^*$, with limit η at infinity. $\square$

8.3. **Crooked fundamental domains in** AdS **obtained from strip deformations.** We now deduce Theorem 1.11 from Theorem 1.10.

Let $\rho, j \in \mathrm{Hom}(\Gamma, G)$ be the holonomy representations of two convex cocompact hyperbolic structures on a fixed surface. Assume that $\Gamma^{\rho,j}$ acts properly discontinuously on AdS^3. By [Ka] or [GK], up to switching j and ρ we may assume that j is "uniformly longer" than ρ in the sense that (1.3) is satisfied. This implies [T1] that the surface is not compact. Note that switching j and ρ amounts to conjugating the $\Gamma^{\rho,j}$-action on $\mathrm{AdS}^3 = G_0$ by the orientation-reversing isometry $g \mapsto g^{-1}$, which maps left AdS crooked planes to right AdS crooked planes. Assuming that j is "uniformly longer" than ρ, we shall prove the existence of a fundamental domain in AdS^3 for $\Gamma^{\rho,j}$ that is bounded by finitely many *left* AdS crooked planes.

By Theorem 1.10, we can realize j as a strip deformation of ρ supported on some collection $\mathscr{E}$ of geodesic arcs $\underline{\alpha}$ on the hyperbolic surface $S := \rho(\Gamma)\backslash\mathbb{H}^2$, which cut the surface into topological disks. We proceed as in the infinitesimal case, again using the notation of Section 4. Similarly to Observation 4.4, the strip deformation taking ρ to j is described by an assignment $\Phi : \widetilde{\mathscr{T}} \to G_0$ of motions to the tiles satisfying the following:

- Φ is $(\rho, (\rho, j))$-equivariant: for all $\gamma \in \Gamma$ and $\delta \in \widetilde{\mathscr{T}}$,
$$\Phi(\rho(\gamma) \cdot \delta) = j(\gamma)\, \Phi(\delta)\, \rho(\gamma)^{-1};$$
- for any transversely oriented geodesic $\tilde{\alpha} \in \pm\widetilde{\mathscr{E}}$, bordering tiles δ, δ' on the negative and positive sides respectively, the relative displacement
$$\Psi(\tilde{\alpha}) := \Phi(\delta)^{-1} \Phi(\delta') \in G_0$$
is hyperbolic with translation axis orthogonal to $\tilde{\alpha}$, oriented in the positive direction; equivalently, $\Psi(\tilde{\alpha}) \in \exp(\mathrm{SQ}(\tilde{\alpha}))$ by Lemma 4.1.(3'). The translation length of $\Psi(\tilde{\alpha})$ is the width of the strip to be inserted in S along the projection of $\tilde{\alpha}$.

To any $\tilde{\alpha} \in \widetilde{\mathscr{E}}$ we associate the AdS crooked plane $\mathsf{D}_{\tilde{\alpha}} := g_{\tilde{\alpha}} \mathsf{C}(\tilde{\alpha})$ where

$$g_{\tilde{\alpha}} := \Phi(\delta) \sqrt{\Phi(\delta)^{-1}\, \Phi(\delta')} = \Phi(\delta') \sqrt{\Phi(\delta')^{-1}\, \Phi(\delta)}. \tag{8.2}$$

Here the square root of a hyperbolic element denotes the hyperbolic element with the same oriented axis but half the translation length. One could think of $g_{\tilde{\alpha}}$ as the motion of the edge $\tilde{\alpha}$ under the strip deformation, which we take to be the average of the motions of the adjacent tiles. Since Φ is $(\rho,(\rho,j))$-equivariant, Ψ is $(\rho,(\rho,\rho))$-equivariant and

$$\mathsf{D}_{\rho(\gamma)\cdot\tilde{\alpha}} = j(\gamma)\,\mathsf{D}_{\tilde{\alpha}}\,\rho(\gamma)^{-1}$$

for all $\gamma \in \Gamma$ and $\tilde{\alpha} \in \widetilde{\mathscr{E}}$. We claim that the $\mathsf{D}_{\tilde{\alpha}}$, for $\tilde{\alpha} \in \widetilde{\mathscr{E}}$, are pairwise disjoint. Indeed, consider two adjacent edges $\tilde{\alpha}, \tilde{\alpha}'$ bounding tiles $\delta, \delta', \delta''$ as in Figure 9. Transversely orient $\tilde{\alpha}$ from δ to δ', and $\tilde{\alpha}'$ from δ' to δ'' so that the positive half-plane of $\tilde{\alpha}$ in $\mathbb{H}^2$ (i.e. the connected component of $\mathbb{H}^2 \smallsetminus \tilde{\alpha}$ lying on the positive side of $\tilde{\alpha}$ for the transverse orientation) contains that of $\tilde{\alpha}'$. Then

$$\Phi(\delta')^{-1} g_{\tilde{\alpha}} = \sqrt{\Psi(\tilde{\alpha})^{-1}} = \sqrt{\Psi(-\tilde{\alpha})} \quad \text{and} \quad \Phi(\delta')^{-1} g_{\tilde{\alpha}'} = \sqrt{\Psi(\tilde{\alpha}')}.$$

Since $\Psi(-\tilde{\alpha}) \in \exp(\mathrm{SQ}(-\tilde{\alpha}))$ and $\Psi(\tilde{\alpha}') \in \exp(\mathrm{SQ}(\tilde{\alpha}'))$, Proposition 8.2 implies

$$\Phi(\delta')^{-1} g_{\tilde{\alpha}'} \overline{\mathsf{H}^+(\tilde{\alpha}')} \subset \Phi(\delta')^{-1} g_{\tilde{\alpha}} \mathsf{H}^-(-\tilde{\alpha}),$$

hence

$$g_{\tilde{\alpha}'} \overline{\mathsf{H}^+(\tilde{\alpha}')} \subset g_{\tilde{\alpha}} \mathsf{H}^-(-\tilde{\alpha}) = g_{\tilde{\alpha}} \mathsf{H}^+(\tilde{\alpha}). \tag{8.3}$$

This shows in particular that the crooked planes $\mathsf{D}_{\tilde{\alpha}}$ and $\mathsf{D}_{\tilde{\alpha}'}$ are disjoint whenever $\tilde{\alpha}, \tilde{\alpha}'$ border the same tile. As in the Minkowski setting, induction and (8.3) allow us to conclude that for any edges $\tilde{\alpha}, \tilde{\alpha}' \in \pm\widetilde{\mathscr{E}}$, transversely oriented so that the positive half-plane of $\tilde{\alpha}$ in $\mathbb{H}^2$ contains that of $\tilde{\alpha}'$,

$$\overline{\mathsf{H}^+(\tilde{\alpha}')} \subset \mathsf{H}^+(\tilde{\alpha}). \tag{8.4}$$

A candidate fundamental domain R is then defined as the intersection of the crooked half-spaces corresponding to the half-planes defining a fundamental domain in $\mathbb{H}^2$ for $\rho(\Gamma)$. The proof concludes by showing that $\Gamma^{\rho,j} \cdot \mathsf{R} = \mathrm{AdS}^3$. This is implied by the following analogue of Lemma 7.6, which shows that the crooked planes $\mathsf{D}_{\tilde{\alpha}}$, for $\tilde{\alpha} \in \widetilde{\mathscr{E}}$, do not accumulate on any set.

Lemma 8.5. *For any $p \in \mathbb{H}^2$ and any sequence $(\tilde{\alpha}_n) \in \widetilde{\mathscr{E}}^{\mathbb{N}}$ going to infinity,*

$$\inf\big\{d(p, h_n \cdot p) \mid h_n \in \mathsf{D}_{\tilde{\alpha}_n}\big\} \underset{n\to+\infty}{\longrightarrow} +\infty.$$

Proof. Let $\tilde{\alpha}, \tilde{\alpha}' \in \widetilde{\mathscr{E}}$ be distinct edges bounding a common tile δ', transversely oriented as in Figure 9. By construction, $g_{\tilde{\alpha}}^{-1} g_{\tilde{\alpha}'} = \sqrt{\Psi(\tilde{\alpha})}\sqrt{\Psi(\tilde{\alpha}')}$. Applying (8.1) twice, we see that for any $p \in \tilde{\alpha}$ and $p' \in \tilde{\alpha}'$,

$$d(g_{\tilde{\alpha}} \cdot p, g_{\tilde{\alpha}'} \cdot p') \geq d(p, p') + \eta(\tilde{\alpha}) + \eta(\tilde{\alpha}'), \tag{8.5}$$

where $\eta(\tilde{\alpha}) := 2\log\cosh(\lambda(\sqrt{\Psi(\tilde{\alpha})})/2) > 0$ depends only on the width of the strip along $\tilde{\alpha}$. The inequality remains true when p is either a point of $\tilde{\alpha}$ or a horoball centered at an endpoint of $\tilde{\alpha}$, and p' is either a point of $\tilde{\alpha}'$ or a horoball centerered at an endpoint of $\tilde{\alpha}'$, with p and p' disjoint.

To prove the lemma, it is enough to treat the case that $p \in \tilde{\alpha}_0$ and there exists a sequence $(\delta_n)_{n\in\mathbb{N}\cup\{-1\}}$ of distinct tiles in $\widetilde{\mathscr{T}}$ such that $\tilde{\alpha}_n$ is adjacent to δ_n and δ_{n-1} for all $n \geq 0$. For $n \geq 1$, let p_n be any point of $\tilde{\alpha}_n$ or any horoball centered at an endpoint of $\tilde{\alpha}_n$, disjoint from p_0. The shortest path from $g_{\tilde{\alpha}_0} \cdot p_0$ to $g_{\tilde{\alpha}_n} \cdot p_n$ intersects each $g_{\tilde{\alpha}_i} \cdot \tilde{\alpha}_i$ at a point $g_{\tilde{\alpha}_i} \cdot p_i$. Applying (8.5) to each subsegment, we find

$$\begin{aligned} d(p_0, p_n) &\leq \sum_{i=1}^{n} d(p_{i-1}, p_i) \leq \sum_{i=1}^{n} \big(d(g_{\tilde{\alpha}_{i-1}} \cdot p_{i-1}, g_{\tilde{\alpha}_i} \cdot p_i) - 2\eta_0 \big) \\ &= d(g_{\tilde{\alpha}_0} \cdot p_0, g_{\tilde{\alpha}_n} \cdot p_n) - 2n\eta_0, \end{aligned}$$

where $\eta_0 > 0$ is the smallest of the finitely many values $\eta(\tilde{\alpha})$. Up to conjugation we may assume $g_{\tilde{\alpha}_0} = e$. Let $h = g_{\tilde{\alpha}_n} h' \in \mathsf{D}_{\tilde{\alpha}_n}$, where $h' \in \mathsf{C}(\tilde{\alpha}_n)$. By definition of $\mathsf{C}(\tilde{\alpha}_n)$, there is a singleton $p_n \subset \tilde{\alpha}_n$ or a horoball p_n centered at an endpoint of $\tilde{\alpha}_n$ such that $h' \cdot p_n \subset p_n$. Then

$$d(p_0, h \cdot p_n) \geq d(p_0, g_{\tilde{\alpha}_n} \cdot p_n) \geq d(p_0, p_n) + 2n\eta_0.$$

By the triangle inequality, $d(p_n, h \cdot p_n) \geq 2n\eta_0$. The result follows. $\square$

Remark 8.6. In (8.2), we could have taken

$$g_{\tilde{\alpha}} := \Phi(\delta) \left(\Phi(\delta)^{-1} \, \Phi(\delta') \right)^t = \Phi(\delta') \left(\Phi(\delta')^{-1} \, \Phi(\delta) \right)^{1-t} \in G_0$$

for an arbitrary fixed $t \in (0, 1)$, not necessarily $t = 1/2$; the proof would have worked the same way. (Here $g^t = \exp(t \log g)$.)

Appendix A. Realizing the zero cocycle

This appendix is a complement to the discussion of Section 5.2, whose notation and setup we continue with (in particular, we refer to Figure 6). It is not needed for the proofs of Theorems 1.7 and 1.10.

In Section 5.2 we gave a realization, through the map $\mathscr{L}$ of (4.4), of the zero cocycle as a linear combination of the infinitesimal strip deformation maps $\psi_\alpha, \psi_{\alpha'}, \psi_{\beta_i}$ by choosing specific geodesic representatives, waists, and widths. In this appendix, we keep the same geodesic representatives (whose extensions to $\mathbb{P}(\mathbb{R}^{2,1})$ intersect in triples as in Figure 6) but vary the waists; we determine all possible realizations of the zero cocycle supported on the arcs $\alpha, \alpha', \beta_1, \beta_2, \beta_3, \beta_4$ and discuss their geometric significance in relation with Conjecture 1.8.

A.1. Generalized infinitesimal strip deformations.

Let $\underline{\Delta}$ be a geodesic hyperideal triangulation of S, with set of edges $\mathscr{E}$. Recall the notation $\Psi(\pm\widetilde{\mathscr{E}}, \mathfrak{g})$ from Section 4.2.

Definition A.1. A relative motion map $\psi \in \Psi(\pm\widetilde{\mathscr{E}}, \mathfrak{g})$ is called a *generalized infinitesimal strip deformation* if $\psi(\tilde{\alpha}) \in \operatorname{span}(\tilde{\alpha})$ for any transversely oriented edge $\tilde{\alpha} \in \pm\widetilde{\mathscr{E}}$. The *support* of ψ is the set of arcs $\underline{\alpha} \subset S$ such that ψ is nonzero on the lifts to $\mathbb{H}^2$ of $\underline{\alpha}$. We also refer to the cohomology class in $T_{[\rho]}\mathfrak{F}$ induced by ψ as a generalized infinitesimal strip deformation.

Infinitesimal strip deformations in the sense of Definition 1.4 are generalized infinitesimal strip deformations for which $\psi(\tilde{\alpha})$ lies in one particular spacelike quadrant of the timelike plane $\operatorname{span}(\tilde{\alpha}) \subset \mathbb{R}^{2,1}$. We will call these strip deformations *positive* (and their opposites *negative*) to distinguish them among generalized strip deformations. Generalized infinitesimal strip deformations can also be neither positive nor negative: if for instance $\psi(\tilde{\alpha}) \in \mathfrak{g}$ is timelike, then the relative motion of the two tiles adjacent to $\tilde{\alpha}$ is an infinitesimal rotation centered at a point of $\tilde{\alpha}$.

Generalized infinitesimal strip deformations supported on a single arc $\underline{\alpha}$ form a linear 2-plane in $H^1_\rho(\Gamma, \mathfrak{g}) \simeq T_{[\rho]}\mathfrak{F}$.

A.2. **The point $w \in \mathbb{R}^{2,1}$ associated with a realization of the zero cocycle.** We now work in the setting of Section 5.2: the hyperideal triangulations $\underline{\Delta}, \underline{\Delta}'$ differ by a single diagonal exchange and have a common refinement $\underline{\Delta}''$. We have four spacelike vectors $v_1, v_2, v_3, v_4 \in \mathbb{R}^{2,1}$, scaled so that

$$v_1 + v_3 = v_2 + v_4, \tag{A.1}$$

and our chosen geodesic representatives $\underline{\alpha}, \underline{\alpha}', \underline{\beta}_i \in \underline{\Delta}''$ lift to $\tilde{\alpha}, \tilde{\alpha}', \tilde{\beta}_i$ with

$$\left\{\begin{array}{lcl} \tilde{\alpha} & = & \mathbb{H}^2 \cap (\mathbb{R}_+ v_1 + \mathbb{R}_+ v_3), \\ \tilde{\alpha}' & = & \mathbb{H}^2 \cap (\mathbb{R}_+ v_2 + \mathbb{R}_+ v_4), \\ \tilde{\beta}_i & = & \mathbb{H}^2 \cap (\mathbb{R}_+ v_i + \mathbb{R}_+ v_{i+1}) \end{array}\right.$$

for all $1 \leq i \leq 4$ (with the convention that $v_5 = v_1$). For simplicity, we henceforth assume that $\underline{\beta}_1, \dots, \underline{\beta}_4$ are all distinct in the quotient surface S. Recall that $\tilde{\alpha} = e_1 \cup e_3$ and $\tilde{\alpha}' = e_2 \cup e_4$. All edges e_i and $\tilde{\beta}_i$ carry transverse orientations shown in Figure 6.

For any $w \in \mathbb{R}^{2,1}$, we define a $(\rho, 0)$-equivariant map $\varphi_w : \widetilde{\mathscr{T}}'' \to \mathfrak{g}$ supported on the $\rho(\Gamma)$-orbits of the "small" tiles $\delta_1, \delta_2, \delta_3, \delta_4$ by

$$\varphi_w(\delta_i) := w \wedge (v_i \wedge v_{i+1}) \tag{A.2}$$

for all $1 \leq i \leq 4$, where $\wedge$ is the Minkowski cross-product of Section 4.1. As in Section 4.2, any $(\rho, 0)$-equivariant map $\varphi : \widetilde{\mathscr{T}}'' \to \mathfrak{g}$ defines a relative motion map $\psi : \widetilde{\mathscr{E}}'' \to \mathfrak{g}$ given by

$$\psi(e) = \varphi(\delta') - \varphi(\delta)$$

for any tiles δ, δ' adjacent to an edge $e \in \pm\widetilde{\mathscr{E}}''$ which is transversely oriented from δ to δ'.

Lemma A.2. *Let $\varphi : \widetilde{\mathscr{T}}'' \to \mathfrak{g}$ be a $(\rho, 0)$-equivariant map with $\varphi = 0$ outside the $\rho(\Gamma)$-orbits of $\delta_1, \delta_2, \delta_3, \delta_4$. The associated map $\psi : \widetilde{\mathscr{E}} \to \mathfrak{g}$ is a generalized infinitesimal strip deformation (Definition A.1) if and only if $\varphi = \varphi_w$ for some (unique) $w \in \mathbb{R}^{2,1}$.*

In this case the support of the generalized infinitesimal strip deformation is contained in $\{\underline{\alpha}, \underline{\alpha}', \underline{\beta}_1, \underline{\beta}_2, \underline{\beta}_3, \underline{\beta}_4\}$. Note that Lemma A.2 holds regardless of how the other geodesic representatives $\underline{\eta}$ for $\eta \in (\Delta \cap \Delta') \smallsetminus \{\beta_1, \beta_2, \beta_3, \beta_4\}$ are chosen.

Proof. Since $\varphi = 0$ outside the $\rho(\Gamma)$-orbits of the δ_i, we have $\psi = 0$ outside the $\rho(\Gamma)$-orbits of the $\tilde{\beta}_i$ and e_i for $1 \leq i \leq 4$. By Definition A.1, the fact that ψ is a generalized infinitesimal strip deformations is equivalent to

$$\left\{\begin{array}{l} \psi(\tilde{\beta}_i) \in \operatorname{span}(v_i, v_{i+1}), \\ \psi(e_i) \in \operatorname{span}(v_i, v_{i+2}), \\ \psi(e_i) = -\psi(e_{i+2}) \end{array}\right. \tag{A.3}$$

for all i. We first check that the space of ρ-equivariant maps $\varphi : \widetilde{\mathscr{T}}'' \to \mathfrak{g}$ for which ψ satisfies (A.3) has dimension ≤ 3. By construction, $\psi(\tilde{\beta}_i) = \varphi(\delta_i)$ and $\psi(e_i) = \varphi(\delta_i) - \varphi(\delta_{i-1})$. If ψ satisfies (A.3), then we may write

$$\varphi(\delta_i) = a_i v_i - b_{i+1} v_{i+1} \in \operatorname{span}(v_i, v_{i+1})$$

for all i, where $a_1, b_1, \dots, a_4, b_4 \in \mathbb{R}$. Since (A.1) is the only relation between v_1, v_2, v_3, v_4, the condition $\varphi(\delta_i) - \varphi(\delta_{i-1}) \in \operatorname{span}(v_i, v_{i+2})$ is equivalent to $b_{i+1} = a_{i-1}$, and so we may eliminate the b_i and write

$$\varphi(\delta_i) = a_i v_i - a_{i-1} v_{i+1}$$

where $a_1, a_2, a_3, a_4 \in \mathbb{R}$. The condition $\psi(e_i) = -\psi(e_{i+2})$ is equivalent to $\varphi(\delta_1) + \varphi(\delta_3) = \varphi(\delta_2) + \varphi(\delta_4)$, which amounts to

$$(a_1 + a_3)\, v_1 - (a_2 + a_4)\, v_2 + (a_3 + a_1)\, v_3 - (a_4 + a_2)\, v_4 = 0,$$

i.e. $a_1 + a_3 = a_2 + a_4$ by (A.1). The space of quadruples (a_1, a_2, a_3, a_4) satisfying this condition has dimension 3, as announced.

The map $w \mapsto \varphi_w$ is linear and injective, for its kernel in $\mathbb{R}^{2,1}$ is contained in $\bigcap_{i=1}^4 \operatorname{span}(v_i \wedge v_{i+1}) = \{0\}$. Therefore, it only remains to see that for any $w \in \mathbb{R}^{2,1}$, the map ψ associated with φ_w satisfies (A.3). We have

$$\psi(\tilde{\beta}_i) = \varphi_w(\delta_i) - 0 = w \wedge (v_i \wedge v_{i+1}) \in \operatorname{span}(v_i, v_{i+1}). \tag{A.4}$$

Using (A.1) and the skew-symmetry of $\wedge$, we also have

$$\psi(e_i) = \varphi_w(\delta_i) - \varphi_w(\delta_{i-1}) = w \wedge (v_i \wedge (v_{i+1} + v_{i-1})) = w \wedge (v_i \wedge v_{i+2}), \tag{A.5}$$

hence $\psi(e_i) \in \operatorname{span}(v_i, v_{i+2})$ and $\psi(e_i) = -\psi(e_{i+2})$. $\square$

A.3. **The case of timelike w.**

We now consider the map φ_w of (A.2) when $w \in \mathbb{R}^{2,1}$ is timelike. We see $\mathbb{H}^2$ as a hyperboloid in $\mathbb{R}^{2,1}$ as in (4.1).

Lemma A.3. *If $w \in \mathbb{R}^{2,1}$ is timelike, then the map $\psi : \widetilde{\mathscr{E}} \to \mathfrak{g}$ associated with φ_w is a generalized infinitesimal strip deformation whose support is exactly $\{\underline{\alpha}, \underline{\alpha}', \underline{\beta}_1, \underline{\beta}_2, \underline{\beta}_3, \underline{\beta}_4\}$. The six (nonzero) vectors $\psi(\tilde{\alpha}), \psi(\tilde{\alpha}'), \psi(\tilde{\beta}_i)$ are all spacelike, i.e. correspond to either positive or negative infinitesimal strip deformations. The corresponding waists on $\tilde{\alpha}, \tilde{\alpha}', \tilde{\beta}_i$ are the respective orthogonal projections of $\mathbb{H}^2 \cap \mathbb{R}w$ in $\mathbb{H}^2$.*

Proof. From (A.4) and (A.5) we know that $\psi(\tilde{\beta}_i)$ and $\psi(e_i)$ are orthogonal to w, hence are zero or spacelike if w is timelike. To see that they are nonzero, we note that the planes $\operatorname{span}(v_i, v_{i+1})$ and $\operatorname{span}(v_i, v_{i+2})$ are timelike, hence the vectors $v_i \wedge v_{i+1}$ and $v_i \wedge v_{i+2}$ are spacelike; in particular, these vectors are not collinear to w, and we conclude using (A.4) and (A.5).

By Lemma 4.1, the translation axes of the (positive or negative) infinitesimal strip deformation along $\tilde{\alpha}$, $\tilde{\alpha}'$, $\tilde{\beta}_i$ are the intersections of $\mathbb{H}^2$ with the orthogonal planes in $\mathbb{R}^{2,1}$ to $\psi(e_1)$, $\psi(e_2)$, $\psi(\tilde{\beta}_i)$, respectively. By (A.4) and (A.5), these all go through $\mathbb{H}^2 \cap \mathbb{R}w$. ☐

A.4. **The map φ of Section 5.2.** Recall that the map φ we constructed in Section 5.2 is given by $\varphi(\delta_i) = v_{i+1} - v_i$ for all i. By (5.6), the associated map $\psi : \widetilde{\mathscr{E}} \to \mathfrak{g}$ satisfies the hypotheses of Lemma A.2, hence $\varphi = \varphi_{w_0}$ for some $w_0 \in \mathbb{R}^{2,1}$. We now determine w_0. (This will not be needed afterwards.)

Lemma A.4. *The vector $w_0 \in \mathbb{R}^{2,1}$ is timelike and satisfies $\mathbb{H}^2 \cap \mathbb{R}w_0 = \{p\}$, where $p \in \mathbb{H}^2$ is the intersection point of the common perpendicular to $\tilde{\beta}_1$ and $\tilde{\beta}_3$ in $\mathbb{H}^2$ with the common perpendicular to $\tilde{\beta}_2$ and $\tilde{\beta}_4$.*

Proof. Since $\varphi(\delta_i) = v_{i+1} - v_i$, we have $\varphi(\delta_i) + \varphi(\delta_{i+2}) = 0$ by (A.1). By (A.4) we have $\varphi(\delta_i) = w_0 \wedge (v_i \wedge v_{i+1})$. Therefore, $w_0 \wedge (v_i \wedge v_{i+1} + v_{i+2} \wedge v_{i+3}) = 0$, and so w_0 is a multiple of $v_i \wedge v_{i+1} + v_{i+2} \wedge v_{i+3}$. In particular, w_0 belongs to the plane $\mathbb{R}(v_i \wedge v_{i+1}) + \mathbb{R}(v_{i+2} \wedge v_{i+3})$, whose intersection with $\mathbb{H}^2$ is the common perpendicular to $\tilde{\beta}_i$ and $\tilde{\beta}_{i+2}$. This holds for all $1 \leq i \leq 4$. ☐

In general, $\mathbb{H}^2 \cap \mathbb{R}w_0$ is *not* the point $\tilde{\alpha} \cap \tilde{\alpha}'$.

A.5. **Link with Conjecture 1.8.** The following lemma provides evidence for Conjecture 1.8.

Lemma A.5. *Let the infinitesimal strip map $\boldsymbol{f} : \overline{X} \to H^1_\rho(\Gamma, \mathfrak{g})$ of Section 1.2 be defined with respect to our choice of geodesic representatives of the arcs of $\Delta \cup \Delta'$, with waists on $\tilde{\alpha}, \tilde{\alpha}', \tilde{\beta}_i$ that are all orthogonal projections of a common point $p \in \mathbb{H}^2$, and with infinitesimal widths $m_\alpha, m_{\alpha'}, m_{\beta_i}$ all equal to 1. Then the image of $\boldsymbol{f}$ looks salient at the codimension-2 face shared by $\boldsymbol{f}(\Delta)$ and $\boldsymbol{f}(\Delta')$, when seen from the origin of $H^1_\rho(\Gamma, \mathfrak{g})$.*

Proof. Recall the linear relation (5.3):

$$c_\alpha \boldsymbol{f}(\alpha) + c_{\alpha'} \boldsymbol{f}(\alpha') + \sum_{\substack{\beta \text{ arc of both} \\ \Delta \text{ and } \Delta'}} c_\beta\, \boldsymbol{f}(\beta) \;=\; 0 \;\in H^1_\rho(\Gamma, \mathfrak{g}).$$

By Claim 3.2.(0) the coefficients $c_\alpha, c_{\alpha'}, c_\beta$ are unique up to scale, and by Claim 3.2–(1) we may take c_α and $c_{\alpha'}$ to be positive. The fact that the image of $\boldsymbol{f}$ looks salient at the codimension-2 face shared by $\boldsymbol{f}(\Delta)$ and $\boldsymbol{f}(\Delta')$ is then expressed by

$$c_\alpha + c_{\alpha'} + \sum_{\substack{\beta \text{ arc of both} \\ \Delta \text{ and } \Delta'}} c_\beta < 0. \tag{A.6}$$

Let us prove that this inequality holds.

By definition, the coefficients $c_\alpha, c_{\alpha'}, c_\beta$ encode a realization of the zero cocycle by (positive and negative) infinitesimal strip deformations with the given geodesic representatives and waists. By Lemma A.3 and uniqueness of $c_\alpha, c_{\alpha'}, c_\beta$, this realization is of the form φ_w for some timelike $w \in \mathbb{R}^{2,1}$ with $\mathbb{H}^2 \cap \mathbb{R}w = \{p\}$. (In particular, $c_\beta = 0$ for $\beta \notin \{\alpha, \alpha', \beta_1, \beta_2, \beta_3, \beta_4\}$.) Moreover, if p varies continuously in $\mathbb{H}^2$ while w remains in the same component of the timelike cone of $\mathbb{R}^{2,1}$, then the signs (positive or negative) of the infinitesimal strip deformations defining φ_w remain constant, similar to (5.4). Therefore, since the infinitesimal widths $m_\alpha, m_{\alpha'}, m_{\beta_i}$ are all equal to 1, the relation (5.3) has the form

$$\|x_1 - x_2\| \, \boldsymbol{f}(\alpha') + \|x_2 - x_3\| \, \boldsymbol{f}(\alpha) - \sum_{i=1}^{4} \|x_i\| \, \boldsymbol{f}(\beta_i) = 0,$$

where we set $x_i := \varphi(\delta_i)$, and $\|x\| := \sqrt{\langle x, x\rangle} > 0$ for spacelike $x \in \mathbb{R}^{2,1}$. Note that the points $v_i \wedge v_{i+1}$, for $1 \leq i \leq 4$, form a parallelogram in $\mathbb{R}^{2,1}$: indeed,

$$v_1 \wedge v_2 + v_3 \wedge v_4 - v_2 \wedge v_3 - v_4 \wedge v_1 = (v_1 + v_3) \wedge (v_2 + v_4) = 0$$

by (A.1). By Lemma A.2, it follows that the $x_i = w \wedge (v_i \wedge v_{i+1})$, for $1 \leq i \leq 4$, form a parallelogram in the spacelike plane $w^\perp \subset \mathbb{R}^{2,1}$. As a consequence,

$$\|x_1 - x_2\| + \|x_2 - x_3\| - \sum_{i=1}^{4} \|x_i\| < 0,$$

and so (A.6) holds. ☐

Note, however, that if the waists are chosen arbitrarily, so that they are not all projections of a common point $p \in \mathbb{H}^2$, then typically none of the terms c_β of (5.3) vanish, and the signs of the terms other than c_α and $c_{\alpha'}$ may vary: the conclusion of Lemma A.5 might then fail.

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