FORMAL MARKOFF MAPS ARE POSITIVE

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Abstract. This note defines a family of Laurent polynomials indexed in $\mathbb{P}^1\mathbb{Q}$ which generalize the Markoff numbers and relate to the character variety of the one-cusped torus. We describe which monomials appear in each polynomial and prove all the coefficients are positive integers. We also conjecture a generalization of that positivity result.

1. introduction

The *Laurent phenomenon* is the property of certain inductively defined sequences of rational functions to take only Laurent polynomials for their values. Fomin and Zelevinsky, who christened the phenomenon, have accounted for its surprising ubiquity in commutative algebra by their "caterpillar lemma" in [4]. In all instances studied in that article, they also noticed that the polynomial coefficients seem to be positive integers, and conjectured that this is indeed the case in general. Positivity can be shown only in a very few cases, e.g. through combinatorial interpretations of the coefficients as in [7], where the coefficients count certain colorings of particular graphs. This subject is related to deep algebraic questions and to the topic of cluster algebras.

In this article, we deal with one of the simplest instances of the Laurent phenomenon — included, in particular, in [7]. The novelty here is that we do not only show that the Laurent polynomials that arise have positive integer coefficients, but also say exactly which monomials appear. Our methods are rather pedestrian. The rational fractions (in three variables) that we will be interested in are the coordinates in $V = (\mathbb{C}(X,Y,Z))^3$ of the images of the point (X,Y,Z) under repeated applications of the map

$$\begin{array}{rccc} & V & \rightarrow & V \\ \psi \ : & (a,b,c) & \mapsto & \left(\frac{a^2+b^2}{c}, b, a\right) \end{array}$$

and of permutations of the three coordinates of V. To explain why this family of rational fractions is indexed in $\mathbb{P}^1\mathbb{Q}$, the remainder of this Introduction gives some geometric background.

In [2], Bowditch defined Markoff maps as an appealing way of analyzing the length spectrum of the set $\mathcal{C}$ of simple closed geodesics on a hyperbolic one-cusped torus $S = \mathbb{H}^2/\Gamma$ (homeomorphic to $(\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$). By fixing a covering $\mathbb{H}^2 \rightarrow S$, we can define the *holonomy representation* $\rho : \pi_1(S) \rightarrow \mathrm{Isom}^+(\mathbb{H}^2) = PSL_2(\mathbb{R})$.

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Bowditch noted that $\mathcal{C}$ stands in natural bijection with $\mathbb{P}^1\mathbb{Q} = \mathbb{Q} \cup \{\infty\}$ via the *slope* function

$$\sigma : \mathcal{C} \xrightarrow{\sim} \mathbb{P}^1\mathbb{Q} \ ,$$

and associated (bijectively) to each $c \in \mathcal{C}$ a complementary region R_c of an infinite trivalent tree $\mathcal{T}$ properly embedded in the hyperbolic plane $\mathbb{H}^2$. This tree $\mathcal{T}$ is dual to the Farey triangulation of $\mathbb{H}^2$ (see Section 2 for definitions): namely, R_c is the complementary region of $\mathcal{T}$ whose closure in the compact disc $\mathbb{H}^2 \cup \mathbb{P}^1\mathbb{R}$ contains the ideal point $\sigma(c) \in \mathbb{P}^1\mathbb{Q} \subset \mathbb{P}^1\mathbb{R}$. If $\mathcal{R}$ is the collection of all the regions R_c, the Markoff map

$$\Phi : \mathcal{R} \longrightarrow \mathbb{R}$$

associates to R_c the trace of an element of $SL_2(\mathbb{R})$ representing c (here we choose a lift to $SL_2(\mathbb{R})$ of the holonomy representation $\pi_1(S) \to PSL_2(\mathbb{R})$; note that $\pi_1(S)$ is free). The definition of Φ extends to Kleinian representations $\rho : \pi_1(S) \to SL_2(\mathbb{C})$, and Bowditch studied in particular the relationship between " Φ being proper" and "ρ being quasifuchsian". Markoff maps also provide new proofs and generalizations of McShane's identity [2, 1], and their intriguing analytic properties have not yet been fully explored. See also [6, 8, 9, 10].

Of course, a Markoff map Φ is a very redundant object. It is in fact enough to know $\Phi(R_c)$ for three mutually adjacent regions R_c to reconstruct Φ completely. For instance, denote by R_s the region $R_{\sigma^{-1}(s)}$ for $s \in \mathbb{P}^1\mathbb{Q}$, and consider

$$\Phi(R_0) = X \ ; \ \Phi(R_\infty) = Y \ ; \ \Phi(R_{-1}) = Z. \tag{1}$$

Then, every $\Phi(R_s)$ can be given by an explicit formula $f_s(X, Y, Z)$. There is in fact a non-trivial algebraic relationship between X, Y, Z, so many very different formulas for f_s exist. In [5], we were led to look for expressions of f_s as a Laurent polynomial of degree 1 in X, Y, Z:

$$f_s = \sum_{\alpha,\beta\in\mathbb{Z}} F_s(\alpha, \beta) \frac{X^{1+\alpha}\, Y^{1+\beta}}{Z^{1+\alpha+\beta}} \ \in \mathbb{Z}\left[X^{\pm 1}, Y^{\pm 1}, Z^{\pm 1}\right] \tag{2}$$

In Section 2, we show that such an expression exists, and that furthermore the integer $F_s(\alpha, \beta)$ equals 0 unless (α, β) satisfies a natural parity condition. Our main theorem is

Theorem 1. *The Laurent polynomial f_s has only positive coefficients. Moreover, all monomials in the Newton polygon of f_s which satisfy the parity condition have nonzero coefficients.*

(Recall that the Newton polygon of a Laurent polynomial $P = \sum a_{\nu_1 \dots \nu_n} X_1^{\nu_1} \dots X_n^{\nu_n}$ in n variables is the convex hull in $\mathbb{R}^n$ of the points $(\nu_1, \dots, \nu_n) \in \mathbb{Z}^n$ for which $a_{\nu_1 \dots \nu_n} \neq 0$.) In fact, we describe the Newton polygon of f_s completely (see (4) below). Some examples are shown in Figure 3 page 11. The numbers $f_s(1, 1, 1)$ (or more generally $f_s(a, b, c)$ where (a, b, c) is a Markoff triple) are the usual Markoff numbers from Diophantine approximation theory [3].

The positivity of the coefficients $F_s(\alpha, \beta)$ is already less than trivial when s is a fairly simple rational of $\mathbb{P}^1\mathbb{Q}$, say an integer (that case was used in Section 7 of [5], to establish a certain convergence property in the Teichmüller space of the cusped torus).

Theorem 1 is proved in Section 3 for positive rationals s. The remaining cases ($s < -1$ and $-1 < s < 0$) will follow by a symmetry argument outlined in Section

4. Section 5 describes a conjectural generalization, already given in [4], of Theorem 1, in order to give a taste of the power of the Laurent phenomenon.

2. The functions f_s are Laurent polynomials

Let $S = (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$ be the one-cusped (or once-punctured) torus and $\pi : \mathbb{R}^2 - \mathbb{Z}^2 \to S$ the natural projection. Denote by $\mathcal{C}$ the set of isotopy classes of simple closed curves in S that are not the loop around the cusp. If p, q are coprime integers and ℓ is a line in $\mathbb{R}^2$ of slope $s = q/p$ missing $\mathbb{Z}^2$, then $\pi(\ell)$ defines an element c of $\mathcal{C}$. We call $s \in \mathbb{P}^1\mathbb{Q}$ the *slope* of c, and write $\sigma(c) = s$. It is well-known that σ establishes a bijection $\mathcal{C} \tilde{\to} \mathbb{P}^1\mathbb{Q}$. The curve of slope s is denoted by c_s.

Consider the hyperbolic plane $\mathbb{H}^2$ with its natural boundary $\partial\mathbb{H}^2 = \mathbb{P}^1\mathbb{R}$. Whenever two curves $c, c' \in \mathcal{C}$ have (minimal) intersecion number 1, we connect the rationals $\sigma(c)$ and $\sigma(c')$ of $\mathbb{P}^1\mathbb{R}$ by a line in $\mathbb{H}^2$. The result is the *Farey triangulation* of $\mathbb{H}^2$ into infinitely many ideal *Farey triangles*. It is well-known that the triples of vertices of Farey triangles are exactly those triples of rationals that can be written

$$\left(\frac{q_0}{p_0}\ ,\ \frac{q_0+q_1}{p_0+p_1}\ ,\ \frac{q_1}{p_1}\right) \quad \text{where} \quad \left|\begin{pmatrix} q_0 & q_1 \\ p_0 & p_1 \end{pmatrix}\right| = \pm 1$$

(we agree that $\infty = \frac{\pm 1}{0}$). Geometrically, the Farey triangulation is generated by reflecting the triangle $1\infty 0$ in its sides *ad infinitum*.

Choose a point $p \in S$. Let τ be the trace operator on $SL_2(\mathbb{R})$, and fix a representation $\rho : \pi_1(S, p) \to SL_2(\mathbb{R})$ such that if $\gamma \in \pi_1(S, p)$ is in the conjugacy class of the loop around the puncture, then $\tau \circ \rho(\gamma) = -2$ (we say that ρ is *type-preserving*).

Proposition 2. *The trace τ induces a function, also noted τ, on $\mathcal{C} \simeq \mathbb{P}^1\mathbb{Q}$. If s, s_0, s_1, s' are distinct elements of $\mathbb{P}^1\mathbb{Q}$ such that s_0s_1s and s_0s_1s' are Farey triangles, then $\tau(s)$ and $\tau(s')$ are the roots of the polynomial $X^2 - \tau(s_0)\tau(s_1)X + \tau(s_0)^2 + \tau(s_1)^2$.*

Proof. For completeness we include a short proof of this well-known fact (see e.g. Section 1 of [2]). Defining τ on $\mathcal{C}$ is straightforward, since each curve in $\mathcal{C}$ determines a conjugacy class (together with its inverse) in the image of ρ. We will further omit the slope bijection $\sigma : \mathcal{C} \to \mathbb{P}^1\mathbb{Q}$ and simply consider τ as defined on $\mathbb{P}^1\mathbb{Q}$.

The modular group $SL_2(\mathbb{Z})$ acts naturally on the cusped torus S while preserving the isotopy class of the loop around the cusp. The induced action on $\mathcal{C}$ coincides (*via* σ) with the Möbius action on $\mathbb{P}^1\mathbb{Q} \subset \partial\mathbb{H}^2$, which extends to an action on the Farey triangulation of $\mathbb{H}^2$ that is transitive on the set of all Farey edges s_0s_1.

Endow the two curves $c_{s_0}, c_{s_1} \in \mathcal{C}$ with orientations and arrange c_{s_0} and c_{s_1} in S so that they intersect only at the basepoint $p \in S$. Then c_{s_0}, c_{s_1} define elements g_{s_0}, g_{s_1} of $\pi_1(S, p)$.

Observation: $[g_{s_0}, g_{s_1}]$ determines a simple loop around the puncture, and therefore has trace -2. The curves c_s and $c_{s'}$ determine the conjugacy classes of $g_{s_0}g_{s_1}$ and $g_{s_0}g_{s_1}^{-1}$ (not necessarily in that order, depending on the chosen orientations).

This observation can be checked easily when $(s_0, s_1) = (0, \infty)$ (hence $\{s, s'\} = \{1, -1\}$). The general case follows because the curves in $\mathcal{C}$ which have intersection number 1 with c_{s_0} and c_{s_1} are always exactly c_s and $c_{s'}$, and the $SL_2(\mathbb{Z})$-action (transitive on Farey edges s_0s_1) respects the intersection numbers and the loop around the cusp.

Recall the following trace relations, valid for all $a, b \in SL_2(\mathbb{R})$:

$$\begin{array}{rcl} \tau(ab) + \tau(ab^{-1}) & = & \tau(a)\tau(b) \\ \tau(ab)\tau(ab^{-1}) & = & \tau^2(a) + \tau^2(b) - 2 - \tau([a,b]). \end{array}$$

Setting $a = g_{s_0}$, $b = g_{s_1}$, the Proposition follows. □

In the notation above, we now define $f_s := \tau(s)$ for all $s \in \mathbb{P}^1\mathbb{Q}$. Dual to the Farey triangulation is an infinite 3-valent tree in $\mathbb{H}^2$ whose complementary regions R_s stand in bijection with the Farey vertices $s \in \mathbb{P}^1\mathbb{Q}$. The Markoff map Φ is therefore defined by $\Phi(R_s) = f_s$. By Proposition 2, the variables

$$(X, Y, Z) = (f_0, f_\infty, f_{-1})$$

of (1) satisfy the *Markoff equation*

$$X^2 + Y^2 + Z^2 = XYZ.$$

(This equation defines the character variety, or variety of type-preserving representations.) Moreover, Proposition 2 implies that if $(A, B, C, D) = (f_{s'}, f_{s_0}, f_{s_1}, f_s)$ and A, B, C are known (for example in terms of X, Y, Z), then we can always recover D by either one of the formulas

$$D = BC - A \qquad \text{or} \qquad D = (B^2 + C^2)/A.$$

In fact, these relations allow us to define f_s (and therefore Φ) inductively for all $s \in \mathbb{P}^1\mathbb{Q}$, in terms of X, Y, Z. In order to make each $f_s = \Phi(R_s)$ a homogeneous *Laurent polynomial* of degree 1 in X, Y, Z, we tweak the first induction relation above and use

$$f_s = f_{s_0} f_{s_1} \frac{X^2 + Y^2 + Z^2}{XYZ} - f_{s'} \tag{3}$$

where s, s_0, s_1, s' are as in Proposition 2. For example, $f_1 = \frac{X^2+Y^2}{Z}$. For all $s \in \mathbb{P}^1\mathbb{Q}$, denote by $[s]$ the unique element of $\{0, -1, \infty\}$ such that s and $[s]$ project to the same point of $\mathbb{P}^1(\mathbb{Z}/2\mathbb{Z})$. In particular, $f_{[s]}$ is one of the variables X, Y, Z.

Proposition 3. *If f_s is defined inductively for all $s \in \mathbb{P}^1\mathbb{Q}$ using (3), then f_s is a Laurent polynomial in X, Y, Z. Moreover there is a finitely supported "coefficient" function $F_s : \mathbb{Z}^2 \to \mathbb{Z}$ such that*

$$f_s = \left(\sum_{\alpha, \beta \in \mathbb{Z}} F_s(\alpha, \beta) \frac{X^{1+\alpha}\, Y^{1+\beta}}{Z^{1+\alpha+\beta}} \right) \in f_{[s]} \cdot \mathbb{Z}\left[X^{\pm 2}, Y^{\pm 2}, Z^{\pm 2}\right].$$

Proof. From (3), by induction, f_s is a homogeneous Laurent polynomial of degree 1. The claim on the parity of the degrees also follows by induction from (3), because $\{f_{[s_0]}, f_{[s_1]}, f_{[s]}\} = \{X, Y, Z\} = \{f_{[s_0]}, f_{[s_1]}, f_{[s']}\}$ holds whenever s, s_0, s_1, s' are as in Proposition 2. □

(Section 5 will expose a generalization of our "tweaking" operation (3), and a conjecture extending Theorem 1.) We now prove Theorem 1 for positive rational numbers s.

3. A family of domains and functions

Define $\mathcal{Q} = \mathbb{Q}^{\geq 0} \cup \{\infty\}$. Any point s of $\mathcal{Q}$ can be written in a unique way

$$s = \frac{q}{p} \quad \text{with } p, q \in \mathbb{N} \text{ coprime}$$

(we agree that $\infty = \frac{1}{0}$). For such $s \in \mathcal{Q}$, define

$$J_s := \left\{ (\alpha,\beta) \in \mathbb{Z}^2 \;\middle|\; \begin{array}{l} \alpha \equiv q \ ; \ \beta \equiv p \ [2] \\ \alpha \geq -q \ ; \ \beta \geq -p \\ \alpha + \beta \leq p + q - 2 \\ p\alpha + q\beta \geq 0 \end{array} \right\}. \tag{4}$$

It will turn out (Lemmas 8–9) that F_s is supported exactly on J_s. Observe that $J_0 = \{(0,-1)\}$ and $J_\infty = \{(-1,0)\}$ and $J_1 = \{(-1,1);(1,-1)\}$. Further, define

- $Z_s = (q,p) + 2\mathbb{Z}^2$ so that $J_s \subset Z_s$;
- $P_i^s = (q+2i, -p) \ \in Z_s$ for all $i \in \mathbb{Z}$;
- $Q_j^s = (-q, p+2j) \ \in Z_s$ for all $j \in \mathbb{Z}$;
- $\varphi_s(\alpha,\beta) = p\alpha + q\beta$;
- $\Lambda = \{(0,0);(0,2);(2,0)\}$;
- $n\Lambda = \Lambda + \cdots + \Lambda = \{(2i,2j) \in 2\mathbb{N}^2 | i + j \leq n\}$ for all $n \in \mathbb{N}$;
- If U is a subset of Z_s, then $\langle U \rangle_s$ denotes the intersection with Z_s of the convex hull of U in $\mathbb{R}^2$.

(We will freely use the notation "$A + B$" for the setwise sum of two subsets of a module or vector space).

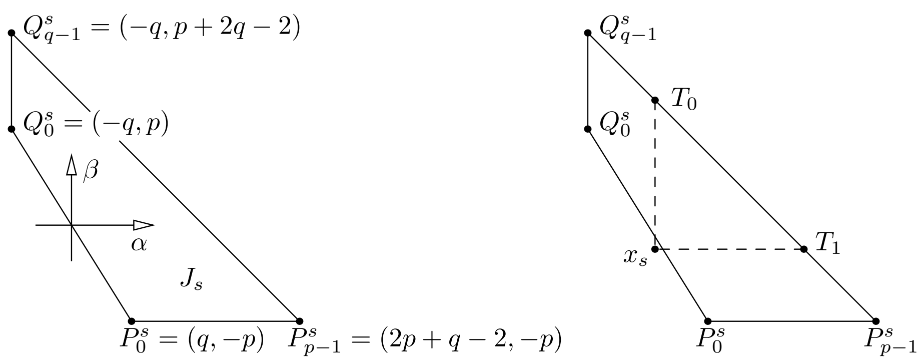

Figure 1. The domain J_s. The points x_s, T_0, T_1 of $\mathbb{Z}^2$ (right) will be defined in Lemma 7.

Lemma 4. *For all $s = \frac{q}{p}$ in $\mathcal{Q}$, one has $P^s_{p-1}, Q^s_{q-1} \in J_s$ and*

$$J_s = \left\langle \{P_i^s \,|\, 0 \leq i < p\} \cup \{Q_j^s \,|\, 0 \leq j < q\} \right\rangle_s .$$

Proof. Having checked the two cases $s = 0, \infty$ separately (one of the families $\{P_i^s\}, \{Q_j^s\}$ is then empty, so the second statement does not imply the first), assume $p, q \geq 1$ and focus on the second statement. Observe that $P^s_{p-1}, P^s_0, Q^s_0, Q^s_{q-1}$ are (in that order) the extremal points of a convex quadrilateral (or triangle, or segment, when $p = 1$ and/or $q = 1$), as shown in Figure 1 (left). The sides of the quadrilateral correspond to the four inequalities defining J_s, hence the result. □

Corollary 5. *For all $s = \frac{q}{p}$ in $\mathcal{Q}$ and n in $\mathbb{N}$, one has*

$$\begin{array}{rcl} J_s + n\Lambda & = & \left\langle \{P_i^s | 0 \leq i < p+n\} \cup \{Q_j^s | 0 \leq j < q+n\} \right\rangle_s \\ J_s + \Lambda & \supset & [P_0^s + p\Lambda] \cup [Q_0^s + q\Lambda]. \end{array}$$

Proof. Again, check the cases $s = 0, \infty$ separately. If $p, q \geq 1$, the first statement follows easily from Lemma 4 (which covers the case $n = 0$), and the second follows from the first (with $n = 1$) by observing that $P_0^s + p\Lambda$ and $Q_0^s + q\Lambda$ are the convex hulls of points of $J_s + \Lambda$: for instance,

$$\begin{array}{rcl} P_0^s + p\Lambda & = & \left\langle \{P_0^s; P_p^s; (q,p)\} \right\rangle_s \\ & = & \left\langle \left\{ P_0^s; P_p^s; \dfrac{qP_p^s + pQ_q^s}{q+p} \right\} \right\rangle_s . \end{array}$$

□

We now redefine the coefficient functions $F_s(\cdot, \cdot)$ of Proposition 3 from a slightly altered point of view. Let $\mathcal{F}$ be the $\mathbb{Z}$-module of functions $F : \mathbb{Z}^2 \to \mathbb{Z}$ having finite support. We can define a convolution law on $\mathcal{F}$ by $F * G(u) = \sum_{x+y=u} F(x)G(y)$. This convolution law mimics products of Laurent polynomials:

$$\sum_{\alpha,\beta} F(\alpha,\beta) \tfrac{X^{1+\alpha}Y^{1+\beta}}{Z^{1+\alpha+\beta}} \cdot \sum_{\alpha,\beta} G(\alpha,\beta) \tfrac{X^{1+\alpha}Y^{1+\beta}}{Z^{1+\alpha+\beta}} = \tfrac{XY}{Z} \sum_{\alpha,\beta} F * G(\alpha,\beta) \tfrac{X^{1+\alpha}Y^{1+\beta}}{Z^{1+\alpha+\beta}}.$$

Also, denoting by $\mathbb{1}_U$ the characteristic function of a set U, define the following elements of $\mathcal{F}$:

$$F_s = \mathbb{1}_{J_s} \text{ for } s \in \{0, 1, \infty\} .$$

It is straightforward to check that the identity of Proposition 3 holds for these F_0, F_1, F_∞. Finally, for $s \in \mathcal{Q} \smallsetminus \{0, 1, \infty\}$, we shall define F_s in an inductive way. In $\mathbb{H}^2$ endowed with the Farey triangulation, consider the line L_s connecting s to the midpoint $\sqrt{-1}$ of the line 0∞. Denote by s_0, s_1 the ends of the first Farey edge encountered by L_s (closest to s). We call s_0 and s_1 the *parents* of s. Up to exchanging indices, we may assume that the parents of s_1 are s_0 and another point $s' \in \mathcal{Q}$ (we agree that the parents of 1 are 0 and ∞). See Figure 2. In particular, one has

$$(5) \qquad \left.\begin{array}{rcl} (p,q) & = & (p_1,q_1) + (p_0,q_0) \\ (p',q') & = & (p_1,q_1) - (p_0,q_0) \end{array}\right\} \text{ for } (s,s',s_0,s_1) = \left(\tfrac{q}{p}, \tfrac{q'}{p'}, \tfrac{q_0}{p_0}, \tfrac{q_1}{p_1}\right) .$$

Definition 6. For each configuration (s, s', s_0, s_1) as above, we set

$$(6) \qquad F_s := (F_{s_0} * F_{s_1} * \mathbb{1}_\Lambda) - F_{s'} \quad \text{where } \Lambda = \{(0,0); (0,2); (2,0)\}.$$

Since the dual of the Farey triangulation is a tree, this definition is easily seen to be self–consistent. Clearly, F_s is in $\mathcal{F}$. It is easy to check that (6) is just a reformulation of (3), so (6) agrees with our first definition (Prop. 3) of F_s (the convolution factor $\mathbb{1}_\Lambda$ corresponds to the multiplication factor $\frac{X^2+Y^2+Z^2}{XYZ}$ of (3)). The following three Lemmas (numbered 7-8-9) are intended to prove that F_s is supported on J_s and $F_s(J_s) \subset \mathbb{Z}^{>0}$, for all $s \in \mathcal{Q}$. The reader is invited to read their three statements first (the three proofs could be written as one vast simultaneous induction on s for the simultaneous three statements).

Lemma 7. *For each configuration (s, s', s_0, s_1) as above where $s \in \mathcal{Q} \smallsetminus \{0, 1, \infty\}$, the set $J_{s'} \smallsetminus J_s$ consists of a unique (extremal) point x_s of $J_{s'}$, and*

$$J_{s_0} + J_{s_1} + \Lambda = J_s \sqcup \{x_s\}$$

Remark: if $s \in \mathcal{Q} \smallsetminus \{0, \infty\}$, following Lemma 4, we call "extremal" the points $P_0^s, P_{p-1}^s, Q_0^s, Q_{q-1}^s$ of J_s (with possible repeats). If $s \in \{0, \infty\}$, then J_s is reduced to an (extremal) point $P_{p-1}^s = Q_{q-1}^s$.

Proof. Let (α, β) be an element of $J_{s'}$. By (5) one has $Z_{s'} = Z_s$ so (α, β) satisfies the congruence conditions of (4). Still by (5), one has $p' \leq p$ and $q' \leq q$ so the first three inequalities of (4) are also satisfied at (α, β). For the fourth inequality, consider the linear form $\varphi_s(\alpha, \beta) = p\alpha + q\beta$. Clearly, $\varphi_s(Z_s) \subset 2\mathbb{Z}$. Furthermore, observe

$$\begin{aligned} \varphi_s(P_i^{s'}) &= pq' - qp' + 2ip \\ \varphi_s(Q_j^{s'}) &= qp' - pq' + 2jq \\ pq' - qp' &= 2(p_0q_1 - p_1q_0) \ = \ \pm 2 \ (s_0, s_1 \text{ Farey neighbors}). \end{aligned}$$

Thus, if $p' = 0$ (resp. $q' = 0$), taking for x_s the only point $Q_0^{s'}$ (resp. $P_0^{s'}$) of $J_{s'}$ yields $\varphi_s(x_s) = -2$. If $p'q' > 0$, we find that exactly one point x_s among $\{P_0^{s'}, Q_0^{s'}\}$ satisfies $\varphi_s(x_s) = -2$ while $\varphi_s(x) \geq 0$ at all other extremal points x of $J_{s'}$. It follows that on $J_{s'} - \{x_s\}$ one has $\varphi_s > -2$ i.e. $\varphi_s \geq 0$. Hence the first statement.

Let us now prove the second statement. For $(y_0, y_1, \lambda) \in J_{s_0} \times J_{s_1} \times \Lambda$, it is again straightforward to check that $(\alpha, \beta) = y_0 + y_1 + \lambda$ satisfies the congruence conditions and the first three inequalities of (4). For the fourth, compute

$$\begin{aligned} \varphi_s(P_i^{s_0}) &= p_1q_0 - p_0q_1 + 2ip \qquad & \varphi_s(P_i^{s_1}) &= p_0q_1 - p_1q_0 + 2ip \\ \varphi_s(Q_j^{s_0}) &= p_0q_1 - p_1q_0 + 2jq \qquad & \varphi_s(Q_j^{s_1}) &= p_1q_0 - p_0q_1 + 2jq. \end{aligned}$$

Again, observe that $p_0q_1 - p_1q_0 = \pm 1$. The same argument as above (involving this time extremal points of J_{s_0}, J_{s_1} instead of $J_{s'}$) shows that φ_s takes the value -1 at exactly one point $y_0 \in \{P_0^{s_0}, Q_0^{s_0}\}$ (resp. $y_1 \in \{P_0^{s_1}, Q_0^{s_1}\}$) and $\varphi_s \geq 1$ holds on $J_{s_0} - \{y_0\}$ (resp. $J_{s_1} - \{y_1\}$). Moreover, y_k belongs to J_{s_k} for $k \in \{0, 1\}$ (this is immediate from Lemma 4, unless $p_kq_k = 0$ where we need to check separately). The following table summarizes the two possible cases for y_0, y_1, x_s.

(7)

	$p_0q_1 - p_1q_0$	y_0	y_1	x_s
Case 1	-1	$Q_0^{s_0}$	$P_0^{s_1}$	$P_0^{s'}$
Case 2	1	$P_0^{s_0}$	$Q_0^{s_1}$	$Q_0^{s'}$

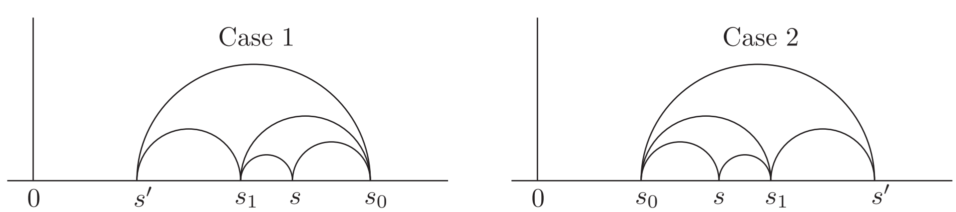

FIGURE 2. Possible relative positions of s, s_0, s_1, s'.

Using Relations (5) and the definitions of P_i^s and Q_j^s, one checks immediately that $y_0 + y_1 = x_s$ in both cases. Since φ_s is linear, x_s turns out to be the only point of $J_{s_0} + J_{s_1} + \Lambda$ where $\varphi_s < 0$. This gives one inclusion of the equality to be proved.

For the other inclusion, $J_s \sqcup \{x_s\} \subset J_{s_0} + J_{s_1} + \Lambda$, we shall restrict to Case 1 above (Case 2 is similar). By Table (7), since $Q_0^{s_0}$ and $P_0^{s_1}$ belong to J_{s_0} and J_{s_1}, one has $q_0, p_1 > 0$. In view of Corollary 5, it is sufficient to prove that

$$J_s \sqcup \{x_s\} \subset (J_{s_0} + P_0^{s_1} + p_1\Lambda) \cup (J_{s_1} + Q_0^{s_0} + q_0\Lambda). \tag{8}$$

Still by Corollary 5, since $P_0^{s_1} + Z_{s_0} = Z_s$, one has

$$\begin{aligned} J_{s_0} + P_0^{s_1} + p_1\Lambda &= \left\langle P_0^{s_1} + \left(\{P_i^{s_0} | 0 \leq i < p_0 + p_1\} \cup \{Q_j^{s_0} | 0 \leq j < q_0 + p_1\}\right)\right\rangle_s \\ &= \left\langle \{P_i^s | 0 \leq i < p\} \cup \{Q_0^{s_0} + P_0^{s_1}, Q_{q_0+p_1-1}^{s_0} + P_0^{s_1}\}\right\rangle_s \\ &= \langle \{P_i^s | 0 \leq i < p\} \cup \{x_s, T_0\}\rangle_s \\ \text{where } T_0 &= (q - 2q_0, p + 2(q_0 - 1)). \end{aligned}$$

(To write the second line, we replaced the collection of the $Q_j^{s_0}$ by its extremal terms: this is justified because $q_0 + p_1 > 0$). Note that the points $\{P_i^s | 0 \leq i < p\}$ are exactly the P_i^s arising in the convex–hull definition of J_s in Lemma 4. Also note that T_0 has the same *abscissa* as $x_s = (q', -p')$ and lies on the edge $E = Q_{q-1}^s P_{p-1}^s$ of J_s (see Figure 1, right).

Similarly,

$$\begin{aligned} J_{s_1} + Q_0^{s_0} + q_0\Lambda &= \left\langle \{Q_j^s | 0 \leq j < q\} \cup \{x_s, T_1\}\right\rangle_s \\ \text{where } T_1 &= (q + 2(p_1 - 1), p - 2p_1). \end{aligned}$$

Here, T_1 has the same *ordinate* as x_s.

We just captured all the P_i^s, Q_j^s which according to Lemma 4 define J_s (Figure 1, right). To finish proving that $J_s \cup x_s$ is contained in the union of the (discretized) polygons $A := \langle \{P_i^s | 0 \leq i < p\} \cup \{x_s, T_0\}\rangle_s$ and $B := \left\langle \{Q_j^s | 0 \leq j < q\} \cup \{x_s, T_1\}\right\rangle_s$, just observe that the points $Q_{q-1}^s, T_0, T_1, P_{p-1}^s$ lie in that order on the edge $E = Q_{q-1}^s P_{p-1}^s$ of J_s, while the edge $P_0^s Q_0^s$ of J_s (defined by "$\varphi_s = 0$") separates x_s from E. Therefore A contains all points of J_s right of the vertical line through x_s, while B contains all points of J_s above the horizontal line through x_s; and every point of J_s satisfies at least one of these two conditions. See the right panel of Figure 1. □

Lemma 8. *The function F_s is supported on a subset of J_s for all $s \in \mathcal{Q}$, and if c is an extremal point of J_s, then $F_s(c) = 1$.*

Proof. We prove both facts by simultaneous induction. They hold for $s \in \{0, 1, \infty\}$ so assume they hold for s_0, s_1, s' and let us prove them for s. By (6), F_s is supported on $(J_{s_0} + J_{s_1} + \Lambda) \cup J_{s'} = J_s \cup \{x_s\}$, with x_s defined as in Lemma 7. Recall the linear form φ_s from the proof of Lemma 7: over $J_{s_0}, J_{s_1}, \Lambda$, the form φ_s achieves its respective minima only at the extremal points $y_0, y_1, 0$; therefore x_s is realized in $J_{s_0} + J_{s_1} + \Lambda$ only as $y_0 + y_1 + (0, 0)$. Hence, by induction, $F_{s_0} * F_{s_1} * \mathbb{1}_\Lambda(x_s) = 1$. But x_s is also an extremal point of $J_{s'}$, so by induction (6) yields $F_s(x_s) = 0$: the function F_s is supported within J_s.

Next, observe that the extremal point P_{p-1}^s of J_s maximizes the first coordinate (a similar statement is true for $J_{s_0}, J_{s_1}, J_{s'}$). Since $P_{p-1}^s = P_{p_0-1}^{s_0} + P_{p_1-1}^{s_1} + (2, 0)$, one has by induction $F_{s_0} * F_{s_1} * \mathbb{1}_\Lambda(P_{p-1}^s) = 1$. Also, P_{p-1}^s does not belong to $J_{s'}$ because all (α, β) in $J_{s'}$ satisfy $\alpha + \beta \leq p' + q' - 2 < p + q - 2$. By (6), we find $F_s(P_{p-1}^s) = 1$.

Similarly, $F_s(Q^s_{q-1}) = 1$. Consider one of the (at most two) remaining extremal points of J_s, say P^s_0. Without loss of generality, one has $p \geq 2$ (otherwise, the point has already been treated as P^s_{p-1}). One cannot have $\{p_0, p_1\} = \{0, p\}$ lest $|p_0 q_1 - p_1 q_0| \geq p > 1$ (recall s_0, s_1 are Farey neighbors). Therefore $p_0, p_1 \geq 1$. Observe that the points $P^s_0, P^{s_0}_0, P^{s_1}_0$ are the minimizers over J_s, J_{s_0}, J_{s_1} of the form $(\alpha, \beta) \mapsto \beta + \varepsilon\alpha$, for very small positive ε. Since $P^s_0 = P^{s_0}_0 + P^{s_1}_0 + (0,0)$, we find that $F_{s_0} * F_{s_1} * \mathbb{1}_\Lambda(P^s_0) = 1$. Finally, P^s_0 cannot belong to $J_{s'}$ because of its second coordinate, $-p < -p'$. By (6), this yields $F_s(P^s_0) = 1$. Similarly, $F_s(Q^s_0) = 1$. □

Lemma 9. *For all $s \in \mathcal{Q}$ one has $F_s(J_s) \subset \mathbb{Z}^{>0}$. If $s \notin \{0, \infty\}$ then*

$$\mathbb{1}_{J_s} \cdot \sup \left\{ \begin{array}{ccc} \mathbb{1}_{\{P^{s_0}_0\}} * F_{s_1} & , & \mathbb{1}_{\{P^{s_1}_0\}} * F_{s_0} \ , \\ \mathbb{1}_{\{Q^{s_0}_0\}} * F_{s_1} & , & \mathbb{1}_{\{Q^{s_1}_0\}} * F_{s_0} \end{array} \right\} \leq F_s.$$

Remark 10. By Corollary 5 and Lemma 7, each function in the curly bracket is supported within $J_s \sqcup \{x_s\}$ (because e.g. $P^{s_0}_0 \in J_{s_0} + \Lambda$ while F_{s_1} is supported within J_{s_1}). In other words, the factor $\mathbb{1}_{J_s}$ in front of the curly bracket can be replaced by $\mathbb{1}_{Z_s - \{x_s\}}$ without altering the strength of the statement.

Proof. Again, both facts are proved by simultaneous induction. They hold for $s \in \{0, 1, \infty\}$; assume they hold for s_0, s_1, s'; let us prove them for s. Recall our convention that the parents of s_1 are s_0 and s' (so in particular, $s_1 \neq 0, \infty$). We saw in the course of proving Lemma 7 that x_s is either $P^{s_0}_0 + Q^{s_1}_0 = Q^{s'}_0$ or $Q^{s_0}_0 + P^{s_1}_0 = P^{s'}_0$. On the other hand, x_{s_1} is either $P^{s_0}_0 + Q^{s'}_0$ or $Q^{s_0}_0 + P^{s'}_0$. In fact, using (5) and the generic characterization $\varphi_\sigma(x_\sigma) = -2$ (for all $\sigma \in \mathbb{P}^1\mathbb{Q}$), it is easy to check that

$$\begin{array}{lllll} x_{s_1} = P^{s_0}_0 + Q^{s'}_0 & \Longleftrightarrow & q_0 p' - p_0 q' = -1 & \Longleftrightarrow & x_s = Q^{s'}_0 \ ; \\ x_{s_1} = Q^{s_0}_0 + P^{s'}_0 & \Longleftrightarrow & p_0 q' - q_0 p' = -1 & \Longleftrightarrow & x_s = P^{s'}_0 \ . \end{array} \tag{9}$$

Define in general $G_s = F_s * \mathbb{1}_\Lambda$. Lemma 8 easily yields $G_\sigma(P^\sigma_0) = G_\sigma(Q^\sigma_0) = 1$ for all $\sigma \in \mathcal{Q}$ (this should again be checked separately for $\sigma = 0, \infty$). By Lemma 7 and the induction hypothesis, we have $F_{s_0} * F_{s_1} * \mathbb{1}_\Lambda > 0$ on J_s. Moreover, by (6),

$$\begin{aligned} F_s + F_{s'} &= F_{s_0} * F_{s_1} * \mathbb{1}_\Lambda \\ &= \sum_{\lambda \in (J_{s_0} + \Lambda)} G_{s_0}(\lambda) \cdot \mathbb{1}_{\{\lambda\}} * F_{s_1} \\ &= \left[\left(\mathbb{1}_{\{P^{s_0}_0\}} + \mathbb{1}_{\{Q^{s_0}_0\}} \right) * F_{s_1} \right] + \sum_{\substack{\lambda \in (J_{s_0} + \Lambda) \\ \lambda \neq P^{s_0}_0, Q^{s_0}_0}} G_{s_0}(\lambda) \cdot \mathbb{1}_{\{\lambda\}} * F_{s_1} \ . \end{aligned}$$

Thus, if we prove

$$\mathbb{1}_{\{P^{s_0}_0\}} * F_{s_1}(x) \geq F_{s'}(x) \ ; \ \mathbb{1}_{\{Q^{s_0}_0\}} * F_{s_1}(x) \geq F_{s'}(x) \ \text{ for all } x \neq x_s \ , \tag{10}$$

then we will have at once $F_s > 0$ on J_s (because $F_s + F_{s'} \geq 2F_{s'}$ and $F_{s'}(J_{s'}) > 0$), and also $F_s \geq \sup \left\{ \mathbb{1}_{\{P^{s_0}_0\}} * F_{s_1}, \mathbb{1}_{\{Q^{s_0}_0\}} * F_{s_1} \right\}$ on J_s. That is half of Lemma 9.

Using the relation $P^{s_0}_0 = -Q^{s_0}_0 \neq 0$ and the identities $\mathbb{1}_{\{\xi\}} * \mathbb{1}_{\{\eta\}} = \mathbb{1}_{\{\xi+\eta\}}$ and $\mathbb{1}_{\{\xi\}} * f(x + \xi) = f(x)$, Equation (10) is equivalent to

$$F_{s_1}(y) \geq \mathbb{1}_{\{Q^{s_0}_0\}} * F_{s'}(y) \ \text{ if } \ y \neq x_s + Q^{s_0}_0 \tag{11}$$

$$F_{s_1}(y) \geq \mathbb{1}_{\{P^{s_0}_0\}} * F_{s'}(y) \ \text{ if } \ y \neq x_s + P^{s_0}_0. \tag{12}$$

For $y \neq x_{s_1}$, both inequalities are already true by induction (s_0, s' are the parents of s_1). For $y = x_{s_1}$, in view of (9), two cases may arise:

- If $x_s = P_0^{s'}$ then $x_{s_1} = x_s + Q_0^{s_0}$ so (11) is true, and (12) need only be checked at $y = x_{s_1}$. One has $F_{s_1}(x_{s_1}) = 0$ and
$$\mathbb{1}_{\{P_0^{s_0}\}} * F_{s'}(x_{s_1}) = F_{s'}(x_{s_1} - P_0^{s_0}) = F_{s'}(P_0^{s'} + 2Q_0^{s_0}).$$
However, (5) yields $\varphi_{s'}(P_0^{s'} + 2Q_0^{s_0}) = 2(p_0 q' - q_0 p') = -2$; hence, the point $(P_0^{s'} + 2Q_0^{s_0})$ does not belong to $J_{s'}$ and $\mathbb{1}_{\{P_0^{s_0}\}} * F_{s'}(x_{s_1}) = 0$.
- Similarly, if $x_s = Q_0^{s'}$ then (12) is true, and for (11) one need only check $\mathbb{1}_{\{Q_0^{s_0}\}} * F_{s'}(x_{s_1}) = F_{s'}(Q_0^{s'} + 2P_0^{s_0}) = 0$ because $\varphi_{s'}(Q_0^{s'} + 2P_0^{s_0}) = -2 < 0$.

It remains to prove $F_s \geq \sup \left\{ \mathbb{1}_{\{P_0^{s_1}\}} * F_{s_0}, \mathbb{1}_{\{Q_0^{s_1}\}} * F_{s_0} \right\}$ on J_s. It is enough to make sure that the contents of the brackets are even lower than the lower bounds on F_s we just established, i.e. to show that

$$\text{(13)} \qquad \mathbb{1}_{\{P_0^{s_1}\}} * F_{s_0} \leq \mathbb{1}_{\{P_0^{s_0}\}} * F_{s_1} \ ; \ \mathbb{1}_{\{Q_0^{s_1}\}} * F_{s_0} \leq \mathbb{1}_{\{Q_0^{s_0}\}} * F_{s_1} \quad \text{on } Z_s - \{x_s\}.$$

We focus only on the first inequality (the second is similar). It is equivalent, by the same method as above, to:

$$\mathbb{1}_{\{P_0^{s'}\}} * F_{s_0}(y) \leq F_{s_1}(y) \quad \text{if } y \neq x_s + Q_0^{s_0}$$

(we used $P_0^{s'} = P_0^{s_1} + Q_0^{s_0}$, a consequence of (5)). But that inequality is true (by induction) as long as $y \neq x_{s_1}$. Again, in view of (9), two cases may arise at $y = x_{s_1}$:

- If $x_s = P_0^{s'}$ then $x_{s_1} = x_s + Q_0^{s_0}$ and there is nothing to do;
- If $x_s = Q_0^{s'}$ we only need check the inequality above at $y = x_{s_1}$. On one hand, $F_{s_1}(x_{s_1}) = 0$; on the other,
$$\mathbb{1}_{\{P_0^{s'}\}} * F_{s_0}(x_{s_1}) = F_{s_0}(x_{s_1} - P_0^{s'}) = F_{s_0}(P_0^{s_0} + 2Q_0^{s'})$$
but, by (5), $\varphi_{s_0}(P_0^{s_0} + 2Q_0^{s'}) = 2(q_0 p' - p_0 q') = -2 < 0$ so the point $(P_0^{s_0} + 2Q_0^{s'})$ does not belong to J_{s_0} and $\mathbb{1}_{\{P_0^{s'}\}} * F_{s_0}(x_{s_1}) = 0$.

Theorem 1 is proved for all $s \in \mathcal{Q}$. □

4. Formal Markoff map

Figure 3 shows the domains J_s and the values of F_s for some of the simplest rationals $s \in \mathcal{Q}$. In each case, the points x of the affine lattice Z_s have been identified with the cells of a honeycomb, carrying the numbers $F_s(x)$. Empty cells carry 0, by convention. Coordinates have been tilted so that the edge $P_0^s Q_0^s$ of J_s is always at the top of J_s, rather than the bottom left as in Figure 1. The left edge of J_s consists of p cells (the P_i^s); the right edge, of q cells (the Q_j^s). The single cells to the bottom left and bottom right of the "root" (dark spot) correspond to the exceptional cases $s = 0$ and $s = \infty$. The single cell above the root corresponds to $s = -1$; the meaning of that convention, already apparent from the Introduction, will be re-emphasized in a moment. Observe the $1's$ in the corners of each J_s, just as in Lemma 8. It is an easy exercise (left to the reader) to prove by induction that the bottom, left, and right edges of each J_s (for $s \in \mathcal{Q} \smallsetminus \{0, \infty\}$) always carry full lines of the Pascal triangle: if $v = (2, -2)$ then

$$F_s(P_i^s) = \binom{p-1}{i} ; \ F_s(Q_j^s) = \binom{q-1}{j} ; \ F_s(Q_{q-1}^s + kv) = \binom{p+q-1}{k}.$$

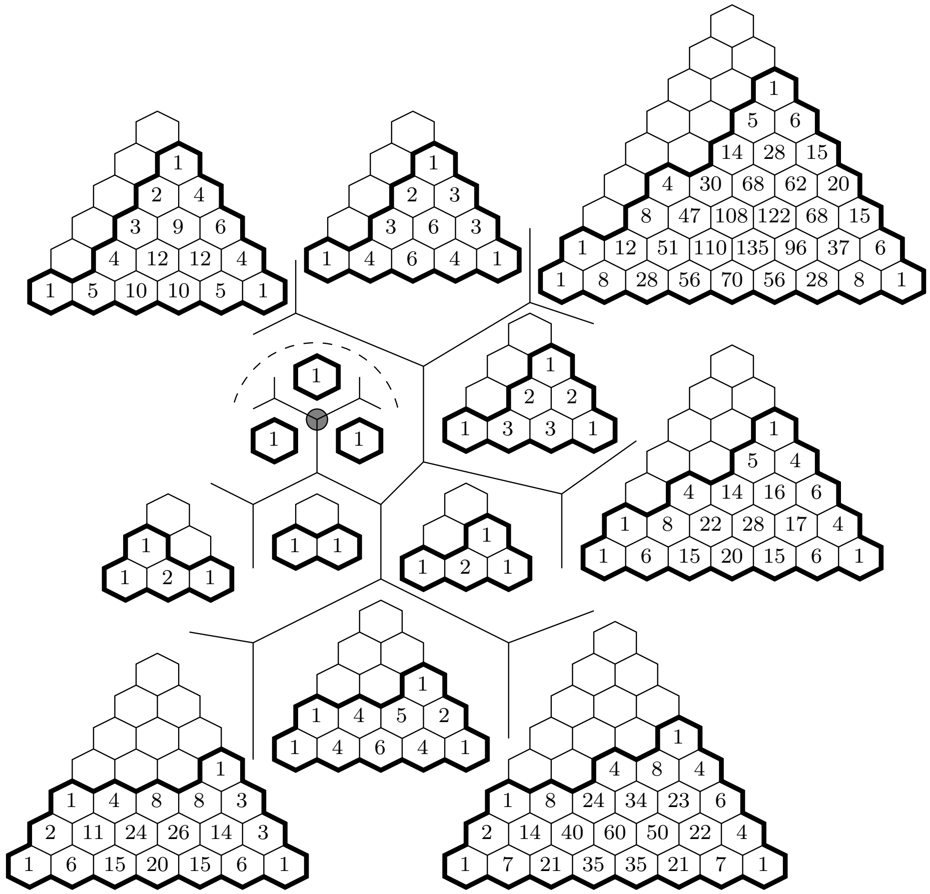

FIGURE 3. The universal (formal) Markoff map. The integers $F_s(\cdot,\cdot)$ inside each "bag" add up to a Markoff number.

Notice the arrangement of the various J_s in the complement U of a planar 3-valent tree: this tree should be seen as the dual of the Farey triangulation of $\mathbb{H}^2$, so each connected component R_s of U corresponds to a horosphere centered at a rational point s. Each configuration s, s_0, s_1, s' as in the previous section corresponds in fact to a pair of edge-adjacent components R_{s_0}, R_{s_1} of U, together with their two common neighbors $R_s, R_{s'}$. Since Formula (6) is symmetric in s, s', one may apply it backwards to define F_s for all s in $\mathbb{P}^1\mathbb{Q}$ (not just $\mathcal{Q}$). This was (very) partially done in Figure 3 by showing $J_{-1} = \{(-1,-1)\}$ just above the root. However, the full picture would exhibit a 6-fold dihedral symmetry around the root (dark spot), so only one sixth of the tree is explored to some depth in Figure 3. This 6-fold symmetry is also the reason why honeycombs were used instead of, say, square cells. As an exercise, the reader may prove the following formulas for the symmetry (true for all $s \in \mathbb{P}^1\mathbb{Q}$) by induction on the tree:

$$F_{\frac{1}{s}}(\alpha,\beta) = F_s(\beta,\alpha)\ ;\ F_{-1-s}(\alpha,\beta) = F_s(-2-\alpha-\beta,\beta)$$

(The Möbius transformations acting on the index s permute the rationals $-1, 0, \infty$ while the affine transformations acting on the argument (α, β) permute the associated singletons J_{-1}, J_0, J_∞, as well as the elements of $-\Lambda \subset \mathbb{Z}^2$).

5. Conjectural generalization

The Markoff polynomial $M = X^2 + Y^2 + Z^2 - XYZ$ encountered in Section 2 has degree 2 in each variable. This is why any solution (X, Y, Z) of the equation $M = 0$ defines many other solutions: by considering M as a polynomial of degree 2 in, say, the variable X, we can always replace X by the conjugate root. Thus, the free product G of three copies of $\mathbb{Z}/2\mathbb{Z}$ acts naturally on the variety $M = 0$ by isomorphisms. An analogous statement holds true if we replace M by *any* polynomial of degree 2 in all its variable (allowing for such monomials as X^2Y^2ZT), and allow for actions by birational isomorphisms.

In this section, we conjecture a generalization of Theorem 1 to all N-variable polynomials M which are *monic of degree* 2 in each variable. Namely, we show that certain expressions for the action of G are Laurent polynomials (as in Proposition 3), and conjecture that the coefficients are positive. The coefficients of M will be considered as variables themselves (noted A_I below), so we work in an algebraic closure $\mathbb{K}$ of the the field of rational fractions over $\mathbb{C}$ in the independent formal variables A_I. (Alternatively, fix any complex values for the A_I and take $\mathbb{K} = \mathbb{C}$.)

Let $N \geq 2$ be an integer, and denote by $[\![N]\!]$ the set of integers $\{1, 2, \ldots, N\}$. For each $I \subset [\![N]\!]$, fix a formal parameter A_I. Consider the Markoff-type equation in N variables $X_1 \ldots, X_N$:

$$\sum_{i=1}^{N} X_i^2 + \sum_{I \subset [\![N]\!]} A_I \prod_{i \in I} X_i = 0 \ . \tag{14}$$

Let $V \subset \mathbb{K}^N$ be the variety defined by (14). For each $k \in [\![N]\!]$ and each point $(x_1, \ldots, x_N)$ of $V \cap \mathbb{K}^{*N}$, define

$$\begin{array}{rcl} E_k(x_1, \ldots, x_N) & := & (x_1, \ldots, x_{k-1}, \overline{x_k}, x_{k+1}, \ldots, x_n) \\ \text{where} \qquad \overline{x_k} & = & \left(\displaystyle\sum_{i \neq k} x_i^2 + \sum_{I \subset [\![N]\!] - \{k\}} A_I \prod_{i \in I} x_i \right) \Big/ x_k \ . \end{array} \tag{15}$$

Then E_k defines a birational $\mathbb{Z}/2\mathbb{Z}$-action on V: indeed, $\overline{x_k} x_k$ is the product of the roots of (14), seen as a monic degree 2 polynomial in the k-th variable. By letting k range over $[\![N]\!]$, we obtain a birational action on V by the free product G of N copies of $\mathbb{Z}/2\mathbb{Z}$.

Observe that the variable $A_{[\![N]\!]}$ is absent from the definition (15) of each generator E_k: therefore, G acts on each "level manifold" of $\mathbb{K}^N$ defined by

$$B(x_1, \ldots, x_N) := \left(\sum_{i=1}^{N} x_i^2 + \sum_{I \subsetneq [\![N]\!]} A_I \prod_{i \in I} x_i \right) \Big/ \prod_{i=1}^{N} x_i \ = \ \text{constant} \ \in \mathbb{K} \tag{16}$$

(indeed, $B(x_1, \ldots, x_N)$ is just the value of $A_{[\![N]\!]}$ for which a given point $(x_1, \ldots, x_N)$ will satisfy (14), when all the $\{A_I\}_{I \subsetneq [\![N]\!]}$ are given). In particular, $B(x_1, \ldots, x_N)$ is invariant under the action of E_k on $\mathbb{K}^N$: therefore, the expression given in (15) for E_k extends to a birational involution of $\mathbb{K}^N$ respecting B. Henceforward, we consider G as acting on $\mathbb{K}^N$ by birational isomorphisms.

Proposition 11. *For each g in G and $x = (x_1, \dots, x_N)$ in $\mathbb{K}^N$, the coordinates of $g \cdot x$ are polynomials in the variables $\{x_i^{\pm 1}\}_{i \in [\![N]\!]}$ and $\{A_I\}_{I \subsetneq [\![N]\!]}$ with integer coefficients depending only on g.*

Proof. We work by induction in G, using the generators E_k. When g is the identity of G, we are done. Suppose the proposition is true for g, so that $g \cdot (x_1, \dots, x_N) = (y_1, \dots, y_N)$ where each y_j is a polynomial in the $\{x_i^{\pm 1}\}_{i \in [\![N]\!]}$ and $\{A_I\}_{I \subsetneq [\![N]\!]}$ with integer coefficients. We must prove that the coordinates of

$$E_k(y_1, \dots, y_N) = (y_1, \dots, \overline{y_k}, \dots, y_N)$$

are polynomials as well, where $\overline{y_k}$ is given as in (15). We saw that the left member $B(x_1, \dots, x_N)$ of (16) is (formally) E_k-invariant for each $k \in [\![N]\!]$; therefore we must have $B(x_1, \dots, x_N) = B(y_1, \dots, y_N)$. Using (15), note that

$$\begin{aligned} \overline{y_k} &= \left(\sum_{i \neq k} y_i^2 + \sum_{I \subset [\![N]\!] - \{k\}} A_I \prod_{i \in I} y_i \right) \Big/ y_k \\ &= \left(\left(B(y_1, \dots, y_N) \prod_{i=1}^{N} y_i \right) - y_k^2 - \sum_{\substack{I \subsetneq [\![N]\!] \\ k \in I}} A_I \prod_{i \in I} y_i \right) \Big/ y_k \\ &= B(x_1, \dots, x_N) \left(\prod_{i \in [\![N]\!] - \{k\}} y_i \right) - y_k - \sum_{\substack{I \subsetneq [\![N]\!] \\ k \in I}} A_I \prod_{i \in I - \{k\}} y_i \ . \end{aligned}$$

Using the formula (16) for $B(x_1, \dots, x_N)$, the last expression is clearly a polynomial in the variables $\{x_i^{\pm 1}\}_{i \in [\![N]\!]}$ and $\{A_I\}_{I \subsetneq [\![N]\!]}$ with integer coefficients. This is a direct analogue of (3). □

Conjecture 12. Trusting computerized experiments, we conjecture that these integers are positive. Theorem 1 corresponds to $N = 3$ under the specialization $A_I \equiv 0$: for example, $E_1(x, y, z) = (\frac{y^2 + z^2}{x}, y, z)$.

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